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TECHNICAL PHYSICS – THE COLLECTION OF SOLVED PROBLEMS

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Preface

Technical physics — The collection of solved problems is a textbook for students of the first and second year of bachelor's study at the Faculty of Mechanical Engineering of the Slovak University of Technology in Bratislava. The textbook is used to practice and supplement the curriculum of the compulsory courses Technical Physics I and Technical Physics II, as well as the elective courses Physics Seminar II and Physics Seminar III. The textbook contains 90 solved problems from classical physics, covering the areas of mechanics, thermodynamics, electricity, magnetism, oscillations, waves and optics. The selection of the problems is in line with the content of the courses mentioned above, thus, they correspond in their difficulty to an introductory university course in the scope of two semesters at a technical university. The Slovak edition [1] of this textbook was published in 2023. After two years of successful use among Slovak students, there was a need to create the English edition for students studying in English. The textbook responds to the strong need for students to have a collection of sample solutions to problems to facilitate their study and preparation for exams.

In preparing the textbook, it was necessary to decide whether to include a theoretical introduction at the beginning of each chapter that would summarise the basic theoretical knowledge needed to solve the problems. Since the explanation of the theory is sufficiently covered in university textbooks [2, 3, 4], students can also use some of the world-famous physics textbooks [5, 6, 7, 8, 9, 10] or refer to the evergreen physics classics [11], theoretical introductions were not included in this textbook. The second question was whether the problems would be general or also include specific numerical values. Given that this is a collection of problems intended for engineering students, it is preferable that numerical values are included in the problems and that the problems are solved with a finite numerical result. There is also the advantage of practising numerical calculations, as this ability has been declining among students in recent years, it is common to find that "as many results as students" when more difficult mathematical expressions are added. The general decline in mathematical ability is also the reason why intermediate steps are given for more complex mathematical steps.

It is advisable to use the textbook in such a way that after reading the assignment of the problem, the students first try to solve the problem independently. Solving the example independently, i.e. without looking at the model solution, allows the students to check that they understand the problem, can apply the physics knowledge to the solution, and finally can solve similar problems independently in an exam. Only when the students conclude that solving the problem is beyond their capabilities is it a good time to look at the typical solution given in this textbook. It is usual for physics problems that there is not just one correct way to solve the problem. For example, some problems in mechanics can be solved using Newton's force law or using the law of conservation of mechanical energy. Another example is some problems in magnetism, which can be solved using the Biot-Savart-Laplace law or Ampère's law of total current. Therefore, even if a student solves a problem by a procedure other than that given in this textbook, his solution may be correct.

If students want to practice and test their knowledge and skills on other problems, it is advisable to reach for a more comprehensive collection of problems from university physics [12, 13, 14], or try to solve other problems that are published on the website of the Institute of Mathematics and Physics of the Faculty of Mechanical Engineering STU [15].

The authors are aware that, despite their best efforts and repeated checking, many errors have made their way into the textbook. Therefore, they will be very grateful to anyone who brings these errors to their attention and sends them information about the errors to the e-mail address: jozef.leja@stuba.sk

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1 Kinematics of a point particle

1.1 *The position vector of a point particle depends on time according to the relation* $\vec{r} = \vec{i}\,A\cos bt + \vec{j}\,A\sin bt$, *where* $A = 5\,\mathrm{m}$, $b = \pi/4\ \mathrm{s}^{-1}$. *Express its components, coordinates, magnitude and direction cosines at any time and at time* $t = 2\,\mathrm{s}$.

The position vector can be decomposed into components

$$\vec{r} = \vec{x} + \vec{y}\ ,$$

where the components of the position vector are

$$\vec{x} = \vec{i}\,A\cos bt\ ,$$

$$\vec{y} = \vec{j}\,A\sin bt\ .$$

At time $t = 2\,\mathrm{s}$, the components of the position vector have the values

$$\vec{x} = \vec{i}\,5\,\mathrm{m}\ .\cos\!\big(\pi/4\ \mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\big) = 0\,\vec{i}\ ,$$

$$\vec{y} = \vec{j}\,5\,\mathrm{m}\ .\sin\!\big(\pi/4\ \mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\big) = 5\,\mathrm{m}\,\vec{j}\ .$$

The position vector can be written using coordinates

$$\vec{r} = x\vec{i} + y\vec{j}\ ,$$

where the coordinates of the position vector are

$$x = A\cos bt\ ,$$

$$y = A\sin bt\ .$$

At time $t = 2\,\mathrm{s}$, the coordinates of the position vector have the values

$$x = 5\,\mathrm{m}\ .\cos\!\big(\pi/4\ \mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\big) = 0\ ,$$

$$y = 5\,\mathrm{m}\,.\,\sin\bigl(\pi/4\,\mathrm{s}^{-1}\,.\,2\,\mathrm{s}\bigr) = 5\,\mathrm{m}\,.$$

The magnitude of the position vector is constant

$$r = \sqrt{x^2+y^2} = \sqrt{(A\sin bt)^2 + (A\cos bt)^2} = A = 5\,\mathrm{m}\,.$$

The direction cosines of the position vector are

$$\cos\alpha = \frac{x}{r} = \frac{A\cos bt}{A} = \cos bt\,,$$

$$\cos\beta = \frac{y}{r} = \frac{A\sin bt}{A} = \sin bt\,.$$

At time $t = 2\,\mathrm{s}$, the direction cosines have values

$$\cos\alpha = \cos\bigl(\pi/4\,\mathrm{s}^{-1}\,.\,2\,\mathrm{s}\bigr) = 0\,,$$

$$\cos\beta = \sin\bigl(\pi/4\,\mathrm{s}^{-1}\,.\,2\,\mathrm{s}\bigr) = 1\,.$$

1.2 *Two bodies that are $d = 100\,\mathrm{m}$ apart started moving in a straight line opposite each other. The first body is moving uniformly with velocity $v = 3\,\mathrm{m\,s^{-1}}$. The second body is moving uniformly accelerated with an initial velocity $v_0 = 7\,\mathrm{m\,s^{-1}}$ and acceleration $a = 4\,\mathrm{m\,s^{-2}}$. Find the time and place of their meeting.*

The distance travelled by the first body in uniform motion will be

$$s_1 = vt\,.$$

The distance travelled by the second body in uniformly accelerated motion will be

$$s_2 = v_0 t + \frac{at^2}{2}\,.$$

The bodies meet when the sum of the paths they have travelled equals their initial distance

$$s_1 + s_2 = d\,,$$

$$vt + v_0 t + \frac{at^2}{2} = d\,.$$

By adding the numerical values, the quadratic equation can be obtained

$$3\,\mathrm{m\,s^{-1}}\,t + 7\,\mathrm{m\,s^{-1}}\,t + \frac{4\,\mathrm{m\,s^{-2}}\,t^2}{2} = 100\,\mathrm{m}\,,$$

the time of the meeting of the bodies is thus the root of the quadratic equation

$$2t^2 + 10t - 100 = 0 \ ,$$

which has two solutions

$$t_1 = 5\,\mathrm{s} \ ,$$

$$t_2 = -10\,\mathrm{s} \ .$$

The physically meaningful solution of the problem corresponds to the positive solution of the quadratic equation

$$t = 5\,\mathrm{s} \ .$$

The point at which the bodies meet will be distant from the first body

$$s_1 = vt = 3\,\mathrm{m\,s^{-1}} \ . \ 5\,\mathrm{s} = 15\,\mathrm{m}$$

and will be distant from the other body

$$s_2 = d - s_1 = 100\,\mathrm{m} - 15\,\mathrm{m} = 85\,\mathrm{m} \ .$$

1.3 *The train starts from rest with a uniformly accelerated motion so that in time $t_1 = 30\,\mathrm{s}$ it passes a path $s_1 = 90\,\mathrm{m}$. What path will it pass, what its instantaneous and average velocity will be in time $t_2 = 60\,\mathrm{s}$?*

For the path s_1 that the train passes in time t_1 in uniformly accelerated motion, the following holds

$$s_1 = \frac{at_1^2}{2} \ ,$$

from which the acceleration of the train can be calculated

$$a = \frac{2s_1}{t_1^2} = \frac{2 \ . \ 90\,\mathrm{m}}{(30\,\mathrm{s})^2} = 0.2\,\mathrm{m\,s^{-2}} \ .$$

The path of the train at time t_2 will be

$$s_2 = \frac{at_2^2}{2} = \frac{0.2\,\mathrm{m\,s^{-2}} \ . \ (60s)^2}{2} = 360\,\mathrm{m} \ .$$

The instantaneous velocity of the train at time t_2 will be

$$v_2 = at_2 = 0.2\,\mathrm{m\,s^{-2}} \ . \ 60\,\mathrm{s} = 12\,\mathrm{m\,s^{-1}} \ .$$

The average speed of the train over time t_2 will be

$$v_p = \frac{s_2}{t_2} = \frac{360\,\mathrm{m}}{60\,\mathrm{s}} = 6\,\mathrm{m\,s^{-1}} \ .$$

1.4 *The position vector of a point particle has the form* $\vec{r} = (A_1t^2 + B_1)\vec{i} + (A_2t^2 + B_2)\vec{j}$, *where* $A_1 = 0.2\,\mathrm{m\,s^{-2}}$, $B_1 = 0.05\,\mathrm{m}$, $A_2 = 0.15\,\mathrm{m\,s^{-2}}$, $B_2 = -0.03\,\mathrm{m}$. *Find the magnitude and direction of the velocity and acceleration of the point particle at time* $t_1 = 2\,\mathrm{s}$. *Express the direction using the angle to the x-axis.*

The coordinates of the position vector are

$$x = A_1t^2 + B_1 \ ,$$

$$y = A_2t^2 + B_2 \ .$$

For the velocity vector, it is stated

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \ .$$

The coordinates of the velocity vector will be

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{d(A_1t^2 + B_1)}{dt} = 2A_1t \ ,$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{d(A_2t^2 + B_2)}{dt} = 2A_2t \ .$$

The magnitude of the velocity vector will be

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2A_1t)^2 + (2A_2t)^2} = 2t\sqrt{A_1^2 + A_2^2} \ .$$

The magnitude of the velocity vector at time $t_1 = 2\,\mathrm{s}$ will be

$$v = 2 \ . \ 2\,\mathrm{s} \ . \ \sqrt{(0.2\,\mathrm{m\,s^{-2}})^2 + (0.15\,\mathrm{m\,s^{-2}})^2} = 1\,\mathrm{m\,s^{-1}} \ .$$

The direction cosine of the velocity vector will be constant

$$\cos\alpha_v = \frac{v_x}{v} = \frac{2A_1t}{2t\sqrt{A_1^2 + A_2^2}} = \frac{A_1}{\sqrt{A_1^2 + A_2^2}}$$

and its value will be

$$\cos\alpha_v = \frac{0.2\,\mathrm{m\,s^{-2}}}{\sqrt{(0.2\,\mathrm{m\,s^{-2}})^2 + (0.15\,\mathrm{m\,s^{-2}})^2}} = 0.8 \ ,$$

which implies that the angle between the velocity vector and the x-axis will be

$$\alpha_v = \arccos 0.8 = 36.6^\circ \ .$$

For the acceleration vector, it is stated

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \ .$$

The coordinates of the acceleration vector will be

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{d(2A_1t)}{dt} = 2A_1 \ ,$$

$$a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{d(2A_2t)}{dt} = 2A_2 \ .$$

The magnitude of the acceleration vector will be constant

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(2A_1)^2 + (2A_2)^2} = 2\sqrt{A_1^2 + A_2^2}$$

and its value will be

$$a = 2 \ . \ \sqrt{(0.2\,\mathrm{m\,s^{-2}})^2 + (0.15\,\mathrm{m\,s^{-2}})^2} = 0.5\,\mathrm{m\,s^{-2}} \ .$$

The direction cosine of the acceleration vector will be constant

$$\cos\alpha_a = \frac{a_x}{a} = \frac{2A_1}{2\sqrt{A_1^2 + A_2^2}} = \frac{A_1}{\sqrt{A_1^2 + A_2^2}}$$

and its value will be

$$\cos\alpha_a = \frac{0.2\,\mathrm{m\,s^{-2}}}{\sqrt{(0.2\,\mathrm{m\,s^{-2}})^2 + (0.15\,\mathrm{m\,s^{-2}})^2}} = 0.8 \ ,$$

which implies that the angle between the acceleration vector and the x-axis will be

$$\alpha_a = \arccos 0.8 = 36.6^\circ \ .$$

1.5 *The wheel started to rotate from rest with a constant angular acceleration $\alpha = 5\,\mathrm{s^{-2}}$. How many times has the wheel rotated in the time $t_1 = 10\,\mathrm{s}$ since the start of the motion?*

The angular velocity at constant angular acceleration is

$$\omega = \int \alpha \, dt = \alpha t + c_1 \ .$$

If the starting angular velocity is zero, then the integration constant is zero

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c_1 = 0$$

and the angular velocity will be

$$\omega = \alpha t \ .$$

The angular displacement at constant angular acceleration is

$$\varphi = \int \omega \, dt = \int \alpha t \, dt = \frac{\alpha t^2}{2} + c_2 \ .$$

If the starting angular displacement is zero, then the integration constant is zero

$$\varphi(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0$$

and the angular displacement will be

$$\varphi = \frac{\alpha t^2}{2} \ .$$

The angular distance of one revolution is 2π, so the number of revolutions of the wheel will be

$$n = \frac{\varphi}{2\pi} = \frac{\alpha t^2}{4\pi}$$

and the number of revolutions of the wheel in time t_1 will be

$$n_1 = \frac{\alpha t_1^2}{4\pi} = \frac{5\,\mathrm{s}^{-2} \ . \ (10\,\mathrm{s})^2}{4\pi} = 39.8 \ .$$

1.6 *The magnitude of the train speed after leaving the station gradually increased from zero to* $v_1 = 20\,\mathrm{m\,s}^{-1}$ *at time* $t_1 = 180\,\mathrm{s}$. *The track is curved with a radius of curvature* $R = 800\,\mathrm{m}$. *Calculate the magnitudes of the tangential, normal, and total accelerations at time* $t_2 = 120\,\mathrm{s}$.

Tangential acceleration indicates the change in magnitude of the velocity

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} \ ,$$

for constant a_t it is

$$a_t = \frac{v_1}{t_1} = \frac{20\,\mathrm{m\,s}^{-1}}{180\,\mathrm{s}} = 0.111\,\mathrm{m\,s}^{-2} \ .$$

The magnitude of the velocity at time t_2 will be

$$v_2 = a_t t_2 = \frac{v_1 t_2}{t_1} \ .$$

The normal acceleration indicates the change in direction of the velocity

$$a_n = \frac{v^2}{R} \; .$$

At time t_2, the normal acceleration will be

$$a_n = \frac{v_1^2 t_2^2}{R t_1^2} = \frac{(20\,\mathrm{m\,s^{-1}})^2 \, . \, (120\,\mathrm{s})^2}{800\,\mathrm{m} \, . \, (180\,\mathrm{s})^2} = 0.222\,\mathrm{m\,s^{-2}} \; .$$

The total acceleration is the vector sum of the tangential and normal accelerations

$$\vec{a} = \vec{a}_t + \vec{a}_n \; .$$

The magnitude of the total acceleration will be

$$a = \sqrt{a_t^2 + a_n^2} \; ,$$

thus, the total acceleration at time t_2 will be

$$a = \sqrt{\left(\frac{v_1}{t_1}\right)^2 + \left(\frac{v_1^2 t_2^2}{R t_1^2}\right)^2}$$

and its value will be

$$a = \sqrt{\left(\frac{20\,\mathrm{m\,s^{-1}}}{180\,\mathrm{s}}\right)^2 + \left(\frac{(20\,\mathrm{m\,s^{-1}})^2 \, . \, (120\,\mathrm{s})^2}{800\,\mathrm{m} \, . \, (180\,\mathrm{s})^2}\right)^2} = 0.248\,\mathrm{m\,s^{-2}} \; .$$

1.7 *The point particle started moving in a circle with constant angular acceleration* $\alpha = 0.25\,\mathrm{s^{-2}}$. *At what time from the start of the motion will the angle between the particle's acceleration and the particle's velocity be* $\gamma = 45°$*?*

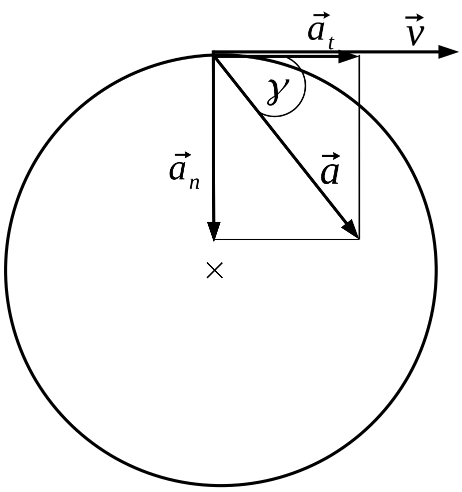

Fig. 1

In circular motion, the tangential acceleration is

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(R\omega)}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t} = R\alpha \ .$$

The angular velocity at constant angular acceleration is

$$\omega = \alpha t \ .$$

The normal acceleration can be expressed as

$$a_n = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = \frac{\alpha^2 t^2 R^2}{R} = \alpha^2 t^2 R \ .$$

The angle between the velocity and acceleration (Fig. 1) is

$$\tan\gamma = \frac{a_n}{a_t} = \frac{\alpha^2 t^2 R}{R\alpha} = \alpha t^2 \ ,$$

which implies for time

$$t = \sqrt{\frac{\tan\gamma}{\alpha}} = \sqrt{\frac{\tan 45^\circ}{0.25\,\mathrm{s}^{-2}}} = 2\,\mathrm{s} \ .$$

1.8 *A wheel with radius $R = 0.1\,\mathrm{m}$ rotates such that the dependence of the angle of rotation on time is given by the function $\varphi = A+Bt+Ct^3$, where $B = 2\,\mathrm{s}^{-1}, C = 1\,\mathrm{s}^{-3}$. For points that lie on the circumference of the wheel, calculate their velocity, angular velocity, angular acceleration, tangential acceleration and normal acceleration at time $t_1 = 2\,\mathrm{s}$.*

The magnitude of the angular velocity can be calculated using the definition

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \ ,$$

which implies

$$\omega = \frac{\mathrm{d}(A+Bt+Ct^3)}{\mathrm{d}t} = B + 3Ct^2 = \left[2\,\mathrm{s}^{-1} + 3\,.\,(1\,\mathrm{s}^{-3})\,.\,(2\,\mathrm{s})^2\right] = 14\,\mathrm{s}^{-1} \ .$$

The magnitude of the angular acceleration can be calculated using the definition

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \ ,$$

which implies

$$\alpha = \frac{\mathrm{d}(B+3Ct^2)}{\mathrm{d}t} = 6Ct = 6\,.\,(1\,\mathrm{s}^{-3})\,.\,(2\,\mathrm{s}) = 12\,\mathrm{s}^{-2} \ .$$

The magnitude of the velocity is

$$v = \omega R = (B + 3Ct^2)R = \left[2\,\mathrm{s}^{-1} + 3\,.\,(1\,\mathrm{s}^{-3})\,.\,(2\,\mathrm{s})^2\right]\,.\,0.1\,\mathrm{m} = 1.4\,\mathrm{m\,s^{-1}}\,.$$

