2-5596 Mechanika viazaných mechanických systémov (VMS)
pre špecializáciu Aplikovaná mechanika, 4.roč. zimný sem.
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Pojmy pre maticové metódy kinematickej analýzy VMS

Position of point L

Position $L \in b \equiv 2$ (see Fig.1) of point L from part (link) $b \equiv 2$ relative to reference part $a \equiv 1$ (ground) can be described by radius vector $\overline{r}_{aL} = \lceil x_{aL}, y_{aL}, z_{aL} \rceil$ with Cartesian position coordinates $L(x_{aL}, y_{aL}, z_{aL})$, or by radius vector in matrix notation $r_{aL} = \lceil x_{aL}, y_{aL}, z_{aL}, 1 \rceil$ with homogeneous position cordinates $L(x_1, x_2, x_3, x_4)$ scaled by factor $x_4 = 1$ in relations

$$x_{aL} = \frac{x_1}{x_4},\; y_{aL} = \frac{x_2}{x_4},\; z_{aL} = \frac{x_3}{x_4}$$

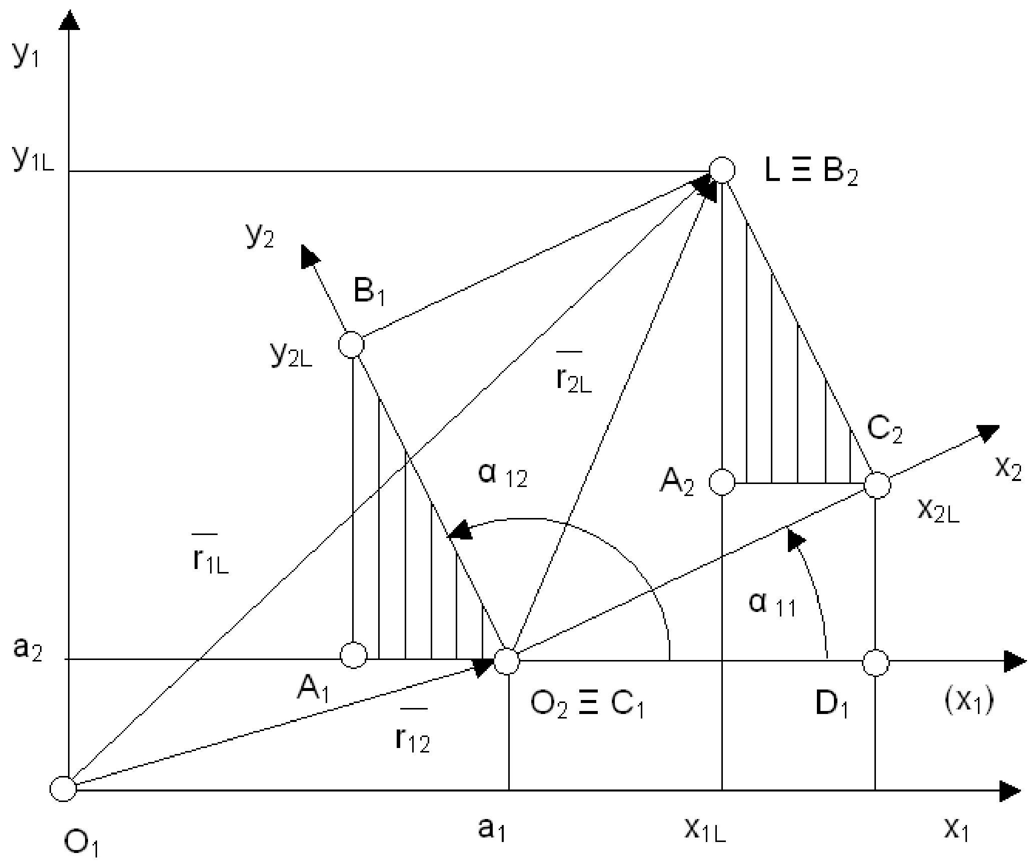

Fig.1 Position $L \in b \equiv 2$ of point L from part (link) $b \equiv 2$ relative to reference part $a \equiv 1$ (ground)

Transformations

The relation between projections of local position radius vector $\overline{r}_{bL}$ and global local position radius vector $\overline{r}_{aL}$ in matrix notation

