Problems in gravitational field

1. *How far is Mars from the Sun if its orbital period is* $T_M = 1{,}9\,\mathrm{y}$ *and the distance between the Sun and the Earth is* $a_E = 1\,\mathrm{AU}$*?*

Kepler's third law can be used

$$\frac{T_E^2}{T_M^2} = \frac{a_E^3}{a_M^3}\ ,$$

which implies

$$a_M^3 = a_E^3 \frac{T_M^2}{T_E^2}\ .$$

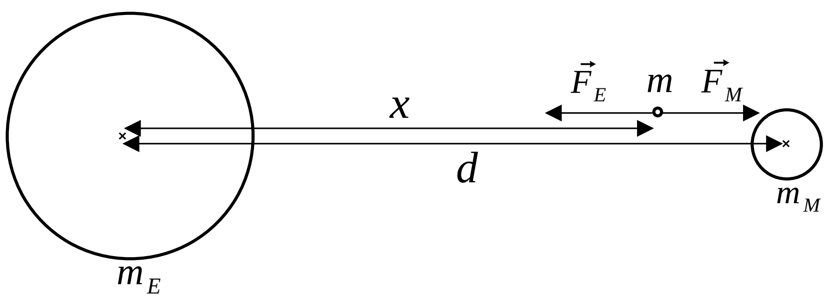
Therefore, the distance of Mars from the Sun is

$$a_M = a_E \sqrt[3]{\frac{T_M^2}{T_E^2}}\ ,$$

when substituted

$$a_M = 1\,\mathrm{AU} \sqrt[3]{\frac{(1{,}9\,\mathrm{y})^2}{(1\,\mathrm{y})^2}} = 1\,\mathrm{AU}\sqrt[3]{1{,}9^2} = 1{,}53\,\mathrm{AU}\ .$$

2. *The spacecraft is located between the Earth and the Moon, how far away from the Earth should the spacecraft be so that the resulting gravitational force on it from the Earth and the Moon is zero? The distance between the Earth and the Moon is* $d = 384\,000\,\mathrm{km}$ *and the mass of the Earth is 81 times the mass of the Moon.*



Obr. 1

Newton's law of gravity can be used

$$\vec{F} = -\kappa \frac{m_1 m_2}{r^3} \vec{r} \, .$$

The resulting gravitational force must be zero

$$\vec{F}_E + \vec{F}_M = 0 \, ,$$

because the forces are in opposite directions, their magnitudes must be equal

$$F_E = F_M \, ,$$

thus, for forces acting on a spacecraft of mass m located at a distance x from the Earth, the following must be true

$$\kappa \frac{m_E m}{x^2} = \kappa \frac{m_M m}{(d-x)^2} \, .$$

If the ratio between the masses of the Earth and the Moon is

$$m_E = 81 m_M \, ,$$

then it applies

$$\frac{81}{x^2} = \frac{1}{(d-x)^2}$$

and the distance of the spacecraft from Earth will be

$$x = \frac{9}{10} d \, ,$$

when substituted

$$x = \frac{9}{10} \, . \, 384\,000\,\mathrm{km} = 345\,600\,\mathrm{km} \, .$$

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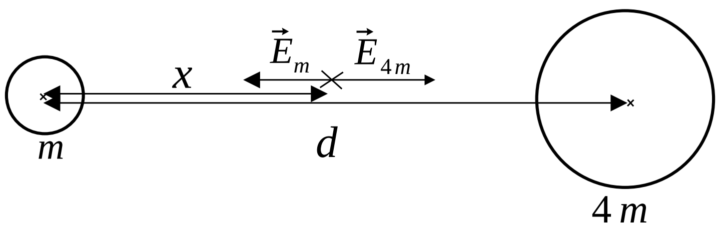
3. *Two spherical bodies with masses m and $4m$ are at a distance d from each other. At what point between them will the resulting gravitational field be zero, and what will be the potential of the gravitational field at that point?*