The magnitude of the tangential acceleration is

$$a_t = \alpha R = 12\,\mathrm{s}^{-2}\,0.1\,\mathrm{m} = 1.2\,\mathrm{m\,s^{-2}}\,.$$

The magnitude of the normal acceleration is

$$a_n = \frac{v^2}{R} = \frac{(1.4\,\mathrm{m\,s^{-1}})^2}{0.1\,\mathrm{m}} = 19.6\,\mathrm{m\,s^{-2}}\,.$$

1.9 *A point particle moves in a straight line so that its acceleration increases uniformly with time, and in time $t_1 = 10\,\mathrm{s}$ it increases from zero to $a_1 = 5\,\mathrm{m\,s^{-2}}$. What is the speed of the point particle at time $t_2 = 20\,\mathrm{s}$, and what is the path the point particle travelled in this time when it was initially at rest?*

The acceleration of the material point increases uniformly

$$a = kt\,,$$

the acceleration from zero to a_1 increases in time t_1, that is

$$a_1 = kt_1 \implies k = \frac{a_1}{t_1}\,,$$

therefore, the acceleration will be

$$a = \frac{a_1}{t_1}t\,.$$

The speed of the point particle is

$$v = \int a\,dt = \int \frac{a_1}{t_1}t\,dt = \frac{a_1 t^2}{2t_1} + c_1\,.$$

If the speed is initially zero

$$v(t = 0\,\mathrm{s}) = 0 \implies c_1 = 0\,,$$

the speed at time t_2 will be

$$v_2 = \frac{a_1 t_2^2}{2t_1} = \frac{5\,\mathrm{m\,s^{-2}}\,.\,(20\,\mathrm{s})^2}{2\,.\,10\,\mathrm{s}} = 100\,\mathrm{m\,s^{-1}}\,.$$

The path of the point particle is

$$s = \int v\,dt = \int \frac{a_1 t^2}{2t_1}\,dt = \frac{a_1 t^3}{6t_1} + c_2\;,$$

if the initial path is zero

$$s(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0\;.$$

The path at time t_2 will be

$$s_2 = \frac{a_1 t_2^3}{6t_1} = \frac{5\,\mathrm{m\,s^{-2}}\;.\;(20\,\mathrm{s})^3}{6\;.\;10\,\mathrm{s}} = 667\,\mathrm{m}\;.$$

1.10 *The particle moves on a circle with angular deceleration that increases with time according to the relation $\alpha = kt$, where $k = -6\,\mathrm{rad\,s^{-3}}$. The initial angular velocity was $\omega_0 = 30\,\mathrm{rad\,s^{-1}}$. Through what angle does the particle rotate in time $t_1 = 5\,\mathrm{s}$?*

The angular deceleration of the particle increases uniformly

$$\alpha = kt\;.$$

The angular velocity of the particle is

$$\omega = \int \alpha\,dt = \int kt\,dt = \frac{kt^2}{2} + c_1\;,$$

because the initial angular velocity of the particle was ω_0

$$\omega(t = 0\,\mathrm{s}) = \omega_0 \implies c_1 = \omega_0\;,$$

the angular velocity will be

$$\omega = \frac{kt^2}{2} + \omega_0\;.$$

The angular displacement of the particle is

$$\varphi = \int \omega\,dt = \int \left(\frac{kt^2}{2} + \omega_0\right)\,dt = \frac{kt^3}{6} + \omega_0 t + c_2\;,$$

because the initial angular displacement of the particle was zero

$$\varphi(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0\;,$$

the angular path will be

$$\varphi = \frac{kt^3}{6} + \omega_0 t\;.$$

The angular displacement of the particle at time t_1 will be

$$\varphi_1 = \frac{kt_1^3}{6} + \omega_0 t_1 = \frac{-6\,\mathrm{rad\,s^{-3}}\;.\;(5\,\mathrm{s})^3}{6} + 30\,\mathrm{rad\,s^{-1}}\;.\;5\,\mathrm{s} = 25\,\mathrm{rad}\;.$$

2 Dynamics of a point particle

2.1 *Three bodies with masses* $m_A = 10\,\mathrm{kg}$, $m_B = 15\,\mathrm{kg}$, $m_C = 20\,\mathrm{kg}$, *lying on a horizontal support and connected by a wire, are subject to a force* $F = 100\,\mathrm{N}$ *in the horizontal direction. The mass of the wire and the friction between the bodies and the support are negligible. Calculate the acceleration of the system and the force acting at each joint.*

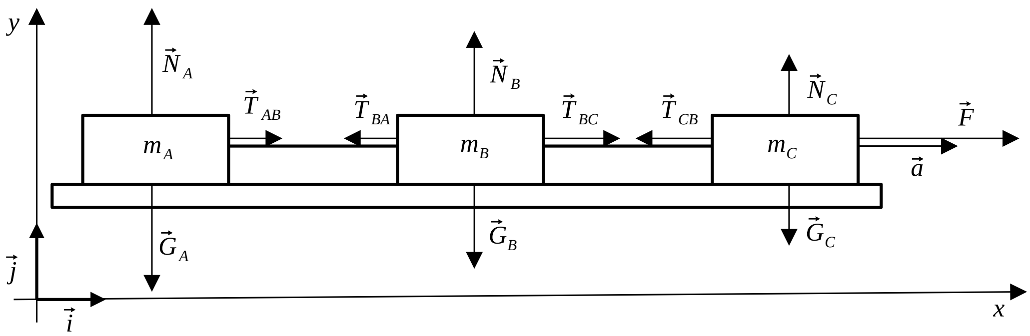

Fig. 2

The motion of bodies is described by Newton's law of force

$$\sum_{i=1}^{n} \vec{F}_i = m\vec{a}\ .$$

If the distance between the bodies does not change, the acceleration of all the bodies is equal

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}\ .$$

The bodies (Fig. 2) are subject to gravitational forces $\vec{G}_A$, $\vec{G}_B$, $\vec{G}_C$, normal forces $\vec{N}_A$, $\vec{N}_B$, $\vec{N}_C$, the tensile forces of the wire $\vec{T}_{AB}$, $\vec{T}_{BA}$, $\vec{T}_{BC}$, $\vec{T}_{CB}$ and the external force $\vec{F}$, therefore, Newton's force law will be

$$\vec{T}_{AB} + \vec{G}_A + \vec{N}_A = m_A\vec{a}\ ,$$

$$\vec{T}_{BA} + \vec{T}_{BC} + \vec{G}_B + \vec{N}_B = m_B \vec{a} \ ,$$

$$\vec{F} + \vec{T}_{CB} + \vec{G}_C + \vec{N}_C = m_C \vec{a} \ .$$

After scalar multiplication of the equations by the unit vectors $\vec{i}$ and $\vec{j}$ and using the equations

$$\vec{i} \cdot \vec{i} = 1 \ ,$$

$$\vec{j} \cdot \vec{j} = 1 \ ,$$

$$\vec{i} \cdot \vec{j} = 0 \ ,$$

the equations take the scalar form

$$T_{AB} = m_A a \ ,$$

$$N_A - G_A = 0 \ ,$$

$$-T_{BA} + T_{BC} = m_B a \ ,$$

$$N_B - G_B = 0 \ ,$$

$$F - T_{CB} = m_C a \ ,$$

$$N_C - G_C = 0 \ .$$

In the vertical direction, the acceleration is zero, therefore

$$N_A = G_A \ ,$$

$$N_B = G_B \ ,$$

$$N_C = G_C \ .$$

Newton's law of action and reaction implies

$$\vec{T}_{AB} = -\vec{T}_{BA} \ ,$$

$$T_{AB} = T_{BA} \ ,$$

$$\vec{T}_{BC} = -\vec{T}_{CB} \ ,$$

$$T_{BC} = T_{CB} \ .$$

Therefore, in the horizontal direction

$$T_{AB} = m_A a \ ,$$

$$-T_{AB} + T_{BC} = m_B a \ ,$$

$$F - T_{BC} = m_C a \ .$$

By summing all the equations, the acceleration of the system can be expressed

$$a = \frac{F}{m_A + m_B + m_C}$$

and then modifying the individual equations of the force between the bodies

$$T_{AB} = \frac{F m_A}{m_A + m_B + m_C} \ ,$$

$$T_{BC} = \frac{F(m_A + m_B)}{m_A + m_B + m_C} \ .$$

After substituting the values, the numerical solution will be

$$a = \frac{100\,\mathrm{N}}{10\,\mathrm{kg} + 15\,\mathrm{kg} + 20\,\mathrm{kg}} = 2.2\,\mathrm{m\,s^{-2}} \ ,$$

$$T_{AB} = \frac{100\,\mathrm{N}\ .\ 10\,\mathrm{kg}}{10\,\mathrm{kg} + 15\,\mathrm{kg} + 20\,\mathrm{kg}} = 22.2\,\mathrm{N} \ ,$$

$$T_{BC} = \frac{100\,\mathrm{N}\ .\ (10\,\mathrm{kg} + 15\,\mathrm{kg})}{10\,\mathrm{kg} + 15\,\mathrm{kg} + 20\,\mathrm{kg}} = 55.6\,\mathrm{N} \ .$$

2.2 *Two bodies with equal masses* $m = 5\,\mathrm{kg}$ *are connected by a wire passing through a freely rotating pulley. The first body hangs freely on the wire, the second lies on an inclined plane which makes an angle* $\alpha = 30°$ *with the horizontal plane. Calculate the acceleration of the bodies and the force acting on the wire if there is no friction between the body and the inclined plane, and if there is friction between the body and the inclined plane, and the friction factor is* $\mu = 0.2$.

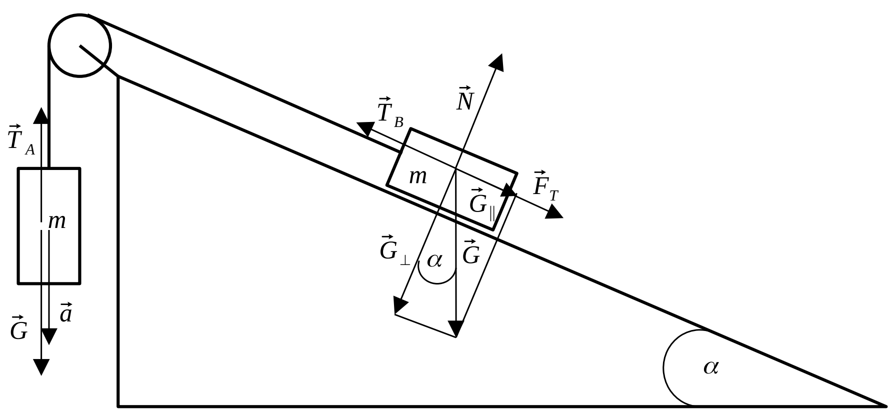

Fig. 3

The gravitational force of a body on an inclined plane (Fig. 3) can be decomposed into a component parallel to the inclined plane and a component perpendicular to the inclined plane

$$\vec{G} = \vec{G}_{\parallel} + \vec{G}_{\perp} \; .$$

The component parallel to the inclined plane will have the magnitude

$$G_{\parallel} = G \sin\alpha$$

and the component perpendicular to the inclined plane

$$G_{\perp} = G \cos\alpha \; .$$

The magnitude of the frictional force between the body and the inclined plane is

$$F_T = \mu G_{\perp} = \mu G \cos\alpha \; .$$

Newton's law of force

$$\sum_{i=1}^{n} \vec{F}_i = m\vec{a} \; ,$$

for a hanging body is

$$\vec{G} + \vec{T}_A = m\vec{a}_A$$

and for a body on an inclined plane is

$$\vec{G}_{\parallel} + \vec{G}_{\perp} + \vec{N} + \vec{T}_B + \vec{F}_T = m\vec{a}_B \; .$$

Newton's law of action and reaction implies

$$T_A = T_B = T \; .$$

The length of the wire does not change, therefore,

$$a_A = a_B = a \; .$$

The force of gravity can be calculated using the acceleration of gravity as

$$\vec{G} = m\vec{g} \; .$$

Newton's law of force for bodies has a scalar form

$$mg - T = ma \ ,$$

$$-mg \sin\alpha + T - \mu mg \cos\alpha = ma \ .$$

By solving the system of equations, it is possible to express the acceleration of the system

$$a = \frac{g(1 - \sin\alpha - \mu\cos\alpha)}{2} \ ,$$

and the force acting on the wire

$$mg - T = ma \implies T = mg - ma \ ,$$

$$T = \frac{mg(1 + \sin\alpha + \mu\cos\alpha)}{2} \ .$$

If the friction between the body and the inclined plane is negligible

$$\mu = 0 \ ,$$

the acceleration of the system will be

$$a = \frac{g(1 - \sin\alpha)}{2}$$

and the force acting on the wire will be

$$T = \frac{mg(1 + \sin\alpha)}{2} \ .$$

After substituting the numerical values

$$a = \frac{9.81\,\mathrm{m\,s^{-2}} \ . \ (1 - \sin 30°)}{2} = 2.45\,\mathrm{m\,s^{-2}} \ ,$$

$$T = \frac{5\,\mathrm{kg} \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ (1 + \sin 30°)}{2} = 36.79\,\mathrm{N} \ .$$

If the friction between the body and the inclined plane is not negligible

$$\mu = 0.1 \ ,$$

the acceleration of the system will be

$$a = \frac{g(1 - \sin\alpha - \mu\cos\alpha)}{2}$$

and the force acting on the wire will be

$$T = \frac{mg(1 + \sin\alpha + \mu\cos\alpha)}{2} \ .$$

After substituting the numerical values

$$a = \frac{9.81\,\mathrm{m\,s^{-2}} \ . \ (1 - \sin 30° - 0.2 \ . \ \cos 30°)}{2} = 1.60\,\mathrm{m\,s^{-2}} \ ,$$

$$T = \frac{5\,\mathrm{kg} \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ (1 + \sin 30° + 0.2 \ . \ \cos 30°)}{2} = 41.04\,\mathrm{N} \ .$$

2.3 *A weight of mass* $m = 5\,\mathrm{kg}$ *hanging on a wire of length* $l = 1\,\mathrm{m}$ *swings with a maximum angular deflection* $\alpha = 60°$. *What force* F_1 *is acting on the wire at the extreme positions and what force* F_2 *is acting on the wire at the lowest position?*

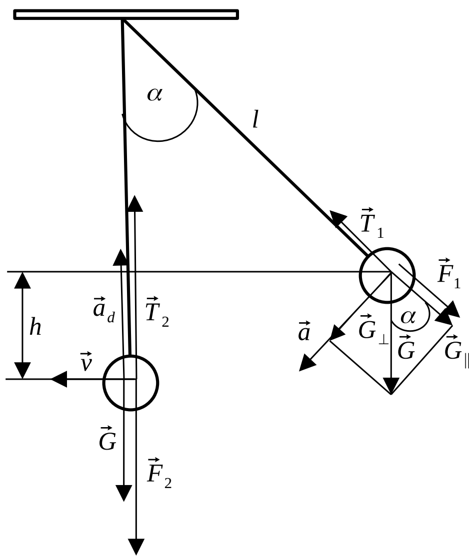

Fig. 4

The weight in the extreme position (Fig. 4) is acted upon by the gravitational force $\vec{G}$ and the force of the wire $\vec{T}_1$. Newton's law of force will therefore have the form

$$\vec{G} + \vec{T}_1 = m\vec{a}\ .$$

The gravitational force can be resolved into a component parallel to the direction of the wire and a component perpendicular to the direction of the wire

$$\vec{G} = \vec{G}_\parallel + \vec{G}_\perp\ .$$

The component of the gravitational force acting in the direction parallel to the direction of the wire is

$$G_\parallel = G\cos\alpha\ ,$$

from Newton's law of forces for components in the direction of the wire implies

$$G_\parallel - T_1 = 0\ ,$$

$$mg\cos\alpha - T_1 = 0\ ,$$

from which it is possible to express the force that the wire acts on the weight

$$T_1 = mg\cos\alpha \ .$$

According to Newton's law of action and reaction, the force that the weight acts on the wire is equal in magnitude and in the opposite direction to the force that the wire acts on the weight

$$\vec{F}_1 = -\vec{T}_1 \ ,$$

therefore, the magnitude of the force that acts on the wire at its extreme position will be

$$F_1 = mg\cos\alpha \ ,$$

after substituting numerical values

$$F_1 = 5\,\mathrm{kg}\ .\ 9.81\,\mathrm{m\,s^{-2}}\ .\ \cos 60^\circ = 24.53\,\mathrm{N} \ .$$

The height of the weight at its extreme position relative to the lowest position can be expressed as

$$h = l - l\cos\alpha \ .$$

The law of conservation of mechanical energy for a weight in the extreme and lowest position has the form

$$E_{p1} + E_{k1} = E_{p2} + E_{k2} \ ,$$

if in the extreme position, the kinetic energy is zero

$$E_{k1} = 0$$

and in the lowest position, the potential energy is zero

$$E_{p2} = 0 \ ,$$

the law of conservation of mechanical energy takes the form

$$E_{p1} = E_{k2} \ ,$$

$$mgh = \frac{mv^2}{2} \ ,$$

from which it is possible to express the velocity of the weight in the lowest position as

$$v = \sqrt{2gh} \ .$$

The weight will move in a circle with radius l and will be acted upon by the gravitational force $\vec{G}$ and the force of the wire $\vec{T}_2$. Newton's law of force for the body at its lowest position will be

$$\vec{G} + \vec{T}_2 = m\vec{a}_d \ ,$$

at the lowest position in the direction of the wire, it states

$$mg - T_2 = -ma_d \ .$$

The magnitude of the centripetal acceleration at its lowest position can be expressed using the velocity of the weight

$$a_d = \frac{v^2}{R} = \frac{2gh}{l} = 2g(1 - \cos\alpha) \ .$$

The force stretching the wire at its lowest point will be

$$T_2 = mg + ma_d = mg(3 - 2\cos\alpha) \ .$$

The force exerted by the weight on the wire is equal in magnitude and opposite in direction to the force exerted by the wire on the weight.

$$\vec{F}_2 = -\vec{T}_2 \ ,$$

therefore, the magnitude of the force that acts on the wire in the lowest position will be

$$F_2 = mg(3 - 2\cos\alpha) \ ,$$

after substituting numerical values

$$F_2 = 5\,\mathrm{kg} \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ (3 - 2 \ . \ \cos 60°) = 98.1\,\mathrm{N} \ .$$

2.4 *What impulse will the wall give to an elastic ball with mass* $m = 1\,\mathrm{kg}$ *and velocity* $v_0 = 10\,\mathrm{ms^{-1}}$ *that hits the wall in a direction making an angle* $\alpha = 60°$ *with the normal?*