$r_{\mathrm{bL}} = \lceil \mathrm{x}_{\mathrm{bL}}, \mathrm{y}_{\mathrm{bL}}, \mathrm{z}_{\mathrm{bL}}, 1 \rceil$, $r_{\mathrm{aL}} = \lceil \mathrm{x}_{\mathrm{aL}}, \mathrm{y}_{\mathrm{aL}}, \mathrm{z}_{\mathrm{aL}}, 1 \rceil$ can be expressed by transformation equation

$$\begin{bmatrix} \mathrm{x}_{\mathrm{aL}} \\ \mathrm{y}_{\mathrm{aL}} \\ \mathrm{z}_{\mathrm{aL}} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \mathrm{a}_{13} & \mathrm{a}_{1} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \mathrm{a}_{23} & \mathrm{a}_{2} \\ \mathrm{a}_{31} & \mathrm{a}_{32} & \mathrm{a}_{33} & \mathrm{a}_{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{x}_{\mathrm{bL}} \\ \mathrm{y}_{\mathrm{bL}} \\ \mathrm{z}_{\mathrm{bL}} \\ 1 \end{bmatrix} \tag{4.4}$$

where $\overline{\mathrm{r}}_{\mathrm{ab}} = \lceil \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3 \rceil$ is radius vector with Cartesian position coordinates of the origin O_{b} relative to the origin O_{a}.

Equation (4.4) can be written in more concise forms

$$\begin{bmatrix} \overline{\mathrm{r}}_{\mathrm{aL}} \\ 1 \end{bmatrix} = \begin{bmatrix} S_{\mathrm{ab}} & \overline{\mathrm{r}}_{\mathrm{ab}} \\ O & 1 \end{bmatrix} \begin{bmatrix} \overline{\mathrm{r}}_{\mathrm{bL}} \\ 1 \end{bmatrix} \tag{4.7}$$

where S_{ab} is orthonormal and intrinsic matrix of directional cosines (shortly matrix S_{ab} of directional cosines) in which elements are given by dot products of corresponding unit vectors. For example $\mathrm{a}_{11} = \cos\alpha_{11} = \mathrm{c}\alpha_{11} = \overline{\mathrm{i}}_{\mathrm{a}}.\overline{\mathrm{i}}_{\mathrm{b}}$.

Let we denote T_{ab} as a transformation matrix (transformation operator) of mutual position of local position frames of part b relative to reference part a (ground)

$$T_{\mathrm{ab}} = \begin{bmatrix} S_{\mathrm{ab}} & \overline{\mathrm{r}}_{\mathrm{ab}} \\ O & 1 \end{bmatrix} \tag{4.8}$$

Transformation equation (4.4) has a final form of a symbolic matrix equation

$$r_{\mathrm{aL}} = T_{\mathrm{ab}} r_{\mathrm{bL}} \tag{4.9}$$

with transformation of local position radius vector r_{bL} into global position radius vector r_{aL} with homogeneous position cordinates.

Multiplying matrices in Eq. (4.7) we obtain a equation of trajectory of point point L from part b relative to reference part a

$$\overline{\mathrm{r}}_{\mathrm{aL}} = S_{\mathrm{ab}} \overline{\mathrm{r}}_{\mathrm{bL}} + \overline{\mathrm{r}}_{\mathrm{ab}} \tag{4.10}$$

according to theory of decomposition of general motion $b : a$ into fictive translational motion of part b represented by reference point O_b and by fictive spherical motion of part b about the origin O_b. Trajectory of reference point O_b moving relative to reference part a is described by radius vector $\overline{r}_{ab}$. Trajectory of point L during fictive spherical motion of part b relative to reference part b is described by radius vector $\overline{r}_{bL}$ and relative to reference part a by term $S_{ab}\overline{r}_{bL}$.

Euler parameters

Euler angles (ψ, θ, φ) representation degenerates near $\theta = 0$ so to avoid of singularity it is more advantageous to describe spherical motion of part b relative to reference part a $(O_b \equiv O_a)$ by quaternion $\hat{e}_{ab}$ with four Euler parameters

$$\hat{e}_{ab} = e_0 + e_1\overline{i}_a + e_2\overline{j}_a + e_3\overline{k}_a \qquad (4.11)$$