Obr. 2

The gravitational field of a spherical body of mass m at a point with position vector $\vec{r}$ is

$$\vec{E} = -\kappa \frac{m}{r^3} \vec{r} \, .$$

If the resulting gravitational field at the point between the two bodies is zero

$$\vec{E}_m + \vec{E}_{4m} = 0 \, ,$$

since the intensities are in opposite directions, their magnitudes must be equal

$$E_m = E_{4m} \, ,$$

therefore must apply

$$\kappa \frac{m}{x^2} = \kappa \frac{4m}{(d-x)^2} \, ,$$

from where the distance to a body of mass m can be expressed

$$x = \frac{d}{3} \, .$$

The gravitational potential

$$V = -\kappa \frac{m}{r} \, ,$$

is a scalar quantity and the resulting potential is the sum of the potentials at a given location from the individual bodies

$$V = V_m + V_{4m} \, ,$$

therefore, the resulting potential will be

$$V = -\kappa \frac{m}{x} - \kappa \frac{4m}{d-x} = -\kappa \frac{3m}{d} - \kappa \frac{12m}{2d} = -\kappa \frac{9m}{d} \, .$$

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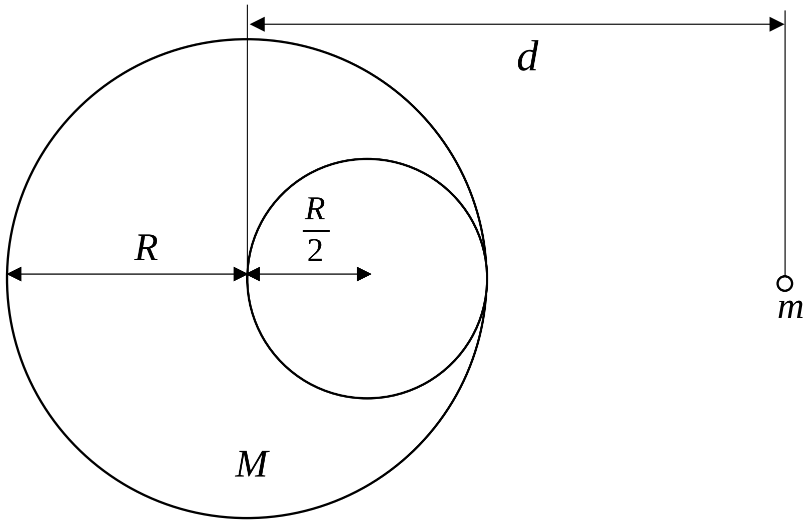
4. *From a homogeneous sphere of radius R and mass M, a new body was created by drilling a spherical cavity into the sphere with radius $R/2$ and centered at a distance $R/2$ from the center of the original sphere. What will be the gravitational force exerted by the new body on a point of mass m located in the direction of the cavity at a distance d from the centre of the original sphere?*



Obr. 3

The gravitational force exerted on the mass point by the original sphere $\vec{F}_0$ was the sum of the gravitational force exerted by the new body $\vec{F}$ and the gravitational force exerted by the drilled part $\vec{F}'$, therefore

$$\vec{F}_0 = \vec{F} + \vec{F}' \,.$$

The magnitude of the gravitational force exerted by the original sphere can be expressed from Newton's law of gravitation as

$$F_0 = \kappa \frac{mM}{d^2} \,.$$

The density of the material is

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} \,,$$

therefore, the weight of the drilled part was

$$m' = \rho V' = \frac{M}{\frac{4}{3}\pi R^3} \, \frac{4}{3}\pi \left(\frac{R}{2}\right)^3 = \frac{M}{8}$$

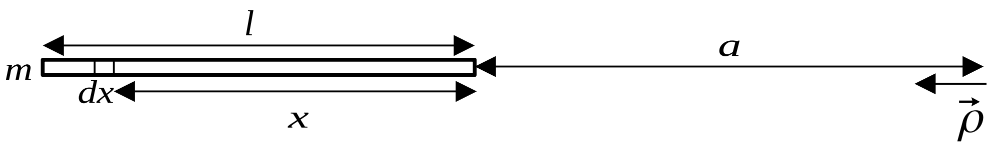
and the magnitude of the gravitational force exerted by the drilled part was

$$F' = \kappa \frac{mm'}{\left(d - \frac{R}{2}\right)^2} = \frac{mM}{8\left(d - \frac{R}{2}\right)^2} \,.$$