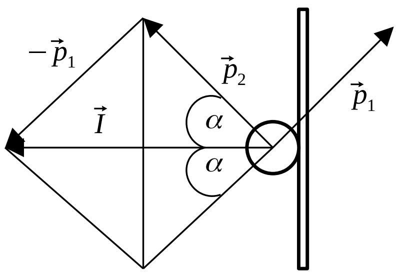

Fig. 5

By definition, the impulse of a force expresses the time effect of a force

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}(t)\, dt \ ,$$

to calculate it, the impulse theorem can also be used

$$\vec{I} = \vec{p}_2 - \vec{p}_1 \ .$$

In an elastic collision, the kinetic energy of the ball does not change

$$E_{k2} = E_{k1} \ ,$$

$$\frac{mv_2^2}{2} = \frac{mv_1^2}{2} \ ,$$

therefore, the magnitude of the velocity does not change

$$v_2 = v_1 = v$$

and therefore the magnitude of the momentum does not change either

$$p_2 = p_1 = p = mv \ .$$

From the figure (Fig. 5) it follows

$$\cos\alpha = \frac{I}{2p} \ ,$$

from which it is possible to express the impulse of the force

$$I = 2p\cos\alpha = 2mv\cos\alpha \ ,$$

after substituting numerical values

$$I = 2 \ . \ 1\,\mathrm{kg} \ . \ 10\,\mathrm{ms}^{-1} \ . \ \cos 60^\circ = 10\,\mathrm{N\,s} \ .$$

2.5 *A body lying on a horizontal surface is acted upon in the horizontal direction by a force whose time dependence is $F(t) = A + Bt + Ct^2$, where $A = 0.2\,\mathrm{N}$, $B = 0.4\,\mathrm{N\,s^{-1}}$, $C = 0.6\,\mathrm{N\,s^{-2}}$. Calculate the impulse of the force for the time $t_0 = 5\,\mathrm{s}$?*

The impulse of a force expresses the time effect of a force

$$\vec{I} = \int_0^{t_0} \vec{F}(t)\, dt \,.$$

In the case of linear motion, the magnitude of the impulse of a force will be

$$I = \int_0^{t_0} F(t)\, dt = \int_0^{t_0} (A+Bt+Ct^2)\, dt = \left[At + B\frac{t^2}{2} + C\frac{t^3}{3}\right]_0^{t_0} = At_0+B\frac{t_0^2}{2}+C\frac{t_0^3}{3} \,.$$

Substituting numerical values

$$I = 0.2\,\mathrm{N} \,.\, 5\,\mathrm{s} + 0.4\,\mathrm{N\,s^{-1}} \,.\, \frac{(5\,\mathrm{s})^2}{2} + 0.6\,\mathrm{N\,s^{-2}} \,.\, \frac{(5\,\mathrm{s})^3}{3} = 31\,\mathrm{N\,s} \,.$$

2.6 *A point particle of mass $m = 5\,\mathrm{kg}$ is moved by a force such that its path changes with time as $x(t) = A + Bt + Ct^2 + Dt^3$, where $C = 2\,\mathrm{m\,s^{-2}}$, $D = -0.2\,\mathrm{m\,s^{-3}}$. Calculate the magnitude of the force acting on the mass point at time $t_0 = 2\,\mathrm{s}$ and find the time when the force will be zero.*

Velocity in a linear motion can be expressed from its definition as

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \frac{\mathrm{d}(A + Bt + Ct^2 + Dt^3)}{\mathrm{d}t} = B + 2Ct + 3Dt^2 \,.$$

Acceleration in a linear motion can be expressed from its definition as

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}(B + 2Ct + 3Dt^2)}{\mathrm{d}t} = 2C + 6Dt \,.$$

The force can be calculated from Newton's law of force

$$F(t) = ma(t) = m(2C + 6Dt) \,.$$

At time t_0 the force will be

$$F(t = t_0) = m(2C + 6Dt_0) \,.$$

After substituting numerical values

$$F(t = 2\,\mathrm{s}) = 5\,\mathrm{kg} \,.\, \left[2 \,.\, 2\,\mathrm{m\,s^{-2}} + 6 \,.\, (-0.2\,\mathrm{m\,s^{-3}}) \,.\, 2\,\mathrm{s}\right] = 8\,\mathrm{N} \,.$$

From the condition that the force is zero

$$F(t) = 0 \, ,$$

it follows

$$m(2C + 6Dt) = 0 \, ,$$

from which it is possible to express the time when the force will be zero

$$t = \frac{-C}{3D}$$

and after substituting numerical values

$$t = \frac{-2\,\mathrm{m\,s^{-2}}}{3\,.\,(-0.2\,\mathrm{m\,s^{-3}})} = 3.33\,\mathrm{s} \, .$$

2.7 *A force $F = F_0 + kt$ acts on a body of mass $m = 5\,\mathrm{kg}$, where $F_0 = 10\,\mathrm{N}$ and $k = 0.1\,\mathrm{N\,s^{-1}}$ are constants. Express the acceleration, velocity, and position of the body at time $t = 10\,\mathrm{s}$ if the body initially had a velocity $v_0 = 2\,\mathrm{m\,s^{-1}}$ and a the starting position was $x_0 = 2\,\mathrm{m}$.*

From Newton's law of force

$$\vec{F} = m\vec{a} \, ,$$

for the acceleration of a body in linear motion follows

$$a = \frac{F}{m} = \frac{F_0}{m} + \frac{kt}{m} \, ,$$

after substituting numerical values, the acceleration at time $t = 10\,\mathrm{s}$ will be

$$a = \frac{10\,\mathrm{N}}{5\,\mathrm{kg}} + \frac{0.1\,\mathrm{N\,s^{-1}}\,.\,10\,\mathrm{s}}{5\,\mathrm{kg}} = 2.2\,\mathrm{m\,s^{-2}} \, .$$

The velocity of a point particle can be calculated using the acceleration

$$v = \int a\,dt = \int \left(\frac{F_0}{m} + \frac{kt}{m}\right)\,dt = \frac{F_0 t}{m} + \frac{kt^2}{2m} + c_1 \, .$$

If the initial velocity was v_0, the integration constant c_1 will be

$$v(t = 0\,\mathrm{s}) = v_0 \implies c_1 = v_0$$

and the velocity of the point particle will be

$$v = \frac{F_0 t}{m} + \frac{kt^2}{2m} + v_0 \ ,$$

after substituting numerical values, the velocity at time $t = 10\,\mathrm{s}$ will be

$$v = \frac{10\,\mathrm{N}\ .\ 10\,\mathrm{s}}{5\,\mathrm{kg}} + \frac{0.1\,\mathrm{N\,s^{-1}}\ .\ (10\,\mathrm{s})^2}{2\ .\ 5\,\mathrm{kg}} + 2\,\mathrm{m\,s^{-1}} = 23\,\mathrm{m\,s^{-1}} \ .$$

The position of a mass point can be calculated using the velocity

$$x = \int v\,dt = \int \left(\frac{F_0 t}{m} + \frac{kt^2}{2m} + v_0 \right) dt = \frac{F_0 t^2}{2m} + \frac{kt^3}{6m} + v_0 t + c_2 \ ,$$

if the initial position was x_0, the integration constant c_2 will be

$$x(t = 0\,\mathrm{s}) = x_0 \implies c_2 = x_0$$

and the position of the mass point will be

$$x = \frac{F_0 t^2}{2m} + \frac{kt^3}{6m} + v_0 t + x_0 \ ,$$

after substituting numerical values, the position at time $t = 10\,\mathrm{s}$ will be

$$x = \frac{10\,\mathrm{N}\ .\ (10\,\mathrm{s})^2}{2\ .\ 5\,\mathrm{kg}} + \frac{0.1\,\mathrm{N\,s^{-1}}\ .\ (10\,\mathrm{s})^3}{6\ .\ 5\,\mathrm{kg}} + 2\,\mathrm{m\,s^{-1}}\ .\ 10\,\mathrm{s} + 2\,\mathrm{m} = 125.33\,\mathrm{m} \ .$$

2.8 *What work must be done to compress the buffer spring of a wagon by* $x_0 = 10\,\mathrm{cm}$*, when a force of* $F_1 = 25\,000\,\mathrm{N}$ *is required to compress it by* $x_1 = 1\,\mathrm{cm}$ *and the force is directly proportional to the shortening of the spiral.*

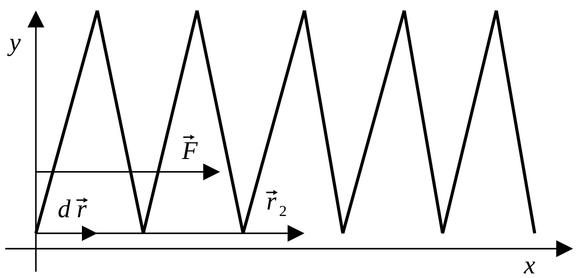

Fig. 6

The magnitude of the external force exerted on a spring (Fig. 6) is directly proportional to the compression of the spring, it means

$$F = kx \ .$$

If a force of F_1 is required to compress a spring by x_1, the spring stiffness will be

$$F_1 = kx_1 \implies k = \frac{F_1}{x_1} \ .$$

Mechanical work expresses the path effect of a force

$$A = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot \vec{dr} \ ,$$

if the displacement vector and the force vector have the same direction, then

$$\vec{F} \cdot \vec{dr} = F dr \cos 0^\circ = F dr \ .$$

If the x-axis has its origin at the point where the uncompressed spring is located and the direction of the x-axis is the same as the direction of compression of the spring, then

$$dr = dx \ ,$$

$$r_1 = 0 \ ,$$

$$r_2 = x_0 \ .$$

The mechanical work will be

$$A = \int_{r_1}^{r_2} F\, dr = \int_0^{x_0} F\, dx = \int_0^{x_0} kx\, dx = \frac{kx_0^2}{2}$$

and after substituting, the spring stiffness will be

$$A = \frac{F_1 x_0^2}{2x_1} \ .$$

After substituting numerical values

$$A = \frac{25\,000\,\mathrm{N} \ . \ (0.10\,\mathrm{m})^2}{2 \ . \ 0.01\,\mathrm{m}} = 12\,500\,\mathrm{J} = 12.5\,\mathrm{kJ} \ .$$

2.9 *A ball is suspended on a wire of length $l = 0.5\,\mathrm{m}$. What is the smallest horizontal velocity that must be given to it so that it can be deflected to its highest position while keeping the string taut?*

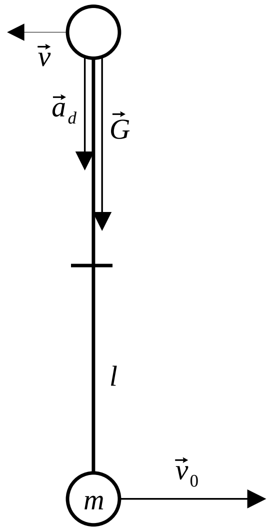

Fig. 7

If the ball at the highest position (Fig. 7) has a velocity of $\vec{v}$ and moves in a circle with radius l, it will have a centripetal acceleration of the magnitude

$$a = m\frac{v^2}{l} \ .$$

If it is also acted upon by a gravitational force

$$\vec{G} = m\vec{g} \ ,$$

then Newton's law of force will have the form

$$\vec{G} = m\vec{a} \ ,$$

in scalar form

$$mg = m\frac{v^2}{l} \ ,$$

from which it is possible to express the velocity of the ball at the highest position

$$v = \sqrt{gl} \ .$$

If the lowest position is a place with zero potential energy, then the total mechanical energy of the ball at the lowest position will be

$$E_{p0} + E_{k0} = \frac{mv_0^2}{2}$$

and at the highest position the total mechanical energy of the ball will be

$$E_p + E_k = mgh + \frac{mv^2}{2} \ ,$$

where the height of the ball will be

$$h = 2l \ .$$

From the law of conservation of mechanical energy, it follows

$$E_{p0} + E_{k0} = E_p + E_k \ ,$$

$$\frac{mv_0^2}{2} = 2mgl + \frac{mv^2}{2} \ ,$$

from which it is possible to calculate the velocity of the ball in the lowest position

$$v_0 = \sqrt{5gl} \ ,$$

after substituting numerical values

$$v_0 = \sqrt{5 \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ 0.5\,\mathrm{m}} = 4.95\,\mathrm{m\,s^{-1}} \ .$$

2.10 *Calculate the power of a car engine with a mass of* $m = 1200\,\mathrm{kg}$ *when the car is moving at a constant speed of* $v = 50\,\mathrm{km\,h^{-1}}$ *on a road with a five percent gradient.*

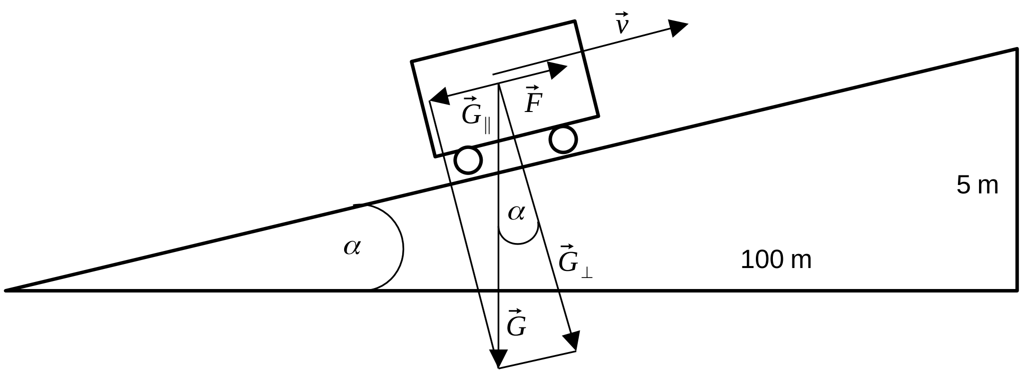

Fig. 8

The speed of the car is

$$v = 50\,\mathrm{km\,h^{-1}} = 13.9\,\mathrm{m\,s^{-1}} \ .$$

A five percent road gradient means that over a distance of $100\,\mathrm{m}$ the road will rise by $5\,\mathrm{m}$, therefore (Fig. 8)

$$\tan\alpha = \frac{5\,\mathrm{m}}{100\,\mathrm{m}} = 0.05$$

and the angle of a road inclination will be

$$\alpha = \arctan 0.05 = 2.86^\circ \ .$$

The engine power expresses the rate of work

$$P = \frac{\mathrm{d}A}{\mathrm{d}t} \ ,$$

if the force is constant, the engine power can be calculated as

$$P = \frac{\mathrm{d}(\vec{F} \cdot \vec{s})}{\mathrm{d}t} = \vec{F} \cdot \frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = \vec{F} \cdot \vec{v} \ .$$

The force of the engine and the speed of car have the same direction, so

$$P = Fv \cos 0^\circ = Fv \ .$$

If the car is moving uniformly, the resulting force acting on the car must be zero, so the force of the engine must be equal to the component of gravity that is parallel to the road

$$F = G_{\parallel} = mg \sin \alpha \ .$$

The power of engine will therefore be

$$P = Fv = mgv \sin \alpha \ ,$$

after substituting numerical values

$$P = 1200\,\mathrm{kg} \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ 13.9\,\mathrm{m\,s^{-1}} \ . \ \sin 2.86^\circ = 8164\,\mathrm{W} \ .$$

3 Mechanics of a rigid body

3.1 *Find the position of the centre of gravity of a body formed by cutting a semicircle with radius $b/2$ from a homogeneous rectangle with sides a, b, on a side of length b and attaching it to the opposite side of the rectangle.*

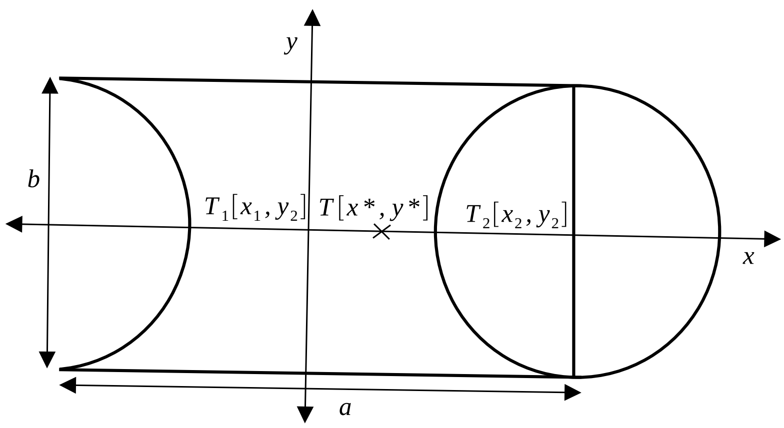

Fig. 9

It is advisable to choose the coordinate system so that its origin is located in the centre of the original rectangle, the x-axis is parallel to the side a and the y-axis to the side b of the rectangle. The position of the centre of gravity of the body can be determined as the common centre of gravity of two symmetrical bodies, whose centres of gravity are located at their centres of symmetry (Fig. 9). The first body will have the shape of a rectangle with semicircular cutouts on the sides with a length of b. The coordinates of the centre of gravity of the first body will be

$$x_1 = 0 \,,$$

$$y_1 = 0 \,.$$

The mass of the first body can be calculated as

$$m_1 = S_1 h\rho = \left[ab - \pi \left(\frac{b}{2}\right)^2\right] h\rho \,,$$

where h denotes the thickness of the body and ρ its density. The second body will be a circle with radius $b/2$. The coordinates of the centre of mass of the second body are

$$x_2 = \frac{a}{2} \,,$$

$$y_2 = 0 \,.$$

The mass of the second body can be calculated as

$$m_2 = S_2 h\rho = \pi \left(\frac{b}{2}\right)^2 h\rho \,.$$

For the position of the common centre of gravity of two bodies

$$\vec{r^*} = \frac{\sum_i m_i \vec{r_i}}{\sum_i m_i} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2} \,,$$

therefore, the coordinates of the common centre of gravity of these two bodies will be

$$x^* = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\pi \left(\frac{b}{2}\right)^2 h\rho \frac{a}{2}}{\left[ab - \pi \left(\frac{b}{2}\right)^2\right] h\rho + \pi \left(\frac{b}{2}\right)^2 h\rho} = \frac{\pi b}{8} \,,$$

$$y^* = 0 \,.$$

3.2 *Find the position of the centre of gravity of a homogeneous hemisphere with radius* $R = 10\,\mathrm{cm}$.