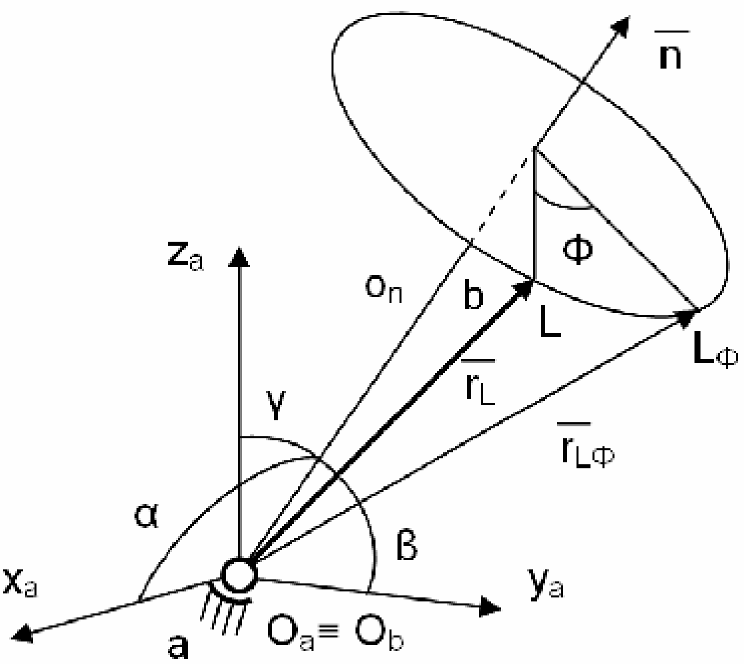

Fig.2 The radius vector $\overline{r}_L$, rep. $\overline{r}_{L\Phi}$ in the initial, resp. final position $\overline{r}_L$ of the point L during of spherical motion of part b relative to reference part a $(O_b \equiv O_a)$.

Quaternion $\hat{e}_{ab}$

The idea of quaternion $\hat{e}_{ab}$ is based on finite rotation of part b from initial position (Fig.2), represented by radius vector $\overline{r}_L$ of the point L to the final position $\overline{r}_{L\Phi}$ by an angle Φ around an axis o_n which position is described wrt global coordinate system of reference part a by directional angles (α, β, γ). The axis o_n is passing a fixed point O_a and it is the line of action of unit vector $\overline{n}$.

The quaternion $\hat{e}_{ab}$ can be expressed as sum:

$$\hat{e}_{ab} = e_0 + \bar{e} = \cos\frac{\Phi}{2} + \sin\frac{\Phi}{2}\bar{n} \tag{4.12}$$

where e_0 is scalar and $\bar{e}$ is vector quantity.

The quaternion $\hat{e}_{ab}$ can be also expressed in matrix notation as

$$\boldsymbol{E}_{ab} = \lceil e_0, \quad e_1, \quad e_2, \quad e_3 \rceil = \left\lceil c\frac{\Phi}{2}, c\alpha\, s\frac{\Phi}{2}, c\beta\, s\frac{\Phi}{2}, c\gamma\, s\frac{\Phi}{2} \right\rceil \tag{4.13}$$

and it can be calculated from matrix $\boldsymbol{S}_{ab}$ of directional cosines.

The general spatial motion of $b : a$ may be described by shortened dual quaternion

$$\hat{\boldsymbol{E}}_{ab} = [\hat{e}_{ab}, \bar{r}_{ab}] = [e_0, \quad e_1, \quad e_2, \quad e_3, a_1, a_2, a_3] \tag{4.14}$$

where $\bar{r}_{ab} = \lceil a_1, a_2, a_3 \rceil$ is radius vector of the origin O_b relative to the origin O_a.

Basic motions

Let local coordinate systems of parts b and a be coincident in the initial mutual position, then translations of $b : a$ along axes x_a, y_a, z_a designated by indices $Z1, Z2, Z3$ and rotations $b : a$ about axes x_a, y_a, z_a designated by indices $Z4, Z5, Z6$ are caled basic motions.

Basic matrices

Transformation matrices $\boldsymbol{T}_{Z1}(x)$, $\boldsymbol{T}_{Z2}(y)$, $\boldsymbol{T}_{Z3}(z)$, $\boldsymbol{T}_{Z4}(\varphi_x)$, $\boldsymbol{T}_{Z5}(\varphi_y)$, $\boldsymbol{T}_{Z6}(\varphi_z)$ of the basic motions (displacements) are caled a basic transformation matrices.