The gravitational force exerted by a new body can be expressed as

$$F = F_0 - F' = \kappa \frac{mM}{d^2} - \kappa \frac{mM}{8\left(d - \frac{R}{2}\right)^2} = \kappa m M \left[\frac{1}{d^2} - \frac{1}{8\left(d - \frac{R}{2}\right)^2}\right] \,.$$

5. *Calculate the potential and the gravitational field of a rod of mass m and length l at a point lying on an extension of the rod at a distance a from its end.*



Obr. 4

The length density of the rod is

$$\lambda = \frac{m}{l} \ ,$$

therefore the mass element will be

$$\mathrm{d}m = \lambda \mathrm{d}x = \frac{m}{l}\mathrm{d}x$$

and the potential of the mass element will be

$$\mathrm{d}V = -\kappa \frac{\mathrm{d}m}{x+a} = -\kappa \frac{m}{l}\frac{\mathrm{d}x}{x+a} \ .$$

The potential of the whole rod can be calculated by integration over the whole mass of the rod

$$V = \int_m \mathrm{d}V = -\kappa \frac{m}{l}\int\limits_0^l \frac{\mathrm{d}x}{x+a} = -\kappa \frac{m}{l}\left[\ln\left(x+a\right)\right]_0^l = -\kappa \frac{m}{l}\ln \frac{l+a}{a} \ .$$

The relationship between the gravitational field and the gravitational potential is

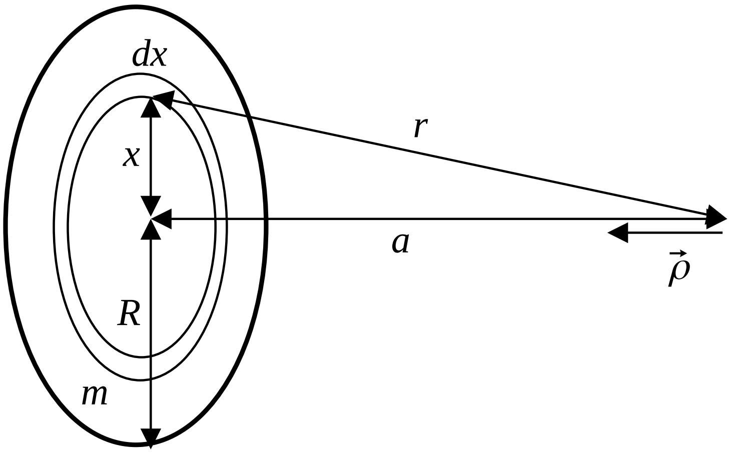
$$\vec{E} = -\mathrm{grad}\, V \ ,$$

therefore the gravitational field of the whole bar will be

$$\vec{E} = -\frac{\mathrm{d}V}{\mathrm{d}a}\vec{\rho} = -\kappa \frac{m}{l}\frac{a}{l+a}\frac{a-l-a}{a^2}\vec{\rho} = \kappa \frac{m}{a(l+a)}\vec{\rho} \ .$$

where $\vec{\rho}$ is the unit vector in the direction of the gravitational field. The rules for the derivative of the composite function and the derivative of the fraction of functions were used.

6. *Calculate the potential and the gravitational field of a disk of mass m and radius R at a point on the axis of the disk at a distance a from its centre.*



Obr. 5

The areal density of the disk is

$$\sigma = \frac{m}{S} = \frac{m}{\pi R^2}$$

and the element of the area of the intermediate circle is

$$\mathrm{d}S = 2\pi x \mathrm{d}x \ ,$$

then the mass element will be

$$dm = \sigma \mathrm{d}S = \frac{m}{\pi R^2} 2\pi x \mathrm{d}x = \frac{2m}{R^2} x \mathrm{d}x \ .$$

The magnitude of the position vector can be expressed using the Pythagorean theorem

$$r^2 = x^2 + a^2 \ ,$$

the potential of the mass element will be

$$dV = -\kappa \frac{dm}{r} = -\kappa \frac{2m}{R^2} x \frac{\mathrm{d}x}{r} = -\kappa \frac{2m}{R^2} x \frac{\mathrm{d}x}{\sqrt{x^2 + a^2}} \ .$$