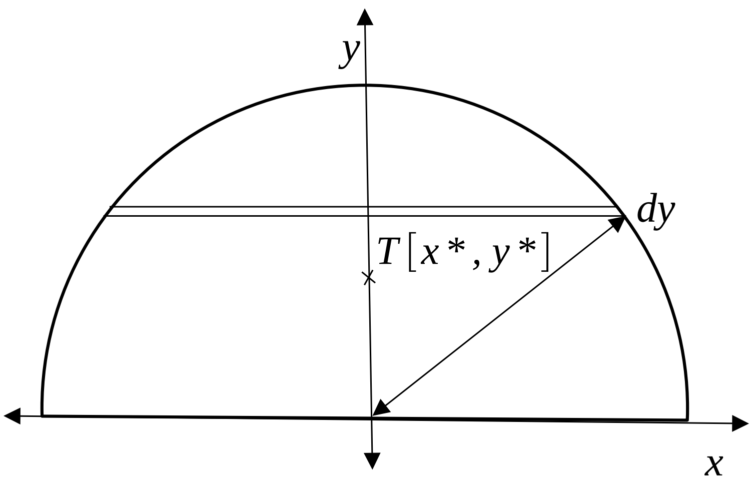

Fig. 10

It is advisable to choose the coordinate system so that its origin is located in the centre of the circular base of the hemisphere and the x and z axes lie in its plane (Fig. 10). The y-axis is the axis of symmetry of the body, therefore, the centre of gravity of the body will be located on this axis and it will be valid for it

$$x^* = 0 \; ,$$

$$z^* = 0 \; .$$

The position of the centre of gravity of the body is

$$\vec{r^*} = \frac{1}{M} \int\limits_M \vec{r} \, dm \; ,$$

where M is the mass of the body. For the coordinate of the centre of gravity in the direction of the y-axis, therefore

$$y^* = \frac{1}{M} \int\limits_M y \, dm = \frac{1}{V\rho} \int\limits_V y\rho \, dV = \frac{1}{V} \int\limits_V y \, dV \; ,$$

where ρ is the density of the hemisphere and the volume of the hemisphere is

$$V = \frac{2}{3}\pi R^3 \; .$$

The volume element of the hemisphere has the shape of a circular disc with radius x and thickness $\mathrm{d}y$

$$dV = \pi x^2 dy = \pi(R^2 - y^2) dy \; ,$$

where the Pythagorean theorem was used

$$R^2 = x^2 + y^2 \; .$$

The position of the centre of mass in the direction of the y-axis will therefore be

$$y^* = \frac{3}{2\pi R^3} \int\limits_0^R y\pi(R^2 - y^2) dy = \frac{3}{2\pi R^3} \left[\pi R^2 \frac{y^2}{2} - \pi \frac{y^4}{4} \right]_0^R =$$

$$= \frac{3}{2\pi R^3} \left(\pi \frac{R^4}{2} - \pi \frac{R^4}{4} \right) = \frac{3}{8} R \; .$$

After substituting numerical values

$$y^* = \frac{3}{8} \, 10 \, \mathrm{cm} = 3.78 \, \mathrm{cm} \; .$$

3.3 *A father and a son carry a load on a rod of length $l = 2\,\mathrm{m}$. How far from the father's end of the rod should the load be hung so that the father carries three times as much force as the son? Compared to the mass of the load, the mass of the rod is negligible.*

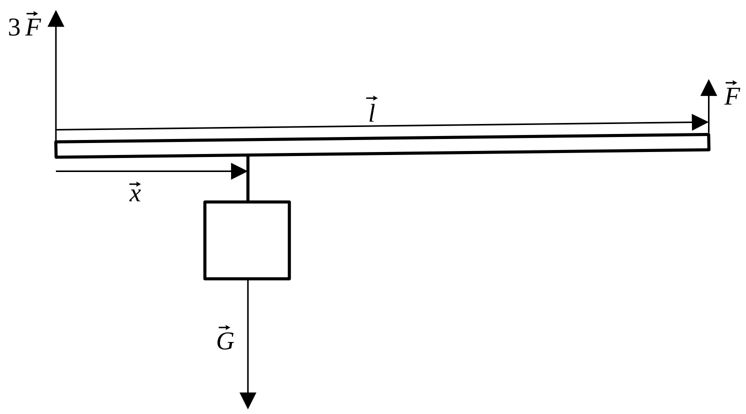

Fig. 11

A body is in equilibrium if the vector sum of all external forces acting on the body is equal to zero

$$\sum_i \vec{F}_i = 0$$

and simultaneously the vector sum of the moments of all external forces with respect to an arbitrary point is equal to zero

$$\sum_i \vec{M}_i = 0 \ ,$$

$$\sum_i \vec{r}_i \times \vec{F}_i = 0 \ .$$

The external forces acting on the rod (Fig. 11) are the gravity of the load $\vec{G}$, the force of the son $\vec{F}$, and the force of the father $3\vec{F}$. The equilibrium condition for these forces implies

$$\vec{G} + \vec{F} + 3\vec{F} = 0 \ .$$

Since the forces of the son and father have opposite directions to the gravitational force, the magnitudes of the forces will be

$$G - 4F = 0 \ ,$$

which implies

$$F = \frac{G}{4} \ .$$

The equilibrium condition for the moments of the forces with respect to the point at the father will be

$$\vec{x} \times \vec{G} + \vec{l} \times \vec{F} = 0 \ ,$$

the magnitudes of the moments of the forces will be

$$xG \sin 90^\circ + lF \sin(-90^\circ) = 0 \ ,$$

$$xG - lF = 0 \ .$$

After substituting from the first equilibrium condition

$$xG - l\frac{G}{4} = 0 \ ,$$

it is possible to express the distance from the father

$$x = \frac{l}{4} \ .$$

After substituting numerical values

$$x = \frac{2\,\mathrm{m}}{4} = 0.5\,\mathrm{m} \ .$$

3.4 *A homogeneous narrow board with length* $l = 5\,\mathrm{m}$ *and mass* $m = 30\,\mathrm{kg}$ *is loaded at one end with a load of mass* $m' = 10\,\mathrm{kg}$. *At what distance from this end should a support be placed so that the plate remains horizontal?*

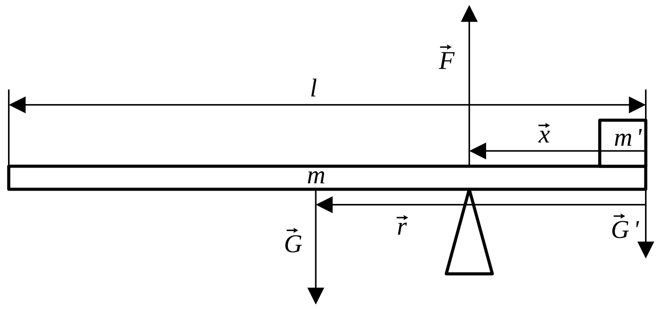

Fig. 12

A body is in equilibrium if the vector sum of all external forces acting on the body is equal to zero

$$\sum_i \vec{F}_i = 0$$

and simultaneously the vector sum of the moments of all external forces with respect to an arbitrary point is equal to zero

$$\sum_i \vec{M}_i = 0 \ ,$$

$$\sum_i \vec{r}_i \times \vec{F}_i = 0 \ ,$$

The external forces acting on the board (Fig. 12) are the gravity of the board $\vec{G}$ and the gravity of the load $\vec{G}'$ and the force of the support F. The equilibrium condition for these forces implies

$$\vec{G} + \vec{G}' + \vec{F} = 0 \ .$$

Since the gravity and the force of the support have opposite directions, the magnitudes of the forces will be

$$G + G' - F = 0 \ ,$$

which implies

$$F = G + G' \ .$$

The equilibrium condition for the moments of the forces with respect to the point at the load will be

$$\vec{r} \times \vec{G} + \vec{x} \times \vec{F} = 0 \ ,$$

for magnitudes of the moments of the forces, follow

$$\frac{l}{2} G \sin 90^\circ + xF \sin(-90^\circ) = 0 \ ,$$

$$\frac{l}{2} G - xF = 0 \ .$$

After substituting from the first equilibrium condition

$$\frac{l}{2} G - x(G + G') = 0 \ ,$$

it is possible to express the distance from the load

$$x = \frac{Gl}{2(G + G')} = \frac{mgl}{2(mg + m'g)} = \frac{ml}{2(m + m')} \ ,$$

after substituting numerical values

$$x = \frac{30\,\mathrm{kg}\ .\ 5\,\mathrm{m}}{2(30\,\mathrm{kg} + 10\,\mathrm{kg})} = 1.875\,\mathrm{m} \ .$$

3.5 *Calculate the moment of inertia of a homogeneous rod of length* $l = 3\,\mathrm{m}$ *and mass* $m = 5\,\mathrm{kg}$ *with respect to an axis passing through the centre of gravity of the rod and with respect to an axis passing through the end of the rod.*

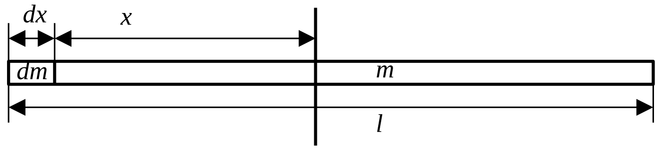

Fig. 13

The moment of inertia for a rigid body can be calculated using the definition

$$J = \int\limits_m r^2 dm \ .$$

If the coordinate system has its origin at the centre of the rod and the direction of the x-axis is the same as the direction of the rod (Fig. 13), the moment of inertia will be

$$J = \int\limits_m x^2 dm \ .$$

Since the rod is homogeneous, its length density is

$$\lambda = \frac{m}{l} \ ,$$

which can be used to express the mass element

$$dm = \lambda dx \ .$$

The moment of inertia of the rod about the axis through the centre of gravity will therefore be

$$J^* = \lambda \int\limits_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx = \lambda \left[\frac{x^3}{3} \right]_{-\frac{l}{2}}^{+\frac{l}{2}} = \lambda \left(\frac{l^3}{24} + \frac{l^3}{24} \right) = \lambda \frac{l^3}{12} \ .$$

After substituting the length density, the moment of inertia will be

$$J^* = \frac{ml^2}{12}$$

and after substituting the numerical values

$$J^* = \frac{5\,\mathrm{kg}\,.\,(3\,\mathrm{m})^2}{12} = 3.75\,\mathrm{kg\,m^2}\,.$$

The moment of inertia about an axis passing through the end of the rod can be calculated using Steiner's theorem

$$J = J^* + ma^2\,,$$

where the distance between the centre of gravity and the end of the rod ist

$$a = \frac{l}{2}\,.$$

The moment of inertia of the rod about the axis passing through the end of the rod will be

$$J = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}\,.$$

After substituting numerical values

$$J^* = \frac{5\,\mathrm{kg}\,.\,(3\,\mathrm{m})^2}{3} = 15\,\mathrm{kg\,m^2}\,.$$

3.6 *Calculate the moment of inertia of a homogeneous cylinder of mass* $m = 5\,\mathrm{kg}$ *with radius* $R = 1\,\mathrm{m}$ *with respect to both the axis identical to the axis of symmetry of the cylinder and the parallel axis passing through the edge of the cylinder.*

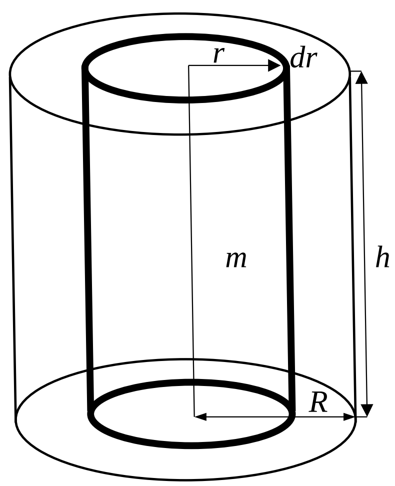

Fig. 14

The moment of inertia of a rigid body can be calculated by the definition

$$J = \int\limits_m r^2 dm \ .$$

If the cylinder is homogeneous, its density can be expressed as

$$\rho = \frac{m}{V} = \frac{m}{\pi R^2 h} \ ,$$

where h is the height of the cylinder. The mass element (Fig. 14) will be the shell of the cylinder with radius r and thickness $\mathrm{d}r$

$$dm = \rho dV = \rho 2\pi r h dr \ .$$

The moment of inertia of the cylinder with respect to the axis passing through the centre of cylinder will be

$$J^* = \rho \int\limits_0^R r^2 2\pi r h \, dr = \rho \left[\pi h \frac{r^4}{2} \right]_0^R = \rho \pi h \frac{R^4}{2} \ ,$$

after substituting the density of the cylinder

$$J^* = \frac{mR^2}{2}$$

and after substituting the numerical values

$$J^* = \frac{5\,\mathrm{kg} \ . \ (1\,\mathrm{m})^2}{2} = 2.5\,\mathrm{kg\,m^2} \ .$$

When calculating the moment of inertia with respect to the axis passing through the edge of the cylinder, it is possible to use Steiner's theorem

$$J = J^* + ma^2 \ ,$$

where a is the distance between the axes of rotation passing through the centre and the edge of the cylinder

$$a = R \ .$$

The moment of inertia with respect to the axis passing through the edge of the cylinder will be

$$J = \frac{mR^2}{2} + mR^2 = \frac{3mR^2}{2} \ ,$$

and after substituting numerical values

$$J^* = \frac{3 \ . \ 5\,\mathrm{kg} \ . \ (1\,\mathrm{m})^2}{2} = 7.5\,\mathrm{kg\,m^2} \ .$$

3.7 *A figure skater rotates with a frequency of $f_1 = 3\,\mathrm{s}^{-1}$. At what frequency will the figure skater rotate if he doubles his moment of inertia by extending his arms?*

In an isolated system, the law of conservation of angular momentum states

$$\vec{L}_1 = \vec{L}_2 \ ,$$

which implies

$$J_1\vec{\omega}_1 = J_2\vec{\omega}_2 \ ,$$

where J_1, J_2 are the moments of inertia and $\vec{\omega}_1, \vec{\omega}_2$ are the angular velocities of the body before and after the arms are extended. The direction of the angular velocity does not change. From the law of conservation of angular momentum follows

$$J_1\omega_1 = J_2\omega_2 \ .$$

The angular velocity can be expressed using the frequency

$$\omega = 2\pi f \ ,$$

then it will be valid

$$J_1 2\pi f_1 = J_2 2\pi f_2 \ ,$$

$$J_1 f_1 = J_2 f_2 \ .$$

If the moment of inertia is doubled

$$J_2 = 2J_1 \ ,$$

the resulting frequency will be halved

$$f_2 = f_1\frac{J_1}{J_2} = \frac{f_1}{2} \ ,$$

after substituting numerical values

$$f_2 = \frac{3\,\mathrm{s}^{-1}}{2} = 1.5\,\mathrm{s}^{-1} \ .$$

3.8 *A gyroscope with a moment of inertia* $J = 10\,\mathrm{kg\,m^2}$ *is rotated from rest by a force, whose moment with respect to the axis of rotation is* $M = 200\,\mathrm{N\,m}$. *In what time will the gyroscope reach a frequency* $f = 8\,\mathrm{s^{-1}}$ *and what will be its kinetic energy then?*

The motion of a gyroscope is described by the equation of motion of a rotating rigid body

$$\vec{M} = J\vec{\alpha}\ ,$$

from which the magnitude of the angular acceleration is

$$\alpha = \frac{M}{J}\ .$$

If the angular acceleration is constant, the angular velocity of the body is

$$\omega = \int \alpha \mathrm{d}t = \alpha t + c\ .$$

Initially, the body was at rest, therefore

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c = 0\ ,$$

so the angular velocity will be

$$\omega = \alpha t\ ,$$

which implies for the time of the rotation

$$t = \frac{\omega}{\alpha}\ ,$$

after substituting the angular acceleration, the time of the rotation is

$$t = \omega \frac{J}{M}\ .$$

The relationship between angular velocity and frequency

$$\omega = 2\pi f\ ,$$

allows to modify the relationship for the time of the rotation to the form

$$t = 2\pi f \frac{J}{M}\ ,$$

after substituting numerical values

$$t = 2\,\pi\,.\,8\,\mathrm{s}^{-1}\,.\,\frac{10\,\mathrm{kg\,m}^2}{200\,\mathrm{N\,m}} = 2.51\,\mathrm{s}\,.$$

The kinetic energy of the rotating gyroscope will be

$$E_k = \frac{J\omega^2}{2} = \frac{J(2\pi f)^2}{2} = 2J\pi^2 f^2\,,$$

after substituting numerical values

$$E_k = 2\,.\,10\,\mathrm{kg\,m}^2\,.\,\pi^2\,.\,(8\,\mathrm{s}^{-1})^2 = 12\,633\,\mathrm{J}\,.$$

3.9 *Calculate the kinetic energy of a cylindrical body with radius* $R = 10\,\mathrm{cm}$ *and mass* $m = 2\,\mathrm{kg}$ *at time* $t = 10\,\mathrm{s}$. *The body began to rotate from rest around its geometric axis with constant angular acceleration* $\alpha = \pi/8\,\mathrm{s}^{-2}$.

The kinetic energy of a rotating rigid body is

$$E_k = \frac{J\omega^2}{2}\,,$$

where J denotes the moment of inertia with respect to the axis of rotation and ω the angular velocity of the body. The moment of inertia of a homogeneous cylindrical body with respect to its geometric axis (Problem 3.6) can be calculated as

$$J = \frac{mR^2}{2}\,.$$

If the angular acceleration is constant, the angular velocity of the body is

$$\omega = \int \alpha \mathrm{d}t = \alpha t + c\,.$$

Initially, the body was at rest, therefore

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c = 0\,,$$

so the angular velocity of the body will be

$$\omega = \alpha t\,.$$

The kinetic energy of the body will be

$$E_k = \frac{1}{2}\frac{mR^2}{2}(\alpha t)^2 = \frac{mR^2\alpha^2 t^2}{4}\,,$$

and after substituting numerical values

$$E_k = \frac{2\,\mathrm{kg}\,.\,(0.1\,\mathrm{m})^2.\left(\frac{\pi}{8}\mathrm{s}^{-2}\right)^2.(10\,\mathrm{s})^2}{4} = 0.077\,\mathrm{J}\,.$$

3.10 *A rod of length $l = 1\,\mathrm{m}$ hangs vertically on an axis passing through its endpoint. What minimum velocity must be given to the free end of the rod to bring it to a horizontal position?*

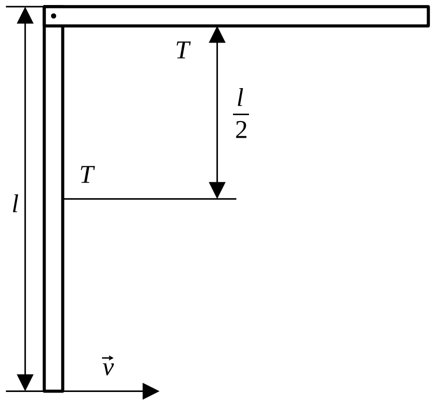

Fig. 15

According to the law of conservation of mechanical energy, the total mechanical energy in an isolated system does not change

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \,.$$

If the rod in the vertical position has zero potential energy and the kinetic energy of the rod in the horizontal position is zero (Fig. 15), the law of conservation of mechanical energy simplifies to the form

$$E_{k1} = E_{p2} \,.$$

The kinetic energy in the vertical position can be calculated as the kinetic energy of a rotating rigid body

$$E_{k1} = \frac{J\omega^2}{2} \,,$$

where the moment of inertia of the rod with respect to the axis passing through its endpoint (Problem 3.5) will be

$$J = \frac{ml^2}{3}$$

and the angular velocity of the rotating rod can be expressed in terms of the velocity of the rod's endpoint as

$$\omega = \frac{v}{l} \,.$$

In the horizontal position, the potential energy of the rod will be

$$E_{p2} = mgh \ ,$$

where the centre of gravity of the rod is raised to a height of

$$h = \frac{l}{2} \ .$$

From the law of conservation of mechanical energy, it follows

$$\frac{1}{2}\frac{ml^2}{3}\left(\frac{v}{l}\right)^2 = mg\frac{l}{2} \ ,$$

from which it is possible to express the velocity of the end point of the rod

$$v = \sqrt{3gl}$$

and after substituting numerical values

$$v = \sqrt{3 \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ 1\,\mathrm{m}} = 5.42\,\mathrm{m\,s^{-1}} \ .$$

4 Gravitational field

4.1 *How far is Mars from the Sun if its orbital period is $T_M = 1.9\,\mathrm{y}$ and the distance between the Sun and the Earth is $a_E = 1\,\mathrm{AU}$?*