Simultaneous motion

Let we consider a simultaneous motions of two movable parts (part2, part3) in a open mechanism with reference part1 (ground) and point $L \in part3$. Using transformation equation (4.9): $\boldsymbol{r}_{aL} = \boldsymbol{T}_{ab}\boldsymbol{r}_{bL}$ we can write

$$\boldsymbol{r}_{1L} = \boldsymbol{T}_{12}\boldsymbol{r}_{2L}, \tag{4.15}$$

$$\boldsymbol{r}_{2L} = \boldsymbol{T}_{23}\boldsymbol{r}_{3L}, \tag{4.16}$$

Substituting (4.16) into (4.15) we obtain

$$\boldsymbol{r}_{1L} = \boldsymbol{T}_{12}\boldsymbol{T}_{23}\boldsymbol{r}_{3L} \tag{4.17}$$

we can also write

$$\boldsymbol{r}_{1L} = \boldsymbol{T}_{13}\boldsymbol{r}_{3L} \tag{4.18}$$

and comparing (4.17) with (4.18) we obtain a transformation matrix $\boldsymbol{T}_{13}$ of resulting motion of part3 : part1

$$\boldsymbol{T}_{13} = \boldsymbol{T}_{12}\boldsymbol{T}_{23} \tag{4.19}$$

where T_{12} is transformation matrix of simultaneous carrying motion and T_{23} is transformation matrix of simultaneous local relative motion.

Number c

Number c is total number of local position coordinates $q_i, i = 1, 2, ..., c$ of the links in the mechanism

$$c = \sum_{t=1}^{t_m} n_t s_t$$

and it is a sum $c = n + z$, where n is number of independet local position coordinates of the links (also n is mobility of mechanism)

$q_{n\,i}$, $i = 1, 2, ..., n$, and

z is number of dependet local position coordinates of the links

$q_{z\,i}$, $i = 1, 2, ..., z$.

Number z_S

The actual number $z_S \le 6$ of unknown dependet local position coordinates of the links in single loop (JM) spatial mechanism can be determined from a system of maximum number $z = 6$ mutually linear independet of constraint equations.

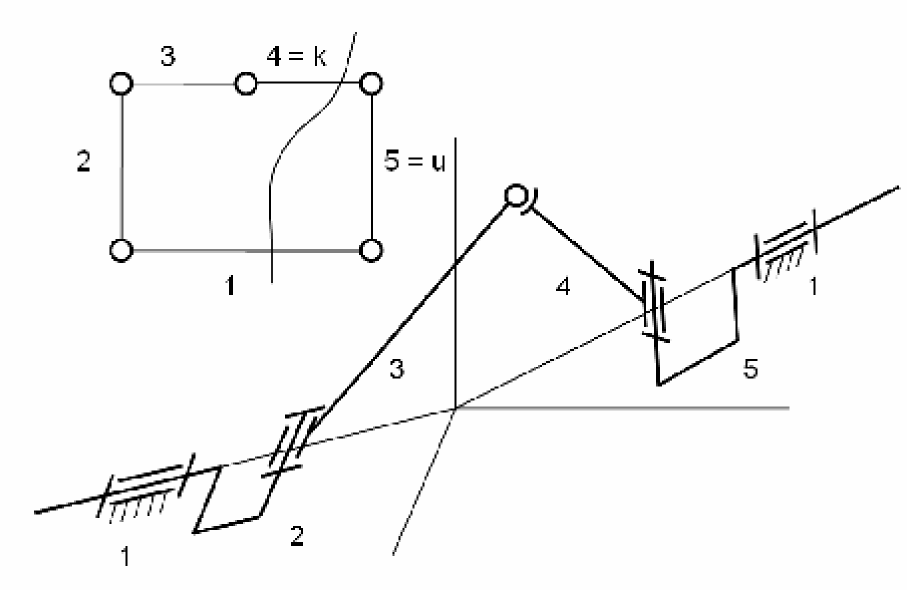

Fig.3 The single loop (JM) spatial mechanism

Virtual cut of JM

When the closed loop of links $1, 2, .., k, .., u$ in single loop (JM) spatial mechanism from Fig.3 is splitted by a virtual cut symbolically in space of link k into two open submechanisms OM1: $1, 2, .., k\text{-}1, k$ and OM2: $1, u, ..., k\text{+}1, k$, they both have the same end link k.