The potential of the whole disk can be calculated by integrating over the whole mass of the disk

$$V = \int_m \mathrm{d}V = -\kappa \frac{2m}{R^2} \int\limits_0^R \frac{x \mathrm{d}x}{\sqrt{x^2 + a^2}} = -\kappa \frac{2m}{R^2} [\sqrt{x^2 + a^2}]_0^R = -\kappa \frac{2m}{R^2} \left(\sqrt{R^2 + a^2} - a \right) \ .$$

The relationship between the gravitational field and the gravitational potential is

$$\vec{E} = -\mathrm{grad}\, V \ ,$$

therefore the gravitational field of the whole disk will be

$$\vec{E} = -\frac{\mathrm{d}V}{\mathrm{d}a}\vec{\rho} = -\kappa\frac{2m}{R^2}\left(\frac{a}{\sqrt{R^2+a^2}} - 1\right)\vec{\rho} = \kappa\frac{2m}{R^2}\left(1 - \frac{a}{\sqrt{R^2+a^2}}\right)\vec{\rho} \ .$$

where $\vec{\rho}$ is the unit vector in the direction of the gravitational field.

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7. *At what speed must a body be thrown from the surface of the Earth to fly beyond the range of the Earth's gravitational pull?*

When a body flies out of the Earth's gravitational pull and comes to rest, it will have both zero gravitational potential energy and zero kinetic energy. The law of conservation of mechanical energy will therefore take the form

$$E_p + E_k = 0 \ .$$

The gravitational potential energy of a body of mass m on the surface of the Earth is

$$E_p = -\kappa\frac{mM_E}{R_E} \ ,$$

where M_E is the mass of the Earth and R_E is the radius of the Earth. The kinetic energy of a body ejected at velocity v from the surface of the Earth is

$$E_k = \frac{mv^2}{2} \ ,$$

therefore the law of conservation of mechanical energy implies

$$-\kappa\frac{mM_E}{R_E} + \frac{mv^2}{2} = 0 \ ,$$

from which it is possible to express the velocity of the body as

$$v = \sqrt{\frac{2\kappa M_E}{R_E}} \ .$$

The result can also be expressed using the gravitational acceleration

$$g = \kappa\frac{M_E}{R_E^2} \ ,$$

which for the velocity of the body implies

$$v = \sqrt{2gR_E}$$

and after inserting the numerical values

$$v = \sqrt{2 \,.\, 9{,}81\,\mathrm{m\,s^{-2}} \,.\, 6\,378\,000\,\mathrm{m}} = 11\,186\,\mathrm{m\,s^{-1}} \;.$$

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8. *The projectile was fired from the Earth's surface at a velocity of $v = 1600\,\mathrm{m\,s^{-1}}$. Calculate the difference in altitudes the body would have reached assuming the gravitational field is homogeneous and assuming the gravitational field is radial.*

The solution can be found using the law of conservation of mechanical energy

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \;.$$

In a homogeneous gravitational field, the potential energy at the Earth's surface E_{p1} can be chosen as the point with zero potential energy, and the body's velocity decreases until the body comes to rest, so its kinetic energy E_{k2} will be zero. The law of conservation of mechanical energy will therefore take the form

$$E_{k1} = E_{p2} \;,$$

thus applies

$$\frac{mv^2}{2} = mgh_h \;,$$

from which it is possible to express the altitude of the projectile in a homogeneous gravitational field as

$$h_h = \frac{v^2}{2g} \;.$$

In a radial gravitational field, the potential energy E_{p1} will be at the surface of the Earth and at h_r the potential energy E_{p2} will be at the height h_r. Therefore, the law of conservation of mechanical energy in a radial gravitational field will be

$$E_{k1} + E_{p1} = E_{p2} \;,$$

thus applies

$$\frac{mv^2}{2} - \kappa\frac{mM_E}{R_E} = -\kappa\frac{mM_E}{R_E + h_r} \;,$$

using the gravitational acceleration

$$g = \kappa\frac{M_E}{R_E^2} \;,$$

the relation can be modified to the form

$$\frac{mv^2}{2} = mgR_E \frac{h_r}{R_E + h_r} ,$$

from which it is possible to express the altitude of the projectile in the radial gravitational field as

$$h_r = \frac{v^2 R_E}{2gR_E - v^2} .$$

The difference of altitudes in the homogeneous and radial field will therefore be