Kepler's third law can be used

$$\frac{T_E^2}{T_M^2} = \frac{a_E^3}{a_M^3} \ ,$$

which implies

$$a_M^3 = a_E^3 \frac{T_M^2}{T_E^2} \ .$$

Therefore, the distance of Mars from the Sun is

$$a_M = a_E \sqrt[3]{\frac{T_M^2}{T_E^2}} \ ,$$

when substituted

$$a_M = 1\,\mathrm{AU} \sqrt[3]{\frac{(1.9\,\mathrm{y})^2}{(1\,\mathrm{y})^2}} = 1\,\mathrm{AU}\sqrt[3]{1.9^2} = 1.53\,\mathrm{AU} \ .$$

4.2 *The spacecraft is located between the Earth and the Moon, how far away from the Earth should the spacecraft be so that the resulting gravitational force on it from the Earth and the Moon is zero? The distance between the Earth and the Moon is $d = 384\,000\,\mathrm{km}$ and the mass of the Earth is 81 times the mass of the Moon.*

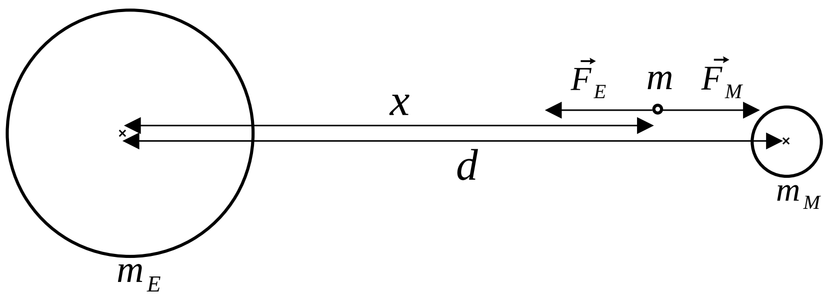

Fig. 16

Newton's law of gravity can be used

$$\vec{F} = -\kappa \frac{m_1 m_2}{r^3} \vec{r} \, .$$

The resulting gravitational force must be zero

$$\vec{F}_E + \vec{F}_M = 0 \, ,$$

because the forces are in opposite directions, their magnitudes must be equal

$$F_E = F_M \, ,$$

thus, for forces acting on a spacecraft of mass m located at a distance x from the Earth (Fig. 16), the following must be true

$$\kappa \frac{m_E m}{x^2} = \kappa \frac{m_M m}{(d-x)^2} \, .$$

If the ratio between the masses of the Earth and the Moon is

$$m_E = 81 m_M \, ,$$

then it applies

$$\frac{81}{x^2} = \frac{1}{(d-x)^2}$$

and the distance of the spacecraft from Earth will be

$$x = \frac{9}{10} d \, ,$$

when substituted

$$x = \frac{9}{10} \, . \, 384\,000\,\mathrm{km} = 345\,600\,\mathrm{km} \, .$$

4.3 *Two spherical bodies with masses m and $4m$ are at a distance d from each other. At what point between them will the resulting gravitational field be zero, and what will be the potential of the gravitational field at that point?*

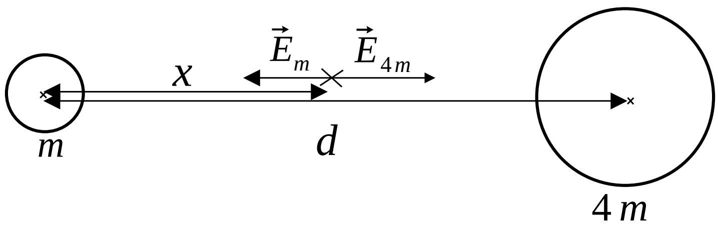

Fig. 17

The gravitational field of a spherical body of mass m at a point with position vector $\vec{r}$ is

$$\vec{E} = -\kappa \frac{m}{r^3} \vec{r} \, .$$

If the resulting gravitational field at the point between the two bodies is zero

$$\vec{E}_m + \vec{E}_{4m} = 0 \, ,$$

since the intensities are in opposite directions, their magnitudes must be equal (Fig. 17)

$$E_m = E_{4m} \, ,$$

therefore must apply

$$\kappa \frac{m}{x^2} = \kappa \frac{4m}{(d-x)^2} \, ,$$

from where the distance to a body of mass m can be expressed

$$x = \frac{d}{3} \, .$$

The gravitational potential

$$V = -\kappa \frac{m}{r} \, ,$$

is a scalar quantity and the resulting potential is the sum of the potentials at a given location from the individual bodies

$$V = V_m + V_{4m} \, ,$$

therefore, the resulting potential will be

$$V = -\kappa \frac{m}{x} - \kappa \frac{4m}{d-x} = -\kappa \frac{3m}{d} - \kappa \frac{12m}{2d} = -\kappa \frac{9m}{d} \, .$$

4.4 *From a homogeneous sphere of radius R and mass M, a new body was created by drilling a spherical cavity into the sphere with radius $R/2$ and centred at a distance $R/2$ from the centre of the original sphere. What will be the gravitational force exerted by the new body on a point of mass m located in the direction of the cavity at a distance d from the centre of the original sphere?*

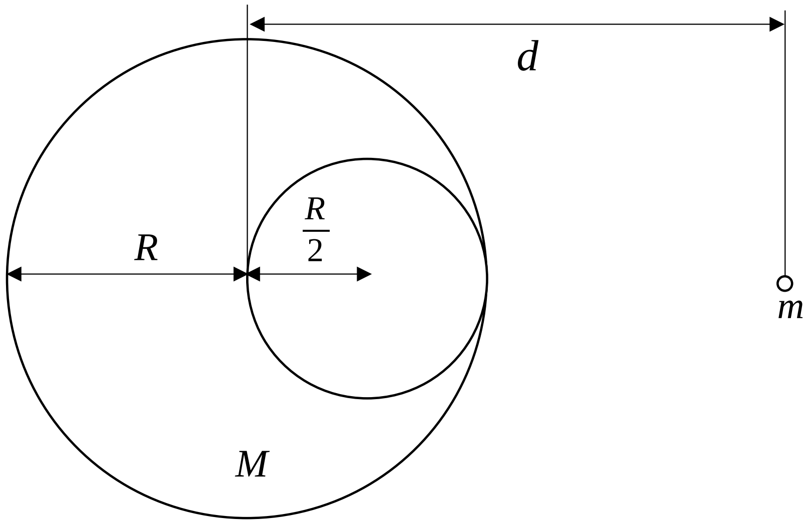

Fig. 18

The gravitational force exerted on the mass point by the original sphere $\vec{F}_0$ was the sum of the gravitational force exerted by the new body $\vec{F}$ and the gravitational force exerted by the drilled part $\vec{F}'$ (Fig. 18), therefore

$$\vec{F}_0 = \vec{F} + \vec{F}' \, .$$

The magnitude of the gravitational force exerted by the original sphere can be expressed from Newton's law of gravitation as

$$F_0 = \kappa \frac{mM}{d^2} \, .$$

The density of the material is

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} \, ,$$

therefore, the weight of the drilled part was

$$m' = \rho V' = \frac{M}{\frac{4}{3}\pi R^3} \, \frac{4}{3}\pi \left(\frac{R}{2}\right)^3 = \frac{M}{8}$$

and the magnitude of the gravitational force exerted by the drilled part was

$$F' = \kappa \frac{mm'}{\left(d - \frac{R}{2}\right)^2} = \kappa \frac{mM}{8\left(d - \frac{R}{2}\right)^2} \, .$$

The gravitational force exerted by a new body can be expressed as

$$F = F_0 - F' = \kappa \frac{mM}{d^2} - \kappa \frac{mM}{8\left(d - \frac{R}{2}\right)^2} = \kappa m M \left[\frac{1}{d^2} - \frac{1}{8\left(d - \frac{R}{2}\right)^2}\right] \, .$$

4.5 *Calculate the potential and the gravitational field of a rod of mass m and length l at a point lying on an extension of the rod at a distance a from its end.*

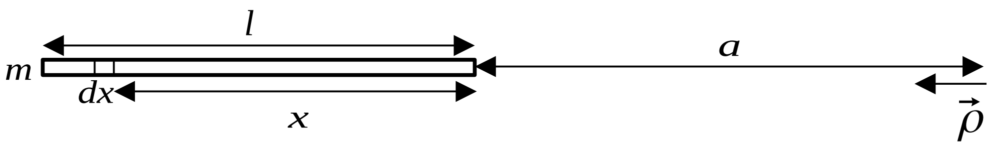

Fig. 19

The length density of the rod is

$$\lambda = \frac{m}{l} \, ,$$

therefore the mass element will be

$$\mathrm{d}m = \lambda \mathrm{d}x = \frac{m}{l}\mathrm{d}x$$

and the potential of the mass element (Fig. 19) will be

$$\mathrm{d}V = -\kappa \frac{\mathrm{d}m}{x+a} = -\kappa \frac{m}{l} \frac{\mathrm{d}x}{x+a} \, .$$

The potential of the whole rod can be calculated by integration over the whole mass of the rod

$$V = \int_m \mathrm{d}V = -\kappa \frac{m}{l} \int\limits_0^l \frac{\mathrm{d}x}{x+a} = -\kappa \frac{m}{l} \left[\ln\left(x+a\right)\right]_0^l = -\kappa \frac{m}{l} \ln \frac{l+a}{a} \, .$$

The relationship between the gravitational field and the gravitational potential is

$$\vec{E} = -\mathrm{grad}\, V \, ,$$

therefore the gravitational field of the whole bar will be

$$\vec{E} = -\frac{\mathrm{d}V}{\mathrm{d}a}\vec{\rho} = -\kappa \frac{m}{l} \frac{a}{l+a} \frac{a-l-a}{a^2} \vec{\rho} = \kappa \frac{m}{a(l+a)} \vec{\rho} \, .$$

where $\vec{\rho}$ is the unit vector in the direction of the gravitational field. The rules for the derivative of the composite function and the derivative of the fraction of functions were used.

4.6 *Calculate the potential and the gravitational field of a disk of mass m and radius R at a point on the axis of the disk at a distance a from its centre.*

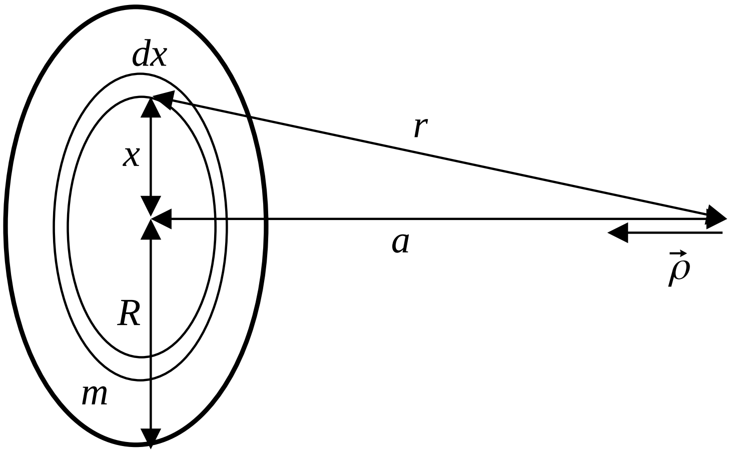

Fig. 20

The areal density of the disk is

$$\sigma = \frac{m}{S} = \frac{m}{\pi R^2}$$

and the element of the area (Fig. 20) of the intermediate circle is

$$\mathrm{d}S = 2\pi x \mathrm{d}x \ ,$$

then the mass element will be

$$dm = \sigma \mathrm{d}S = \frac{m}{\pi R^2} 2\pi x \mathrm{d}x = \frac{2m}{R^2} x \mathrm{d}x \ .$$

The magnitude of the position vector can be expressed using the Pythagorean theorem

$$r^2 = x^2 + a^2 \ ,$$

the potential of the mass element will be

$$dV = -\kappa \frac{dm}{r} = -\kappa \frac{2m}{R^2} x \frac{\mathrm{d}x}{r} = -\kappa \frac{2m}{R^2} x \frac{\mathrm{d}x}{\sqrt{x^2 + a^2}} \ .$$

The potential of the whole disk can be calculated by integrating over the whole mass of the disk

$$V = \int_m \mathrm{d}V = -\kappa \frac{2m}{R^2} \int\limits_0^R \frac{x \mathrm{d}x}{\sqrt{x^2 + a^2}} = -\kappa \frac{2m}{R^2} [\sqrt{x^2 + a^2}]_0^R = -\kappa \frac{2m}{R^2} \left(\sqrt{R^2 + a^2} - a \right) \ .$$

The relationship between the gravitational field and the gravitational potential is

$$\vec{E} = -\mathrm{grad}\, V \ ,$$

therefore the gravitational field of the whole disk will be

$$\vec{E} = -\frac{\mathrm{d}V}{\mathrm{d}a}\vec{\rho} = -\kappa\frac{2m}{R^2}\left(\frac{a}{\sqrt{R^2+a^2}} - 1\right)\vec{\rho} = \kappa\frac{2m}{R^2}\left(1 - \frac{a}{\sqrt{R^2+a^2}}\right)\vec{\rho} \ .$$

where $\vec{\rho}$ is the unit vector in the direction of the gravitational field.

4.7 *At what speed must a body be thrown from the surface of the Earth to fly beyond the range of the Earth's gravitational pull?*

When a body flies out of the Earth's gravitational pull and comes to rest, it will have both zero gravitational potential energy and zero kinetic energy. The law of conservation of mechanical energy will therefore take the form

$$E_p + E_k = 0 \ .$$

The gravitational potential energy of a body of mass m on the surface of the Earth is

$$E_p = -\kappa\frac{mM_E}{R_E} \ ,$$

where M_E is the mass of the Earth and R_E is the radius of the Earth. The kinetic energy of a body ejected at velocity v from the surface of the Earth is

$$E_k = \frac{mv^2}{2} \ ,$$

therefore, the law of conservation of mechanical energy implies

$$-\kappa\frac{mM_E}{R_E} + \frac{mv^2}{2} = 0 \ ,$$

from which it is possible to express the velocity of the body as

$$v = \sqrt{\frac{2\kappa M_E}{R_E}} \ .$$

The result can also be expressed using the gravitational acceleration

$$g = \kappa\frac{M_E}{R_E^2} \ ,$$

which for the velocity of the body implies

$$v = \sqrt{2gR_E}$$

and after inserting the numerical values

$$v = \sqrt{2 \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ 6\,378\,000\,\mathrm{m}} = 11\,186\,\mathrm{m\,s^{-1}} \ .$$

4.8 *The projectile was fired from the Earth's surface at a velocity of* $v = 1600\,\mathrm{m\,s^{-1}}$. *Calculate the difference in altitudes the body would have reached assuming the gravitational field is homogeneous and assuming the gravitational field is radial.*

The solution can be found using the law of conservation of mechanical energy

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \ .$$

In a homogeneous gravitational field, the potential energy at the Earth's surface E_{p1} can be chosen as the point with zero potential energy, and the body's velocity decreases until the body comes to rest, so its kinetic energy E_{k2} will be zero. The law of conservation of mechanical energy will therefore take the form

$$E_{k1} = E_{p2} \ ,$$

thus applies

$$\frac{mv^2}{2} = mgh_h \ ,$$

from which it is possible to express the altitude of the projectile in a homogeneous gravitational field as

$$h_h = \frac{v^2}{2g} \ .$$

In a radial gravitational field, the potential energy E_{p1} will be at the surface of the Earth and at h_r the potential energy E_{p2} will be at the height h_r. Therefore, the law of conservation of mechanical energy in a radial gravitational field will be

$$E_{k1} + E_{p1} = E_{p2} \ ,$$

thus applies

$$\frac{mv^2}{2} - \kappa\frac{mM_E}{R_E} = -\kappa\frac{mM_E}{R_E + h_r} \ ,$$

using the gravitational acceleration

$$g = \kappa\frac{M_E}{R_E^2} \ ,$$

the relation can be modified to the form

$$\frac{mv^2}{2} = mgR_E\frac{h_r}{R_E + h_r} \ ,$$

from which it is possible to express the altitude of the projectile in the radial gravitational field as

$$h_r = \frac{v^2 R_E}{2gR_E - v^2} \ .$$

The difference in altitudes in the homogeneous and radial field will therefore be

$$\Delta h = h_r - h_h = \frac{v^2 R_E}{2gR_E - v^2} - \frac{v^2}{2g} \ ,$$

and after inserting the numerical values

$$\Delta h = \frac{(1600\,\mathrm{m\,s^{-1}})^2 \ . \ 6.378 \cdot 10^6\,\mathrm{m}}{2 \ . \ 9.81\,\mathrm{m\,s^{-2}} \ . \ 6.378 \cdot 10^6\,\mathrm{m} - (1600\,\mathrm{m\,s^{-1}})^2} - \frac{(1600\,\mathrm{m\,s^{-1}})^2}{2 \ . \ 9.81\,\mathrm{m\,s^{-2}}} = 2725\,\mathrm{m} \ .$$

4.9 *Calculate the kinetic energy of a body with mass* $m = 70\,\mathrm{kg}$ *that hits the surface of the Earth from a height* $h = 10\,\mathrm{km}$, *if the Earth's gravitational field is assumed to be radial.*

The solution can be found using the law of conservation of mechanical energy

$$E_{k1} + E_{p1} = E_{k2} + E_{p2} \ .$$

If the initial kinetic energy of the body E_{k1} is zero, the law of conservation of mechanical energy takes the form

$$E_{p1} = E_{k2} + E_{p2} \ .$$

The potential energy of a body at a height h above the surface of the Earth is equal to

$$E_{p1} = -\kappa \frac{mM_E}{R_E + h}$$

and the potential energy of a body on the Earth's surface is equal to

$$E_{p2} = -\kappa \frac{mM_E}{R_E} \ .$$

The law of conservation of mechanical energy will therefore take the form

$$-\kappa \frac{mM_E}{R_E + h} = E_{k2} - \kappa \frac{mM_E}{R_E} \ ,$$

which implies for the kinetic energy at impact

$$E_k = -\kappa m M_E \left(\frac{1}{R_E + h} - \frac{1}{R_E} \right) \ .$$

Using gravitational acceleration

$$g = \kappa \frac{M_E}{R_E^2}$$

the kinetic energy can be expressed as

$$E_k = mgR_E \frac{h}{R_E + h} \ ,$$

after inserting the numerical values

$$E_k = 70\,\mathrm{kg}\ .\ 9.81\,\mathrm{m\,s^{-2}}\ .\ 6.378 \cdot 10^6\,\mathrm{m}\ .\ \frac{10^4\,\mathrm{m}}{6.378 \cdot 10^6\,\mathrm{m} + 10^4\,\mathrm{m}} = 6.856 \cdot 10^6\,\mathrm{J}\ .$$

4.10 *How high does a satellite have to be above the equator to be over the same place all the time as it moves?*