The position equations of arbitrary point $L \in k$ corresponding to open submechanisms OM1, OM2 have equal left sides

$$r_{1L} = T_{12}T_{23}...T_{k-1,k}r_{kL} \;,\; r_{1L} = T_{1u}T_{u,u-1}...T_{k+1,k}r_{kL} \,.$$

From comparison of right sides of equations yield

$$T_{12}T_{23}...T_{k-1,k}r_{kL} = T_{1u}T_{u,u-1}...T_{k+1,k}r_{kL} \tag{4.29}$$

Vanishing r_{kL} in eq. (4.29) we obtain matrix equation (4.30) of mutual position of links in single loop (JM) mechanism

$$T_{12}T_{23}...T_{k-1,k} = T_{1u}T_{u,u-1}...T_{k+1,k} \tag{4.30}$$

After multiplication of transformation matrices we can compare corresponding elements in the matrices of the rank (4x4) on left and right side of equation (4.31a)

$$\begin{bmatrix} a_{11L} & \otimes & \otimes & \otimes \\ a_{21L} & a_{22L} & \otimes & \otimes \\ a_{31L} & a_{32L} & a_{33L} & \otimes \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11R} & \otimes & \otimes & \otimes \\ a_{21R} & a_{22R} & \otimes & \otimes \\ a_{31R} & a_{32R} & a_{33R} & \otimes \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.31a}$$

which yield to a system of four identities and 12 nonlinear constraint algebric equations where q_{zj}, resp. q_{ni} are dependet, resp. independent local position coordinates of links in single loop (JM) mechanism and z_S, resp n_S are their actual numbers.

Proper elements

Because matrix S_{ab} of directional cosines is orthonormal and intrinsic, then in the eq. (4.31a) is possible to compare maximum number 3 of its elements and maximum number 2 elements from single column. By the symbols $\otimes$ is depicted one possible combination of elements selection for process of comparison. To obtain numerically stable solution of a system of maximum 6 nonlinear constraint algebric equations for single loop (JM) mechanism the most advantageous selection is combination with maximum value of corresponding equation determinant.

Velocities in OM

The transformation matrix is given by eq. TM (4.74):

$$T_{ab} = \prod_{i=1}^{m_{ab}} T_Z^i(p_{iab})$$

$$r_{1L} = \prod_{i=1}^{m_p} T_Z^i(p_i) r_{uL}$$

The time derivation of sum of transformation matrices we can generalize into form

$$U_i = T_Z^1 T_Z^2 \dots T_Z^i D_Z^i T_Z^{i+1} \dots T_Z^{m_p} \tag{4.76c}$$

Then the velocity v_{1L} of the point $L \in u$ will be

$$v_{1L} = (\sum_{i=1}^{m_p} U_i \dot{p}_i) r_{uL} \tag{4.77}$$

Accelerations in OM

The accelerations in OM we obtain after differentiation

$$a_{1L} = [(\sum_{i=1}^{m_p} U_i \ddot{p}_i + \sum_{k=1}^{m_p} U_{ik} \dot{p}_i \dot{p}_k)] r_{uL} \tag{4.79}$$

$$U_{ik} = T_Z^1 T_Z^2 \dots T_Z^i D_Z^i T_Z^{i+1} \dots T_Z^k D_Z^k T_Z^{k+1} \dots T_Z^{m_p} \tag{4.80}$$

$$U_{ii} = T_Z^1 T_Z^2 \dots T_Z^i (D_Z^i)^2 T_Z^{i+1} \dots T_Z^{m_p} \tag{4.81}$$

Velocities in JM

The matrix equation of mutual position of links in single loop (JM) mechanism splited by virtul cut is:

$$\prod_{i=1}^{m_p^1} T_Z^i(p_i) = \prod_{i=1}^{m_p} T_Z^i(p_i) \tag{4.83}$$

The matrix equation for velocities will be:

$$\sum_{i=1}^{m_p^1} U_i \dot{p}_i = \sum_{i=m_p^1+1}^{m_p} U_i \dot{p}_i \tag{4.84}$$

Accelerations in JM

The accelerations in JM we obtain after differentiation

$$\sum_{i=1}^{m_p^1} (U_i \ddot{p}_i + \sum_{k=1}^{m_p^1} U_{ik} \dot{p}_i \dot{p}_k) = \sum_{i=1}^{m_p} (U_i \ddot{p}_i + \sum_{i=m_p^1+1}^{m_p} U_{ik} \dot{p}_i \dot{p}_k) \tag{4.85}$$

Velocities in VM

The dveloped matrix equations for velocities of driven links in single loop (JM) mechanism we apply for each basic loop of a multiloop mechanism.