$$\Delta h = h_r - h_h = \frac{v^2 R_E}{2gR_E - v^2} - \frac{v^2}{2g} ,$$

and after inserting the numerical values

$$\Delta h = \frac{(1600\,\mathrm{m\,s^{-1}})^2 \,.\, 6{,}378 \cdot 10^6\,\mathrm{m}}{2 \,.\, 9{,}81\,\mathrm{m\,s^{-2}} \,.\, 6{,}378 \cdot 10^6\,\mathrm{m} - (1600\,\mathrm{m\,s^{-1}})^2} - \frac{(1600\,\mathrm{m\,s^{-1}})^2}{2 \,.\, 9{,}81\,\mathrm{m\,s^{-2}}} = 2725\,\mathrm{m} .$$

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9. *Calculate the kinetic energy of a body with mass* $m = 70\,\mathrm{kg}$ *that hits the surface of the Earth from a height* $h = 10\,\mathrm{km}$*, if the Earth's gravitational field is assumed to be radial.*

The solution can be found using the law of conservation of mechanical energy

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} .$$

If the initial kinetic energy of the body E_{k1} is zero, the law of conservation of mechanical energy takes the form

$$E_{p1} = E_{k2} + E_{p2} .$$

The potential energy of a body at a height h above the surface of the Earth is equal to

$$E_{p1} = -\kappa \frac{mM_E}{R_E + h}$$

and the potential energy of a body on the Earth's surface is equal to

$$E_{p2} = -\kappa \frac{mM_E}{R_E} .$$

The law of conservation of mechanical energy will therefore take the form

$$-\kappa \frac{mM_E}{R_E + h} = E_{k2} - \kappa \frac{mM_E}{R_E} ,$$

which implies for the kinetic energy at impact

$$E_k = -\kappa m M_E \left(\frac{1}{R_E + h} - \frac{1}{R_E} \right) .$$

Using gravitational acceleration

$$g = \kappa \frac{M_E}{R_E^2}$$

the kinetic energy can be expressed as

$$E_k = mgR_E \frac{h}{R_E + h} ,$$

after inserting the numerical values

$$E_k = 70\,\mathrm{kg}\ .\ 9{,}81\,\mathrm{m\,s^{-2}}\ .\ 6{,}378 \cdot 10^6\,\mathrm{m}\ .\ \frac{10^4\,\mathrm{m}}{6{,}378 \cdot 10^6\,\mathrm{m} + 10^4\,\mathrm{m}} = 6{,}856 \cdot 10^6\,\mathrm{J}\ .$$

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10. *How high does a satellite have to be above the equator to be over the same place all the time as it moves?*

A satellite with mass m moves on a circle with radius $R_E + h$, the gravitational force of the Earth acts on the satellite as a centripetal force, thus

$$F_g = F_c ,$$

which can be rewritten into the form

$$\kappa \frac{m M_E}{(R_E + h)^2} = m \frac{v^2}{R_E + h} ,$$

where M_E is the mass and R_E is the radius of the Earth. For the speed of the satellite it follows

$$v = \sqrt{\frac{\kappa M_E}{R_E + h}} .$$

For a satellite to be over the same place all the time, its angular velocity must be the same as the angular velocity of the Earth

$$\omega_s = \omega_E ,$$

therefore must apply

$$\frac{v}{R_E + h} = \frac{2\pi}{T_E} ,$$

after inserting the speed of the satellite

$$\sqrt{\frac{\kappa M_E}{(R_E+h)^3}} = \frac{2\pi}{T_E} \ .$$

For the height of the satellite it follows

$$h = \sqrt[3]{\frac{\kappa M_E T_E^2}{4\pi^2}} - R_E \ ,$$

after inserting the numerical values

$$h = \sqrt[3]{\frac{6{,}67 \cdot 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \ . \ 5{,}972 \cdot 10^{24}\,\mathrm{kg} \ . \ (3{,}1536 \cdot 10^{7}\,\mathrm{s})^2}{4\pi^2}} - 6{,}371 \cdot 10^6\,\mathrm{m} \ ,$$

the height of the satellite will be

$$h = 35{,}8 \cdot 10^6\,\mathrm{m} \ .$$