A satellite with mass m moves on a circle with radius $R_E + h$, the gravitational force of the Earth acts on the satellite as a centripetal force, thus

$$F_g = F_c \ ,$$

which can be rewritten into the form

$$\kappa \frac{mM_E}{(R_E + h)^2} = m \frac{v^2}{R_E + h} \ ,$$

where M_E is the mass and R_E is the radius of the Earth. For the speed of the satellite, it follows

$$v = \sqrt{\frac{\kappa M_E}{R_E + h}} \ .$$

For a satellite to be over the same place all the time, its angular velocity must be the same as the angular velocity of the Earth

$$\omega_s = \omega_E \ ,$$

therefore must apply

$$\frac{v}{R_E + h} = \frac{2\pi}{T_E} \ ,$$

after inserting the speed of the satellite

$$\sqrt{\frac{\kappa M_E}{(R_E + h)^3}} = \frac{2\pi}{T_E} \ .$$

For the height of the satellite, it follows

$$h = \sqrt[3]{\frac{\kappa M_E T_E^2}{4\pi^2}} - R_E \ ,$$

after inserting the numerical values

$$h = \sqrt[3]{\frac{6.67 \cdot 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}\ .\ 5.972 \cdot 10^{24}\,\mathrm{kg}\ .\ (3.1536 \cdot 10^{7}\,\mathrm{s})^2}{4\pi^2}} - 6.371 \cdot 10^6\,\mathrm{m} \ ,$$

the height of the satellite will be

$$h = 35.8 \cdot 10^6\,\mathrm{m} \ .$$

5 Thermodynamics and molecular physics

5.1 *The density of air under normal conditions, that is, pressure* $p_0 = 101\,325\,\mathrm{Pa}$ *and temperature* $t_0 = 0\,°\mathrm{C}$ *is* $\rho_0 = 1.293\,\mathrm{kg\,m^{-3}}$*. What will be the density of air at pressure* $p = 0.5\,\mathrm{MPa}$ *and temperature* $t = 50\,°\mathrm{C}$*?*

The problem can be solved using the equation of state of an ideal gas

$$pV = nRT \ ,$$

where R is the universal gas constant, and the amount of substance is

$$n = \frac{m}{M} \ ,$$

where M denotes the molar mass. The equation of state takes the form

$$pV = \frac{m}{M}RT \ .$$

From the density of air

$$\rho = \frac{m}{V} \ ,$$

it is possible to express the mass of air

$$m = \rho V \ ,$$

and the equation of state will take the form

$$pV = \frac{\rho V}{M}RT \ ,$$

from which the density can be expressed as

$$\rho = \frac{pM}{RT} \ .$$

The density of the gas under normal conditions will be

$$\rho_0 = \frac{p_0 M}{R T_0} \ .$$

By dividing the equations, the density of the gas can be expressed using the relation

$$\rho = \rho_0 \frac{p T_0}{p_0 T} \ .$$

After inserting the numerical values

$$\rho = 1.293\,\mathrm{kg\,m^3} \ . \ \frac{0.5 \cdot 10^6\,\mathrm{Pa} \ . \ 273.15\,\mathrm{K}}{1.013\,25 \cdot 10^5\,\mathrm{Pa} \ . \ 323.15\,\mathrm{K}} = 5.39\,\mathrm{kg\,m^{-3}} \ ,$$

where the gas temperatures in the thermodynamic scale have been converted from the Celsius scale using the relationship

$$T[\mathrm{K}] = t[^\circ\mathrm{C}] + 273.15\,\mathrm{K} \ .$$

5.2 *The pressure in the cylinder of a steam engine with volume* $V = 20\,\mathrm{l}$ *is reduced by* $\Delta p = 0.5\,\mathrm{MPa}$ *when the valve is opened. What mass of steam has been released from the cylinder if the steam temperature* $t = 100\,^\circ\mathrm{C}$ *has not changed?*

The solution of the problem is possible using the equation of state of an ideal gas

$$pV = nRT \ ,$$

where R is the universal gas constant, and the amount of substance is

$$n = \frac{m}{M} \ ,$$

where M denotes the molar mass. The equation of state takes the form

$$pV = \frac{m}{M} RT \ .$$

Before the steam was released, the equation of state described the gas is

$$p_1 V = \frac{m_1}{M} RT \ ,$$

After the steam was released, the equation of state described the gas is

$$p_2 V = \frac{m_2}{M} RT \ .$$

By subtracting the equations of state, it is possible to obtain the equation

$$(p_1 - p_2)V = (m_1 - m_2)\frac{RT}{M} \ ,$$

in which the mass of the released steam is

$$\Delta m = m_1 - m_2 \ ,$$

and the reduction of pressure is

$$\Delta p = p_1 - p_2 \ .$$

The equation can therefore be written as

$$\Delta p V = \Delta m \frac{RT}{M} \ ,$$

from which it is possible to express the mass of the released steam

$$\Delta m = \frac{MV}{RT}\Delta p \ ,$$

and after inserting the numerical values

$$\Delta m = \frac{0.018\,\mathrm{kg\,mol^{-1}}\ .\ 0.02\,\mathrm{m^3}}{8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}\ .\ 373.15\,\mathrm{K}}\ .\ 5\cdot 10^5\,\mathrm{Pa} = 0.058\,\mathrm{kg} = 58\,\mathrm{g} \ ,$$

where the gas temperatures in the thermodynamic scale have been converted from the Celsius scale using the relationship

$$T[\mathrm{K}] = t[^\circ\mathrm{C}] + 273.15\,\mathrm{K} \ .$$

5.3 *The same gas is in two containers separated by a cap. In the first container with volume $V_1 = 2\,\mathrm{l}$, the pressure of the gas is $p_1 = 0.2\,\mathrm{MPa}$. In the second container with volume $V_2 = 4\,\mathrm{l}$, the pressure of the gas is $p_2 = 0.4\,\mathrm{MPa}$. What will be the resulting pressure when the cap is opened, if the temperature in the containers is the same and it will remain the same after joining?*

From the equation of state for the gas in the first container

$$p_1 V_1 = n_1 RT \ ,$$

follows the amount of gas in the first container is

$$n_1 = \frac{p_1 V_1}{RT} \ .$$

From the equation of state for the gas in the second container

$$p_2V_2 = n_2RT \ ,$$

follows the amount of gas in the second container is

$$n_2 = \frac{p_2V_2}{RT} \ .$$

When the cap is opened, the equation of state for the gas will be

$$p(V_1 + V_2) = (n_1 + n_2)RT \ ,$$

when substituted for the amounts of substance, the equation of state takes the form

$$p(V_1 + V_2) = \left(\frac{p_1V_1}{RT} + \frac{p_2V_2}{RT}\right) RT \ ,$$

which can be simplified to the equation

$$p(V_1 + V_2) = p_1V_1 + p_2V_2 \ ,$$

from which the resulting pressure can be calculated as

$$p = \frac{p_1V_1 + p_2V_2}{V_1 + V_2} \ ,$$

and after inserting the numerical values, the resulting pressure is

$$p = \frac{0.2 \cdot 10^6\,\mathrm{Pa}\ .\ 2 \cdot 10^{-3}\,\mathrm{m}^3 + 0.4 \cdot 10^6\,\mathrm{Pa}\ .\ 4 \cdot 10^{-3}\,\mathrm{m}^3}{2 \cdot 10^{-3}\,\mathrm{m}^3 + 4 \cdot 10^{-3}\,\mathrm{m}^3} = 0.33 \cdot 10^6\,\mathrm{Pa} \ .$$

5.4 *What is the internal energy of nitrogen, which at pressure* $p = 0.5\,\mathrm{MPa}$ *occupies volume* $V = 5\,\mathrm{l}$*?*

The internal energy of a gas is the sum of the kinetic energies of all N gas molecules

$$U = N\epsilon_m \ ,$$

where the equipartition theorem for the mean kinetic energy of one molecule implies

$$\epsilon_m = \frac{i}{2}kT \ ,$$

where i denotes the number of degrees of freedom of the molecule and k is the Boltzmann constant. Thus, the kinetic energy of all molecules will be

$$U = \frac{i}{2}NkT \ .$$

The number of gas molecules will be

$$N = nN_A \ ,$$

where N_A is Avogadro's constant, and the expression for Boltzmann's constant is

$$k = \frac{R}{N_A} \ ,$$

therefore, the internal energy of a gas can also be expressed as

$$U = \frac{i}{2} nRT \ .$$

Using the equation of state

$$pV = nRT \ ,$$

the internal energy of a gas can be expressed in terms of pressure and volume as

$$U = \frac{i}{2} pV \ ,$$

Since nitrogen is a diatomic gas, $i = 5$, and the internal energy of the gas will be

$$U = \frac{5}{2} pV \ ,$$

and after inserting the numerical values, the internal energy is

$$U = \frac{5}{2} \ . \ 0.5 \cdot 10^6 \, \mathrm{Pa} \ . \ 5 \cdot 10^{-3} \, \mathrm{m}^3 = 6250 \, \mathrm{J} \ .$$

5.5 *How does the mean kinetic energy of an argon gas molecule with mass* $m = 500\,\mathrm{g}$ *change if we supply the gas with heat* $Q = 5000\,\mathrm{J}$*, and at the same time, the gas does work* A' = $2000\,\mathrm{J}$*? The molar mass of argon is* $M = 39.9\,\mathrm{g\,mol^{-1}}$.

The internal energy of a gas is the sum of the kinetic energies of all N gas molecules

$$U = N\epsilon_m \ ,$$

therefore, the change in the internal energy of the gas will also be the change in the mean kinetic energy of all the molecules

$$\Delta U = N\Delta\epsilon_m \ .$$

The number of molecules can be calculated from the amount of substance

$$N = nN_A \ ,$$

which can be calculated from the mass of the gas

$$n = \frac{m}{M} \ .$$

The change in the internal energy of a gas can therefore be written as

$$\Delta U = \frac{mN_A}{M} \Delta \epsilon_s \ ,$$

from which the change in the mean kinetic energy of the molecule will be

$$\Delta \epsilon_m = \frac{M}{mN_A} \Delta U \ .$$

According to the first law of thermodynamics

$$\Delta U = Q + A \ ,$$

the change in the internal energy of the gas is equal to the sum of the heat input and the external mechanical work. If the work is done by the gas

$$A = -A' \ ,$$

the change in internal energy will be

$$\Delta U = Q - A' \ .$$

The change in the mean kinetic energy of a molecule can thus be expressed as

$$\Delta \epsilon_m = \frac{M}{mN_A} (Q - A')$$

and after inserting the numerical values

$$\Delta \epsilon_m = \frac{39.9 \cdot 10^{-3}\,\mathrm{kg\,mol^{-1}}}{0.5\,\mathrm{kg} \ . \ 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}} \ . \ (5000\,\mathrm{J} - 2000\,\mathrm{J}) = 3.98 \cdot 10^{-22}\,\mathrm{J} \ .$$

5.6 *A container of volume* $V = 0.05\,\mathrm{m}^3$ *contains hydrogen at temperature* $t_0 = 27\,°\mathrm{C}$ *and pressure* $p_0 = 100\,\mathrm{kPa}$. *Calculate the pressure and temperature of the gas if the hydrogen received heat* $Q = 1.5\,\mathrm{kJ}$.

The change in the internal energy of a gas can be expressed using the temperature difference as

$$\Delta U = \frac{i}{2} Nk\Delta T = \frac{i}{2} Nk(T - T_0) ,$$

where the number of particles can be calculated using the amount of substance

$$N = nN_A$$

and the Boltzmann constant is

$$k = \frac{R}{N_A} ,$$

this implies for the change of the internal energy of the gas

$$\Delta U = \frac{i}{2} nR(T - T_0) .$$

Using the equation of state

$$p_0 V = nRT_0 ,$$

$$pV = nRT ,$$

the change of the internal energy can be written in the form

$$\Delta U = \frac{i}{2}(pV - p_0 V) .$$

According to the first law of thermodynamics

$$\Delta U = Q + A ,$$

the change in the internal energy of the gas is equal to the sum of the heat input and the external mechanical work. If the volume of the gas is constant, the mechanical work is zero

$$\mathrm{d}A = -p\mathrm{d}V \implies A = 0$$

and the change in internal energy is equal to the heat input

$$\Delta U = Q .$$

It is therefore valid

$$Q = \frac{i}{2}(pV - p_0 V) ,$$

from which the gas pressure can be calculated

$$p = \frac{Q + \frac{i}{2} p_0 V}{\frac{i}{2} V} \,.$$

Because hydrogen is a diatomic gas, the number of degrees of freedom of its molecule is $i = 5$ and

$$p = \frac{1500\,\mathrm{J} + \frac{5}{2}\,100\,000\,\mathrm{Pa}\,.\,0.05\,\mathrm{m}^3}{\frac{5}{2}\,.\,0.05\,\mathrm{m}^3} = 112\,000\,\mathrm{Pa} = 112\,\mathrm{kPa}\,.$$

For the isochoric process, Charles' law states

$$\frac{p_0}{T_0} = \frac{p}{T}\,,$$

which gives the resulting temperature

$$T = \frac{p}{p_0} T_0$$

and after inserting the numerical values

$$T = \frac{112\,000\,\mathrm{Pa}}{100\,000\,\mathrm{Pa}}\,.\,300.15\,\mathrm{K} = 336.17\,\mathrm{K} = 63.02\,{}^\circ\mathrm{C}\,.$$

5.7 *In nitrogen with mass $m = 200\,\mathrm{g}$, initial temperature $t_1 = 27\,{}^\circ\mathrm{C}$ and pressure $p_1 = 0.4\,\mathrm{MPa}$, a thermodynamic process took place in which the pressure of nitrogen decreased to $p_2 = 0.3\,\mathrm{MPa}$. How much heat was added to the nitrogen, what work did the gas do, and how did its internal energy change if the process was a) isochoric, b) isothermal, c) adiabatic? Draw all these processes in p-V diagrams.*

---

a) For an isochoric process (Fig. 21), the volume of the gas is constant

$$V = \text{const.} \implies \mathrm{d}V = 0\,,$$

which implies that the work of the gas is zero

$$\mathrm{d}A' = p\mathrm{d}V \implies A' = 0\,.$$

According to the first law of thermodynamics, the heat delivered to a gas is equal to the sum of the change in its internal energy and the work done by the gas

$$Q = \Delta U + A'\,,$$

thus, for an isochoric process

$$Q = \Delta U \ .$$

The change in the internal energy of a gas can be expressed in terms of the temperature change as

$$\Delta U = \frac{i}{2} n R \Delta T = \frac{i}{2} \frac{m}{M} R (T_2 - T_1) \ ,$$

where $M = 28\,\mathrm{g\,mol^{-1}}$ is the molar mass of nitrogen. From Charles' law

$$\frac{p_1}{T_1} = \frac{p_2}{T_2} \ ,$$

follows

$$T_2 = \frac{p_2}{p_1} T_1$$

and the change in internal energy can be expressed as

$$\Delta U = \frac{i}{2} \frac{m}{M} R T_1 \left(\frac{p_2}{p_1} - 1 \right) \ ,$$

after inserting the numerical values

$$\Delta U = \frac{5 \ . \ 0.2\,\mathrm{kg}}{2 \ . \ 28 \cdot 10^{-3}\,\mathrm{kg\,mol^{-1}}} \ . \ 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}} \ . \ 300.15\,\mathrm{K} \left(\frac{0.3 \cdot 10^6\,\mathrm{Pa}}{0.4 \cdot 10^6\,\mathrm{Pa}} - 1 \right) =$$
$$= -11\,140\,\mathrm{J} = -11.14\,\mathrm{kJ} \ .$$

The heat added to the gas is equal to the change in internal energy

$$Q = \Delta U = -11.14\,\mathrm{kJ} \ .$$

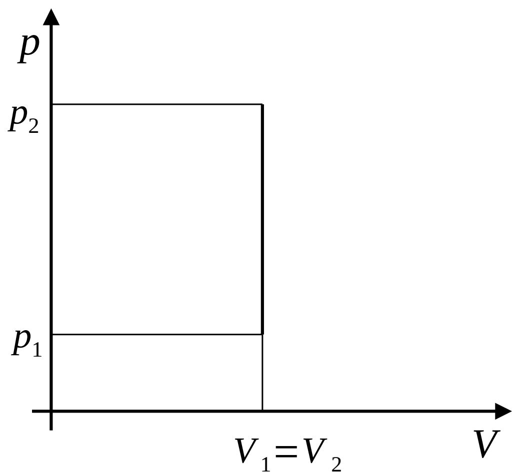

Fig. 21

b) For an isothermal process (Fig. 22), the temperature of the gas is constant

$$T = \text{const.} \implies \Delta T = 0 \ ,$$

therefore, the change in internal energy of the gas is zero

$$\Delta U = \frac{i}{2} nR\Delta T = 0 \ ,$$

and from the first law of thermodynamics

$$Q = \Delta U + A' \ ,$$

follows

$$Q = A' \ .$$

From the Boyle-Mariott law

$$pV = p_1 V_1 \ ,$$

follows

$$p = \frac{p_1 V_1}{V} \ ,$$

and from the equation of state

$$p_1 V_1 = nRT_1 \ ,$$

follows

$$p = \frac{nRT_1}{V} = \frac{m}{M}\frac{RT_1}{V} \ .$$

The work of the gas can be calculated as

$$A' = \int_{V_1}^{V_2} p \mathrm{d}V = \frac{m}{M} RT_1 \int_{V_1}^{V_2} \frac{1}{V} \mathrm{d}V = \frac{m}{M} RT_1 \left[\ln V\right]_{V_1}^{V_2} =$$

$$= \frac{m}{M} RT_1 (\ln V_2 - \ln V_1) = \frac{m}{M} RT_1 \ln \frac{V_2}{V_1} = \frac{m}{M} RT_1 \ln \frac{p_1}{p_2} \ .$$

After inserting the numerical values

$$A' = \frac{0.2\,\mathrm{kg}}{28 \cdot 10^{-3}\,\mathrm{g\,mol^{-1}}} \ . \ 8.314\,\mathrm{J\,K\,mol^{-1}} \ . \ 300.15\,\mathrm{K} \ . \ \ln \frac{0.4 \cdot 10^6\,\mathrm{Pa}}{0.3 \cdot 10^6\,\mathrm{Pa}} =$$

$$= 5128\,\mathrm{J} = 5.128\,\mathrm{kJ} \ .$$

The heat added to the gas is equal to the work of the gas

$$Q = A' = 5.128\,\mathrm{kJ} \ .$$

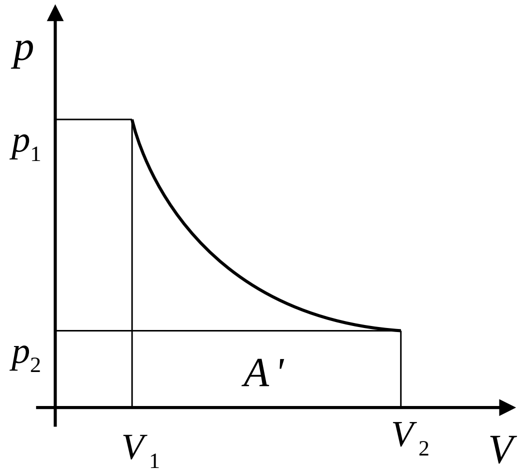

Fig. 22

c) In an adiabatic process (Fig. 23), there is no heat exchange between the gas and the surroundings

$$Q = 0 \; .$$

From the first law of thermodynamics

$$Q = \Delta U + A' \; ,$$

follows

$$\Delta U = -A' \; .$$

The change in the internal energy of a gas can be expressed as

$$\Delta U = \frac{i}{2}\frac{m}{M}R(T_2 - T_1) = \frac{i}{2}\frac{m}{M}RT_1\left(\frac{T_2}{T_1} - 1\right) \; .$$