Accelerations in VM

The dveloped matrix equations for accelerations of driven links in single loop (JM) mechanism we apply for each basic loop of a multiloop mechanism.

Numeric analysis of position in JM

The actual number $z_S \le 6$ of unknown dependet local position coordinates of the links in single loop (JM) spatial mechanism can be determined numericaly from a system of maximum number $z = 6$ mutually linear independet but nonlinear constraint algebric equations.

According to general form of a matrix equation TM (4.30): $T_{12}T_{23}...T_{k-1,k} = T_{1u}T_{u,u-1}...T_{k+1,k}$ of mutual position of links in single loop (JM) mechanism based on idea of virtual cut we can write equation with basic matrices. For numeric solution of position of

$$T_1(s_1)T_2(s_2)...T_k(s_k) = T_{k+1}(s_{k+1})...T_a(s_a) \qquad (4.89)$$

We can substitute each transformation matrix by product of constant C_i and variable P_i matrices.

$$T_{ai}(s_i) = C_i P_i(s_i) \qquad (4.90)$$

$$C_i P_i(s_1) C_2 P_2(s_2)...C_k P_k(s_k) = C_{k+1} P_{i_{k+1}}(s_{k+1})...C_a P_a(s_a) \qquad (4.91)$$

Goal of the position kinematic analysis of the spatial mechanism with known arithmetic vector $\bar{q}_n = \lceil q_{n1},...,q_{nn} \rceil$ of independet global position coordinates of the input link/or links is to determine the time course for number $z = n_v k$ of unknown dependet global position coordinates of ouput links in the arithmetic vector $\bar{q}_z = \lceil q_{z1},...,q_{zz} \rceil$ solving nonlinear algebric constraint equations $f_i(q_{z1},...,q_{zd}) = 0$, $i = 1,2,...,z$ by numerical iteration Newton-Raphson (N-R) method.
In the initial configuration of mechanism the nonlinear algebric constraint equations $\bar{f}$ can be linearized by approximation with the sum of residual functions $\bar{f}_{(r)}$ obtained by introducing the arithmetic vector $\bar{q}_{z(r)} : \bar{s}_{(1)} = \lceil s_1, s_2,...,s_s \rceil_{(1)}$ of estimated dependet global position coordinates (r is number of iteration step) and linear terms of Taylor series $\bar{f} \cong \bar{f}_{(r)} + V_{(r)} \Delta \bar{q}_{z(r)}$, where matrix $V_{(r)}$ is Jacobi matrix of the rank $(z \text{ x } z)$

$V_{(r)} = \left[\dfrac{\partial f_i}{\partial q_{zj}} \right]_{(r)}$, $i = 1,2,...,c$, and $j = 1,2,...,z$, and

$\Delta \bar{q}_{z(r)} = \Delta \bar{s}_{(1)}$ is unknown arithmetic vector of corrections.

$$(C_i P_i C_2 P_2 ... C_k P_k)_{(r)} + \sum_{i=1}^{k} (C_i P_i,...C_i P_i D_Z^i ... C_k P_k)_{(r)} \Delta s_i =$$

$$(C_{k+1} P_{i_{k+1}}(s_{k+1})...C_a P_a(s_a))_{(r)} + \sum_{i=1}^{k} (C_{k+1} P_{i_{k+1}}(s_{k+1})...C_s P_s(s_s))_{(r)} \Delta s_i$$

The arithmetic vector $\bar{f}$ of nonlinear algebric constraint equations is $\bar{f} = 0$ after introducing the solution $\bar{q}_z$, but $\bar{f} \neq \bar{f}_{(r)}$, $\bar{f}_{(r)} \neq 0$ (residuals) after introducing the arithmetic vector $\bar{q}_{z(r)} = \bar{s}_{(1)}$ of estimated dependet global position coordinates. The arithmetic vector $\bar{q}_{z(r)}$ will be the accepted solution, if residuals $\bar{f}_{(r)} \le \bar{f}_{Tj}$, where $\bar{f}_{Tj}$ are prescribed tolerancies. This can be achieved by iteration proces of converging numerical Newton-Raphson method

$$\bar{q}_{z(r+1)} = \bar{q}_{z(r)} + \Delta\bar{q}_{z(r)} \tag{4.99}$$

which will finish, if corrections $\Delta q_{z(r)}$ meet the condition $\Delta q_{z(r)} \le x_{Tj}$ where x_{Tj} are prescribed tolerancies.