From the Poisson's equation

$$p_1 V_1^\kappa = p_2 V_2^\kappa \; ,$$

can be expressed

$$\frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^\kappa \implies \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}} = \frac{V_2}{V_1} \; ,$$

and from the equation of state

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2} \; ,$$

can be expressed

$$\frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1} \,,$$

which can be transformed into the form

$$\frac{T_2}{T_1} = \left(\frac{p_1}{p_2}\right)^{-1} \left(\frac{p_1}{p_2}\right)^{\frac{1}{\kappa}} = \left(\frac{p_1}{p_2}\right)^{\frac{1-\kappa}{\kappa}} \,.$$

Using Mayer's equation

$$C_p = C_v + R \,,$$

the Poisson constant can be expressed in terms of the number of degrees of freedom

$$\kappa = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} = \frac{\frac{i}{2}R + R}{\frac{i}{2}R} = \frac{i+2}{i} \,.$$

The change of internal energy can therefore be calculated as

$$\Delta U = \frac{i}{2}\frac{m}{M}RT_1 \left[\left(\frac{p_1}{p_2}\right)^{-\frac{2}{i+2}} - 1\right] = \frac{i}{2}\frac{m}{M}RT_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{i+2}} - 1\right] \,.$$

Inserting numerical values

$$\Delta U = \frac{5 \,.\, 0.2\,\mathrm{kg}}{2 \,.\, 28 \cdot 10^{-3}\,\mathrm{kg\,mol^{-1}}} \,.\, 8.314\,\mathrm{J\,K^{-1}} \,.\, 300.15\,\mathrm{K} \left[\left(\frac{0.3 \cdot 10^6\,\mathrm{Pa}}{0.4 \cdot 10^6\,\mathrm{Pa}}\right)^{\frac{2}{7}} - 1\right] =$$

$$= -3516\,\mathrm{J} = -3.516\,\mathrm{kJ} \,.$$

The work of the gas will be

$$A' = -\Delta U = 3.516\,\mathrm{kJ} \,.$$

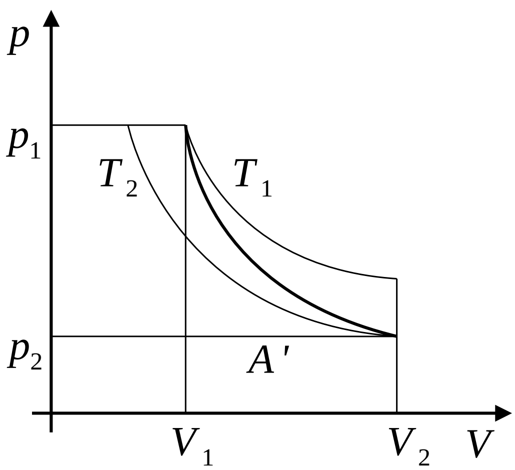

Fig. 23

5.8 *To air of temperature* $t_1 = 20\,°\mathrm{C}$, *which occupies at pressure* $p_1 = 0.1\,\mathrm{MPa}$ *a volume* $V_1 = 2\,\mathrm{m}^3$, *a heat* $Q = 400\,\mathrm{kJ}$ *has been added. Calculate the change in internal energy, external work and final state quantities if the action was a) isochoric b) isobaric c) isothermal. The Poisson constant of air is* $\kappa = 1.4$.

a) In an isochoric process the volume of a gas is constant, therefore, the gas does not produce the mechanical work

$$V = \text{const.} \implies A' = 0$$

and from the first law of thermodynamics

$$Q = \Delta U + A' \ ,$$

follows that the change in internal energy of a gas is equal to the added heat

$$\Delta U = Q = 400\,\mathrm{kJ} \ .$$

From the equation of state

$$p_1 V_1 = nRT_1 \ ,$$

the amount of substance can be expressed as

$$n = \frac{p_1 V_1}{RT_1}$$

and from Poisson's constant

$$\kappa = \frac{i+2}{i} \ ,$$

the number of degrees of freedom is

$$i = \frac{2}{\kappa - 1} \ .$$

The change of the internal energy of a gas can be expressed as

$$\Delta U = \frac{i}{2} nR(T_2 - T_1) = \frac{1}{\kappa - 1} \frac{p_1 V_1}{T_1} (T_2 - T_1) \ ,$$

because

$$Q = \frac{1}{\kappa - 1} \frac{p_1 V_1}{T_1} (T_2 - T_1) \ ,$$

the resulting temperature can be calculated as

$$T_2 = \frac{Q + \frac{1}{\kappa-1}p_1V_1}{\frac{1}{\kappa-1}\frac{p_1V_1}{T_1}} = T_1\left[\frac{(\kappa-1)Q}{p_1V_1} + 1\right]$$

and after inserting the numerical values

$$T_2 = 293.15\,\mathrm{K}\ .\ \left[\frac{(1.4-1)\ .\ 400\cdot 10^3\,\mathrm{J}}{0.1\cdot 10^6\,\mathrm{Pa}\ .\ 2\,\mathrm{m}^3} + 1\right] = 527.67\,\mathrm{K}\ .$$

From Charles's law

$$\frac{p_1}{T_1} = \frac{p_2}{T_2}\ ,$$

follows the resulting pressure

$$p_2 = p_1\frac{T_2}{T_1}\ ,$$

which can be modified to the form

$$p_2 = p_1\left[\frac{(\kappa-1)Q}{p_1V_1} + 1\right]\ ,$$

and after inserting the numerical values

$$p_2 = T_2 = 0.1\cdot 10^6\,\mathrm{Pa}\ .\ \left[\frac{(1.4-1)\ .\ 400\cdot 10^3\,\mathrm{J}}{0.1\cdot 10^6\,\mathrm{Pa}\ .\ 2\,\mathrm{m}^3} + 1\right] = 0.18\cdot 10^6\,\mathrm{Pa}\ .$$

b) In an isobaric process the pressure of the gas is constant

$$p = \text{const.}\ .$$

The amount of substance is

$$n = \frac{p_1V_1}{RT_1}$$

and the number of degrees of freedom is

$$i = \frac{2}{\kappa-1}\ .$$

The change of internal energy of gas can then be expressed as

$$\Delta U = \frac{i}{2}nR(T_2-T_1) = \frac{1}{\kappa-1}\frac{p_1V_1}{RT_1}R(T_2-T_1) = \frac{1}{\kappa-1}p_1V_1\left(\frac{T_2}{T_1}-1\right)\ .$$

From Gay-Lussac's law

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}\ ,$$

follows

$$\frac{T_2}{T_1} = \frac{V_2}{V_1} ,$$

by which the change of internal energy can be calculated as

$$\Delta U = \frac{1}{\kappa - 1} p_1 V_1 \left(\frac{V_2}{V_1} - 1 \right) .$$

The work of the gas at constant pressure is

$$A' = \int_{V_1}^{V_2} p_1 \mathrm{d}V = p_1 (V_2 - V_1) .$$

From the first law of thermodynamics

$$Q = \Delta U + A' ,$$

follows

$$Q = \frac{1}{\kappa - 1} p_1 V_1 \left(\frac{V_2}{V_1} - 1 \right) + p_1 (V_2 - V_1) ,$$

which can be modified to the form

$$Q(\kappa - 1) + p_1 V_1 \kappa = p_1 V_2 \kappa ,$$

from which the resulting volume can be expressed as

$$V_2 = V_1 \left[\frac{Q(\kappa - 1)}{\kappa p_1 V_1} + 1 \right] ,$$

after inserting the numerical values

$$V_2 = 2\,\mathrm{m}^3 \,.\, \left[\frac{400 \cdot 10^3\,\mathrm{J} \,.\, (1.4 - 1)}{1.4 \,.\, 0.1 \cdot 10^6\,\mathrm{Pa} \,.\, 2\,\mathrm{m}^3} + 1 \right] = 3.14\,\mathrm{m}^3 .$$

From Gay-Lussac's law

$$\frac{V_1}{T_1} = \frac{V_2}{T_2} ,$$

for the resulting temperature follows

$$T_2 = T_1 \frac{V_2}{V_1} ,$$

which can be transformed to the form

$$T_2 = T_1 \left[\frac{Q(\kappa - 1)}{\kappa p_1 V_1} + 1 \right]$$

and after inserting the numerical values

$$T_2 = 293.15\,\mathrm{K}\;.\;\left[\frac{400\cdot 10^3\,\mathrm{J}\;.\;(1.4-1)}{1.4\;.\;0.1\cdot 10^6\,\mathrm{Pa}\;.\;2\,\mathrm{m}^3}+1\right] = 460.66\,\mathrm{K}\;.$$

c) In an isothermal process the temperature of the gas is constant

$$T = \text{const.}\;,$$

therefore the change of the internal energy of the gas is zero

$$\Delta U = 0\;.$$

From the first law of thermodynamics

$$Q = \Delta U + A'\;,$$

follows that the work of the gas is equal to the added heat

$$A' = Q = 400\,\mathrm{kJ}\;.$$

From Boyle-Mariott's law

$$p_1 V_1 = p_2 V_2 = pV\;,$$

it is possible to express the pressure of a gas

$$p = \frac{p_1 V_1}{V}\;,$$

which can be used to express the work of the gas

$$A' = \int_{V_1}^{V_2} p\mathrm{d}V = \int_{V_1}^{V_2} \frac{p_1 V_1}{V}\mathrm{d}V = p_1 V_1 [\ln V]_{V_1}^{V_2} = p_1 V_1 \ln\frac{V_2}{V_1}\;.$$

Because it applies

$$p_1 V_1 \ln\frac{V_2}{V_1} = Q\;,$$

the resulting volume will be

$$V_2 = V_1 e^{\frac{Q}{p_1 V_1}}\;,$$

after inserting the numerical values

$$V_2 = 2\,\mathrm{m}^3\;.\;e^{\frac{400\cdot 10^3\,\mathrm{J}}{0.1\cdot 10^6\,\mathrm{Pa}\;.\;2\,\mathrm{m}^3}} = 14.78\,\mathrm{m}^3\;.$$

The resulting pressure will then be

$$p_2 = p_1 \frac{V_1}{V_2} = p_1 e^{-\frac{Q}{p_1 V_1}}$$

and after adding the numerical values

$$p_2 = 0.1 \cdot 10^6\,\mathrm{Pa}\,.\, e^{\frac{-400 \cdot 10^3\,\mathrm{J}}{0.1 \cdot 10^6\,\mathrm{Pa}\,.\,2\,\mathrm{m}^3}} = 13\,533\,\mathrm{Pa}\,.$$

5.9 *Helium with the amount of substance* $n = 2\,\mathrm{kmol}$ *expands isobarically and increases its volume threefold. What is the change of entropy for this action?*

The change of entropy for reversible processes is defined as

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T}\,,$$

for an isobaric event, when the system goes from the 1 state to the 2 state, the total entropy change will be

$$\Delta S = \int_1^2 \frac{\mathrm{d}Q}{T} = \int_1^2 \frac{nC_p \mathrm{d}T}{T}\,.$$

The molar heat capacity at constant volume is

$$C_V = \frac{i}{2} R\,,$$

using Mayer's relation

$$C_p = C_V + R\,,$$

the molar heat capacity at constant pressure can be expressed as

$$C_p = \frac{i}{2} R + R = \frac{i+2}{2} R\,.$$

The entropy change will then be

$$\Delta S = n \frac{i+2}{2} R \int_1^2 \frac{\mathrm{d}T}{T} = n \frac{i+2}{2} R [\ln T]_{T_1}^{T_2} = n \frac{i+2}{2} R \ln \frac{T_2}{T_1}\,.$$

From Gay-Lussac's law

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}\,,$$

follows

$$\frac{T_2}{T_1} = \frac{V_2}{V_1}\,,$$

by which the entropy change can be expressed as

$$\Delta S = n\frac{i+2}{2}R\ln\frac{V_2}{V_1}$$

and after adding the numerical values

$$\Delta S = 2\cdot 10^3\,\mathrm{mol}\;.\;\frac{3+2}{2}\;.\;8.314\,\mathrm{J\,K^{-1}\,mol^{-1}}\;.\;\ln 3 = 45\,669\,\mathrm{J\,K^{-1}}\;.$$

5.10 *Calculate the change of entropy of an ideal gas that is isothermally expanded from a volume $V_0 = 2\,\mathrm{l}$ into a vacuum to a total volume $V_1 = 8\,\mathrm{l}$. The gas is helium with mass $m = 20\,\mathrm{g}$.*

The change of entropy is defined as

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T}\;.$$

From the first law of thermodynamics

$$\mathrm{d}Q = \mathrm{d}U + \mathrm{d}A' = nC_v\mathrm{d}T + p\mathrm{d}V\;,$$

for the isothermal process

$$\mathrm{d}T = 0\;,$$

follows

$$\mathrm{d}Q = p\mathrm{d}V\;.$$

From the equation of state

$$pV = nRT$$

it is possible to express the pressure of the gas

$$p = \frac{nRT}{V}\;,$$

by which the first law of thermodynamics will have the form

$$\mathrm{d}Q = \frac{nRT}{V}\mathrm{d}V$$

and the change of entropy can be expressed as

$$\mathrm{d}S = \frac{nRT}{V}\frac{\mathrm{d}V}{T} = nR\frac{\mathrm{d}V}{V}\;.$$

The amount of substance will be

$$n = \frac{m}{M} \ ,$$

thus, the entropy change can be written as

$$\mathrm{d}S = \frac{m}{M} R \frac{\mathrm{d}V}{V}$$

and the total change of entropy in isothermal expansion will be

$$\Delta S = \frac{m}{M} R \int_{V_0}^{V_1} \frac{\mathrm{d}V}{V} = \frac{m}{M} R \left[\ln V\right]_{V_0}^{V_1} = \frac{m}{M} R \ln \frac{V_1}{V_0} \ ,$$

after inserting the numerical values

$$\Delta S = \frac{0.02\,\mathrm{kg}}{4 \cdot 10^{-3}\,\mathrm{kg\,mol^{-1}}} \ . \ 8.314\,\mathrm{J\,K^{-1}\,mol^{-1}} \ . \ \ln \frac{8 \cdot 10^{-3}\,\mathrm{m^3}}{2 \cdot 10^{-3}\,\mathrm{m^3}} = 57.63\,\mathrm{J\,K^{-1}} \ .$$

6 Electric field and electric current

6.1 *In a vacuum, there are two balls at a distance* $d = 10\,\mathrm{cm}$, *from each other, which have electric charges* $Q_1 = 20 \cdot 10^{-6}\,\mathrm{C}$ *and* $Q_2 = -10 \cdot 10^{-6}\,\mathrm{C}$. *What force are they attracted and what force will they repel each other when they touch and then move apart again to their original distance?*

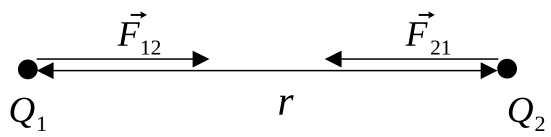

Fig. 24

The force exerted by the ball (Fig. 24) with charge Q_1 on the ball with charge Q_2 is given by Coulomb's law

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^3}\vec{r}_{12}\ ,$$

where $\epsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$ is the electric constant and $\vec{r}_{12}$ is the position vector of the ball with charge Q_2 with respect to the ball with charge Q_1. Likewise, the force exerted by the ball with charge Q_2 on the ball with charge Q_1 is given by Coulomb's law

$$\vec{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^3}\vec{r}_{21}\ ,$$

where $\vec{r}_{21}$ is the position vector of the ball with charge Q_1 with respect to the ball with charge Q_2. The forces have an attractive direction, and from Coulomb's law for their magnitudes follows

$$F_{12} = F_{21} = \frac{1}{4\pi\epsilon_0}\frac{|Q_1||Q_2|}{r^2}\ ,$$

after inserting numerical values

$$F_{12} = F_{21} = \frac{1}{4\pi\ .\ 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}} \cdot \frac{20 \cdot 10^{-6}\,\mathrm{C}\ .\ 10 \cdot 10^{-6}\,\mathrm{C}}{(0.1\,\mathrm{m})^2} = 179.8\,\mathrm{N}\ .$$

If the balls touch, the resulting electric charge will be

$$Q = Q_1 + Q_2 = 20 \cdot 10^{-6}\,\mathrm{C} + (-10 \cdot 10^{-6}\,\mathrm{C}) = 10 \cdot 10^{-6}\,\mathrm{C}$$

and after separating the balls, each ball retains the same electric charge

$$Q^* = \frac{Q}{2} = \frac{10 \cdot 10^{-6}\,\mathrm{C}}{2} = 5 \cdot 10^{-6}\,\mathrm{C}\ .$$

The forces will have a repulsive direction and from Coulomb's law for their magnitudes follows

$$F_{12}^* = F_{21}^* = \frac{1}{4\pi\epsilon_0}\frac{|Q^*||Q^*|}{r^2}\ ,$$

after inserting numerical values

$$F_{12}^* = F_{21}^* = \frac{1}{4\pi\ .\ 8.854 \cdot 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}}\ .\ \frac{5 \cdot 10^{-6}\,\mathrm{C}\ .\ 5 \cdot 10^{-6}\,\mathrm{C}}{(0.1\,\mathrm{m})^2} = 22.5\,\mathrm{N}\ .$$

6.2 *Calculate the electric potential and intensity of the electric field of a rod with an electric charge Q and length l at a point lying on the extension of the rod at a distance a from its end.*

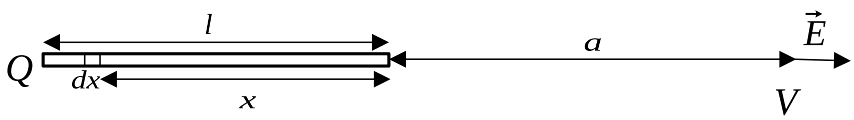

Fig. 25

The linear density of electric charge of the rod is

$$\lambda = \frac{Q}{l}\ ,$$

therefore, the element of electric charge (Fig. 25) of the rod will be

$$\mathrm{d}Q = \lambda \mathrm{d}x = \frac{Q}{l}\mathrm{d}x$$

and the electric potential of this element at a point located at a distance a from the end of the rod will be

$$\mathrm{d}V = \frac{1}{4\pi\epsilon_0}\frac{\mathrm{d}Q}{x+a} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\frac{\mathrm{d}x}{x+a}\ .$$

The electric potential of the entire rod can be calculated by integrating over the entire electric charge of the rod as

$$V = \int_Q \mathrm{d}V = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\int_0^l \frac{\mathrm{d}x}{x+a} = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left[\ln\left(x+a\right)\right]_0^l = \frac{1}{4\pi\epsilon_0}\frac{Q}{l}\ln\frac{l+a}{a}\ .$$

The relationship between the electric field and the electric potential is

$$\vec{E} = -\mathrm{grad}\, V\ ,$$

therefore, the electric field of the rod will be:

$$\vec{E} = -\frac{\mathrm{d}V}{\mathrm{d}a}\vec{\rho} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{l}\frac{a}{l+a}\frac{a-l-a}{a^2}\vec{\rho} = \frac{1}{4\pi\epsilon_0}\frac{Q}{a(l+a)}\vec{\rho}\ ,$$

where $\vec{\rho}$ is a unit vector in the direction of the electric field, and the rules for the derivative of a composite function and the derivative of a fraction of functions were used.