$$M_1 \Delta q_{Z1} + \ldots + M_Z \Delta q_{ZZ} = N_o + N_1 \Delta q_{n1} + \ldots + N_Z \Delta q_{nn} \tag{4.95}$$

$$M \Delta\bar{q}_Z = N_o + N \Delta\bar{q}_n \tag{4.96}$$

For given inputs is known

$$\bar{q}_n = \bar{0} \tag{4.97}$$

then

$$M \Delta\bar{q}_Z = N_o \tag{4.98}$$

This equation is similar to the equation TM(2.7)

$$V_{(r)} \Delta\bar{y}_{z(r)} = -\bar{f}_{(r)}$$

Numeric analysis of of velocities in VM

Velocities $\dot{q}_z$ of links in VM we can determine after time derivation of equation (4.91):

$$C_i P_i(s_1) C_2 P_2(s_2) \ldots C_k P_k(s_k) = C_{k+1} P_{i_{k+1}}(s_{k+1}) \ldots C_a P_a(s_a).$$

We obtain

$$\sum_{i=1}^{k} C_i P_i, \ldots C_i P_i D_Z^i \ldots C_k P_k \dot{s}_i = \sum_{i=1}^{k} C_{k+1} P_{i_{k+1}} \ldots C_i P_i D_z^i \ldots C_s P_s \dot{s}_i \tag{4.101a}$$

Separating velocities $\dot{s}_i$ in the Eq.(4.101a) we obtain

$$M_1 \dot{q}_{Z1} + \ldots + M_Z \dot{q}_{ZZ} = N_1 \dot{q}_{n1} + \ldots + N_Z \dot{q}_{nn} \tag{4.101b}$$

Selecting a proper set of 6 lineary independent scalar equations we obtain equation

$$M \dot{q}_Z = N \dot{q}_n \tag{4.102}$$

from which we can determine unknown arithmetic vector $\dot{q}_z$ of dependet velocities.

Numeric analysis of of accelerations in VM

Accelerations $\ddot{q}_z$ of the links in VM we determine by time derivation of equation (4.102): $M\dot{q}_Z = N\dot{q}_n$

$$M\ddot{q}_z + K\dot{q}_z = N\ddot{q}_n + L\dot{q}_n \tag{4.103}$$

Where

$$K = \sum_{b=1}^{z} \frac{\partial M}{\partial q_z}\dot{q}_z + \sum_{a=1}^{n} \frac{\partial M}{\partial q_n}\dot{q}_n \tag{4.104}$$

$$L = \sum_{b=1}^{z} \frac{\partial N}{\partial q_z}\dot{q}_z + \sum_{a=1}^{n} \frac{\partial N}{\partial q_n}\dot{q}_n \tag{4.105}$$

Rank h of a matrix M

Let us denote by h the rank of a matrix M of type (z, z) from equation (4.102): $M\dot{q}_Z = N\dot{q}_n$ for a mechanism with k of basic loops, mobility n, and number $\mathrm{z} = \mathrm{n_V k}$ of dependet position coordinates of links.

Actual mobility $\mathrm{n_S}$

Mechanisms have actual mobility $\mathrm{n_S} = \mathrm{n}$ if fulfilles condition

$$\mathrm{h} = \mathrm{z} \tag{4.108}$$

Augmented matrix R

Let us denote by R the augmented matrix

$$R = (M, N) \tag{4.109}$$

having the rank $\mathrm{h_R}$

Unremoved DOF

In mechanism is unremoved $\mathrm{n_N}$ DOF if

$$\mathrm{h} = \mathrm{h_R} < \mathrm{z} \tag{4.110}$$

so actual mobility $\mathrm{n_S}$ is then $\mathrm{n_S} = \mathrm{n} + \mathrm{n_N}$. If condition (4.110) is valid for whole cycle of mechanism, then mechanism is in permanent singular state. If condition (4.110) is fulfilled for specific configuration of links, then mechanism is in instantaneous singular state.

Zero natrix N

For VMS with mobility $\mathrm{n_S} \le 0$ is natrix $N = \mathrm{O}$ and actual mobility $\mathrm{n_S}$ can be determined using rank h of a matrix M.