6.3 *A particle with an electric charge* $Q' = 5\,\mu\mathrm{C}$ *is located in a vacuum at a distance* $r = 2\,\mathrm{cm}$ *from an electrically charged straight thin conductor with a linear electric charge density* $\lambda = 3\,\mu\mathrm{C\,m^{-1}}$. *What is the electric force acting on this particle?*

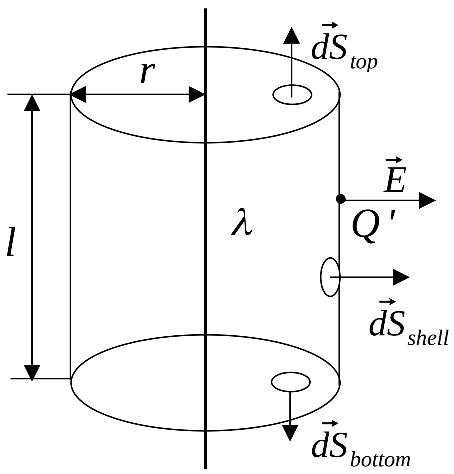

Fig. 26

According to Gauss's law of electrostatics, the electric flux through any closed surface is equal to the ratio of the electric charge inside the closed surface and the electric constant

$$\oint_S \vec{E}\cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_0}\ .$$

If the closed surface is chosen as the surface of a cylinder whose axis of symmetry is located on the electric conductor and the radius of the cylinder is equal to the distance of the particle from the electric conductor (Fig. 26), the total electric flux can be expressed as the sum of the electric fluxes through the upper base, the lower base and the shell of the cylinder

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \int_{S_{\text{top}}} \vec{E} \cdot \mathrm{d}\vec{S}_{\text{top}} + \int_{S_{\text{bottom}}} \vec{E} \cdot \mathrm{d}\vec{S}_{\text{bottom}} + \int_{S_{\text{shell}}} \vec{E} \cdot \mathrm{d}\vec{S}_{\text{shell}} \ ,$$

because both $\mathrm{d}\vec{S}_{\text{top}}$ and $\mathrm{d}\vec{S}_{\text{bottom}}$ are perpendicular to $\vec{E}$ their scalar products are

$$\vec{E} \cdot \mathrm{d}\vec{S}_{\text{top}} = \vec{E} \cdot \mathrm{d}\vec{S}_{\text{bottom}} = 0 \ ,$$

because $\mathrm{d}\vec{S}_{\text{shell}}$ is parallel to $\vec{E}$ their scalar product is

$$\vec{E} \cdot \mathrm{d}\vec{S}_{\text{shell}} = E\mathrm{d}S_{\text{shell}} \ ,$$

because the magnitude of the electric field E at a distance r is constant, it is valid

$$\int_{S_{\text{shell}}} E\mathrm{d}S_{\text{shell}} = E \int_{S_{\text{shell}}} \mathrm{d}S_{\text{shell}} = E2\pi rl$$

and the electric charge inside the cylinder can be expressed as

$$Q = \lambda l \ .$$

From Gauss' law of electrostatics follows

$$E2\pi rl = \frac{\lambda l}{\epsilon_0} \ ,$$

from which it is possible to express the electric field of the conductor in the place where the charged particle is located

$$E = \frac{\lambda}{2\pi r \epsilon_0} \ .$$

The force acting at this location on a particle with charge Q' will be

$$F = EQ' = \frac{\lambda}{2\pi r \epsilon_0} Q' \ ,$$

after substitution of numerical values

$$F = \frac{3 \cdot 10^{-6}\,\mathrm{C\,m^{-1}}}{2\pi \ . \ 0.02\,\mathrm{m} \ . \ 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}} \ . \ 5 \cdot 10^{-6}\,\mathrm{C} = 13.5\,\mathrm{N} \ .$$

6.4 *Two capacitors with electrical capacitances $C_1 = 1\,\mu\mathrm{F}$ and $C_2 = 2\,\mu\mathrm{F}$ are connected in series and connected to a voltage source $U = 600\,\mathrm{V}$. Calculate the charge and the voltage on each of them. We then disconnect the charged capacitors from the source and each other and reconnect them in parallel by connecting the positive and negative electrodes of the capacitors. Calculate the charge and the voltage will be on each of them after the stabilization.*

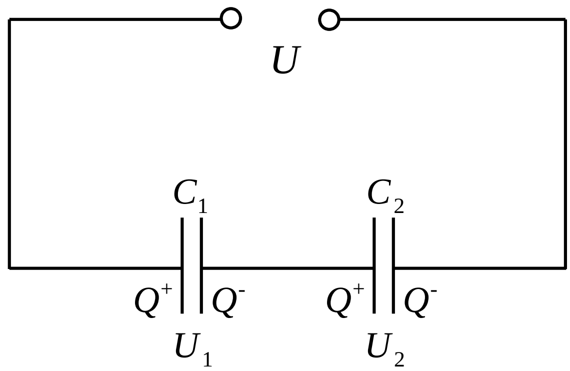

Fig. 27

When capacitors are connected in series (Fig. 27), the charge on the capacitors is equal

$$Q = Q_1 = Q_2$$

and the total voltage on the capacitors is the sum of the voltages on the individual capacitors

$$U = U_1 + U_2\,.$$

From the definition of the capacitance, the voltage follows

$$U = \frac{Q}{C}\,,$$

using this formula, the sum of the voltages can be written in the form

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}\,,$$

from which it is possible to express the resulting capacitance

$$C = \frac{C_1 C_2}{C_1 + C_2}\,,$$

and then calculates the charge on the capacitors

$$Q = CU = \frac{C_1 C_2}{C_1 + C_2} U \ .$$

After substitution

$$Q = \frac{1\,\mu\text{F} \ . \ 2\,\mu\text{F}}{1\,\mu\text{F} + 2\,\mu\text{F}} \ . \ 600\,\text{V} = 400\,\mu\text{C} \ ,$$

the voltage on the first capacitor will be

$$U_1 = \frac{Q}{C_1} \ ,$$

after substitution

$$U_1 = \frac{400\,\mu\text{C}}{1\,\mu\text{F}} = 400\,\text{V}$$

and the voltage on the second capacitor will be

$$U_2 = \frac{Q}{C_2} \ ,$$

after substitution

$$U_2 = \frac{400\,\mu\text{C}}{2\,\mu\text{F}} = 200\,\text{V} \ .$$

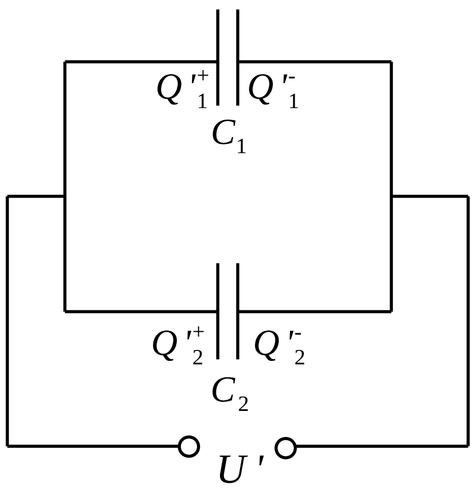

Fig. 28

When the capacitors are connected in parallel (Fig. 28), the voltage on the capacitors is equal

$$U' = U'_1 = U'_2$$

and the total charge on the capacitors is the sum of the charges on the individual capacitors

$$Q' = Q'_1 + Q'_2 = 2Q \ .$$

From the definition of the capacitance for the charge follows

$$Q = CU \ ,$$

using this formula, the sum of the charges can be written in the form

$$C'U' = C_1U' + C_2U' \ ,$$

from which it is possible to express the resulting capacitance

$$C' = C_1 + C_2$$

and then calculates the voltage on the capacitors

$$U' = \frac{Q'}{C'} = \frac{2Q}{C_1 + C_2} \ ,$$

after substitution

$$U' = \frac{2 \ . \ 400\,\mu\mathrm{C}}{1\,\mu\mathrm{F} + 2\,\mu\mathrm{F}} = 266.7\,\mathrm{V} \ ,$$

the charge on the first capacitor will be

$$Q'_1 = C_1U' \ ,$$

after substitution

$$Q'_1 = 1\,\mu\mathrm{F} \ . \ 266.7\,\mathrm{V} = 266.7\,\mu\mathrm{C} \ ,$$

and the charge on the second capacitor will be

$$Q'_2 = C_2U' \ ,$$

after substitution

$$Q'_2 = 2\,\mu\mathrm{F} \ . \ 266.7\,\mathrm{V} = 533.4\,\mu\mathrm{C} \ .$$

6.5 *Calculate the capacitance of the cylindrical capacitor, which is formed by two coaxial conductive cylindrical surfaces in a vacuum. The height of both is* $h = 2\,\mathrm{cm}$*, the radius of the inner electrode is* $r_1 = 0.5\,\mathrm{mm}$*, the radius of the outer electrode is* $r_2 = 5\,\mathrm{mm}$.

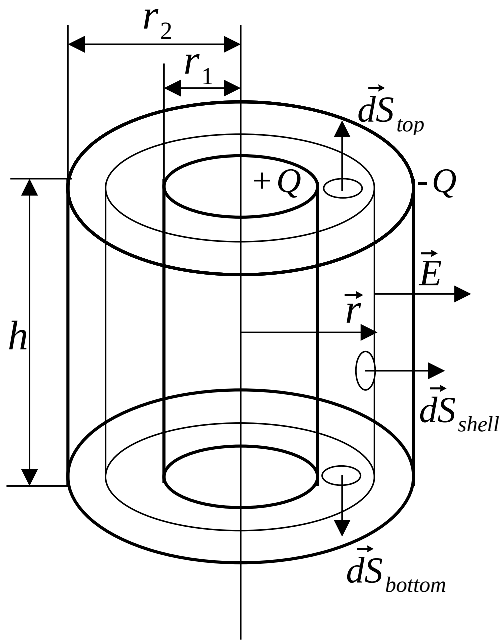

Fig. 29

The electric field between the electrodes of the capacitor can be expressed using the Gauss' law of electrostatics, which states that the electric flux through any closed surface is equal to the ratio of the electric charge inside the closed surface and the electric constant

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_0} \; .$$

If the closed surface is chosen as the surface of a cylinder whose axis of symmetry is located on the axis of the capacitor (Fig. 29), the total electric flux can be expressed as the sum of the electric fluxes through the upper base, the lower base and the shell of the cylinder

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \int_{S_\mathrm{top}} \vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{top} + \int_{S_\mathrm{bottom}} \vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{bottom} + \int_{S_\mathrm{shell}} \vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{shell} \; .$$

Because both $\mathrm{d}\vec{S}_\mathrm{top}$ and $\mathrm{d}\vec{S}_\mathrm{bottom}$ are perpendicular to $\vec{E}$ their scalar products are

$$\vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{top} = \vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{bottom} = 0 \; ,$$

because $\mathrm{d}\vec{S}_\mathrm{shell}$ is parallel to $\vec{E}$ their scalar product is

$$\vec{E} \cdot \mathrm{d}\vec{S}_\mathrm{shell} = E \mathrm{d}S_\mathrm{shell} \; ,$$

because the magnitude of the electric field E at a distance r is constant, it is valid

$$\int\limits_{S_{\text{shell}}} E \mathrm{d}S_{\text{shell}} = E \int\limits_{S_{\text{shell}}} \mathrm{d}S_{\text{shell}} = E 2\pi r h \; .$$

From Gauss' law of electrostatics follows

$$E 2\pi r h = \frac{Q}{\epsilon_0} \; ,$$

from which it is possible to express the electric field as

$$E = \frac{Q}{2\pi r h \epsilon_0} \; .$$

The voltage between the electrodes can be calculated by integrating the electric field

$$U = \int\limits_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_{r_1}^{r_2} E \mathrm{d}r \cos 0^\circ = \int\limits_{r_1}^{r_2} \frac{Q}{2\pi r h \epsilon_0} \mathrm{d}r = \frac{Q}{2\pi h \epsilon_0} \int\limits_{r_1}^{r_2} \frac{1}{r} \mathrm{d}r =$$

$$= \frac{Q}{2\pi h \epsilon_0} \left[\ln r \right]_{r_1}^{r_2} = \frac{Q}{2\pi h \epsilon_0} \ln \frac{r_2}{r_1} \; .$$

The electrical capacitance of the capacitor can now be expressed from the definition

$$C = \frac{Q}{U} = \frac{2\pi h \epsilon_0}{\ln \frac{r_2}{r_1}} \; ,$$

after substitution

$$C = \frac{2\pi \; . \; 0.02\,\mathrm{m} \; . \; 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}}{\ln \frac{5 \cdot 10^{-3}\,\mathrm{m}}{0.5 \cdot 10^{-3}\,\mathrm{m}}} = 4.83 \cdot 10^{-13}\,\mathrm{F} = 0.483\,\mathrm{pF} \; .$$

6.6 *Calculate the capacitance of the spherical capacitor, which consists of two concentric conductive spherical surfaces with radius* $r_1 = 3\,\mathrm{cm}$ *and* $r_2 = 4\,\mathrm{cm}$, *if the medium between them is filled with a dielectric with relative permittivity* $\epsilon_r = 2.6$. *What will be the charge on the electrodes if the capacitor is connected to voltages* $U = 600\,\mathrm{V}$, *and what will be the energy of the capacitor?*

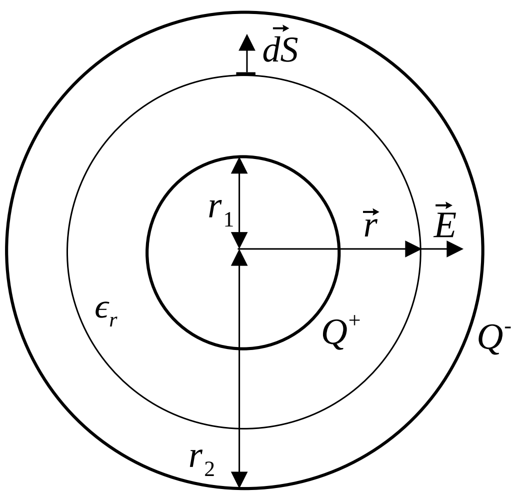

Fig. 30

The electric field between the electrodes of the capacitor can be expressed using the Gauss' law of electrostatics

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_0 \epsilon_r} \, .$$

If the closed surface is chosen so that it is the surface of a sphere with the centre located at the centre of the capacitor (Fig. 30), the electric field vector will have the same direction as the surface element vector, and the magnitude of the electric field will be constant everywhere on this surface. Therefore, it will apply

$$\oint_S \vec{E} \cdot \mathrm{d}\vec{S} = \oint_S E \mathrm{d}S \cos 0^\circ = E \int_S \mathrm{d}S = ES = E 4\pi r^2 \, .$$

Then the Gauss' law of electrostatics will imply

$$E 4\pi r^2 = \frac{Q}{\epsilon_0 \epsilon_r} \, ,$$

from which it is possible to express the electric field

$$E = \frac{Q}{4\pi r^2 \epsilon_0 \epsilon_2} \, ,$$

and the voltage between the electrodes can be calculated by integrating the electric field

$$U = \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \mathrm{d}\vec{r} = \int_{r_1}^{r_2} E \mathrm{d}r \cos 0^\circ = \int_{r_1}^{r_2} \frac{Q}{4\pi r^2 \epsilon_0 \epsilon_r} \mathrm{d}r = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \int_{r_1}^{r_2} \frac{1}{r^2} \mathrm{d}r =$$

$$= \frac{Q}{4\pi \epsilon_0 \epsilon_r} \left[-\frac{1}{r} \right]_{r_1}^{r_2} = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{Q}{4\pi \epsilon_0 \epsilon_r} \frac{r_2 - r_1}{r_1 r_2} \, .$$

The electrical capacitance of the capacitor can now be expressed from the definition

$$C = \frac{Q}{U} = 4\pi \epsilon_0 \epsilon_r \frac{r_1 r_2}{r_2 - r_1} \, ,$$

after substitution

$$C = 4\pi \, . \, 8.854 \cdot 10^{-12} \, \mathrm{C^2 \, N^{-1} \, m^{-2}} \, . \, 2.6 \, . \, \frac{0.03 \, \mathrm{m} \, . \, 0.04 \, \mathrm{m}}{0.04 \, \mathrm{m} - 0.03 \, \mathrm{m}} = 3.47 \cdot 10^{-11} \, \mathrm{F} = 34.7 \, \mathrm{pF} \, .$$

If the capacitor is connected to an electric voltage $U = 300 \, \mathrm{V}$, the charge on its electrodes will be

$$Q = CU = 3.47 \cdot 10^{-11} \, \mathrm{F} \, . \, 600 \, \mathrm{V} = 2.08 \cdot 10^{-8} \, \mathrm{C} = 20.8 \, \mathrm{nC}$$

and the energy of the electric field of the capacitor will be

$$W = \frac{1}{2} C U^2 = \frac{1}{2} \, . \, 3.47 \cdot 10^{-11} \, \mathrm{F} \, . \, (600 \, \mathrm{V})^2 = 6.25 \cdot 10^{-6} \, \mathrm{J} = 6.25 \, \mu\mathrm{J} \, .$$

6.7 *Calculate the electric field in an aluminium conductor in the shape of a straight cylinder with a radius* $r_0 = 2.5\,\mathrm{mm}$ *and a length* $L = 1\,\mathrm{m}$*, when a stationary electric current* $I = 10\,\mathrm{A}$ *flows through it. What will be the voltage at the ends of this conductor, and what electric charge will flow through the conductor during time* $t' = 10\,\mathrm{s}$*? The resistivity of aluminum is* $\rho = 2.828 \cdot 10^{-8}\,\Omega\,\mathrm{m}$.

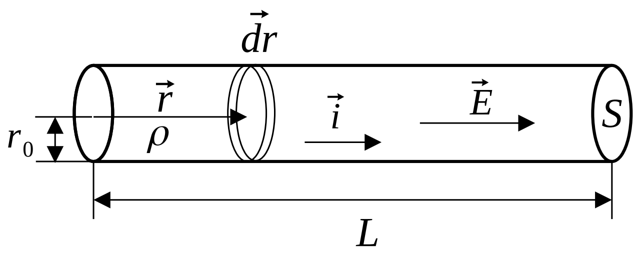

Fig. 31

The solution can be found using Ohm's law in differential form

$$\vec{i} = \gamma \vec{E}\,,$$

where γ is the conductivity of the conductor

$$\gamma = \frac{1}{\rho}\,,$$

and $\vec{i}$ is the electric current density, its magnitude is

$$i = \frac{I}{S} = \frac{I}{\pi r_0^2}\,.$$

In the scalar form, it is possible to write

$$i = \gamma E\,,$$

from which it is possible to express the magnitude of the electric field

$$E = \frac{i}{\gamma} = \frac{1}{\gamma}\frac{I}{\pi r_0^2} = \rho\frac{I}{\pi r_0^2}\,,$$

after substitution

$$E = 2.828 \cdot 10^{-8}\,\Omega\,\mathrm{m}\,.\,\frac{10\,\mathrm{A}}{\pi\,.\,(0.0025\,\mathrm{m})^2} = 1.44 \cdot 10^{-2}\,\mathrm{V\,m^{-1}} = 14.4\,\mathrm{mV\,m^{-1}}\,.$$

The voltage between the ends of the conductor (Fig. 31) will be

$$U = \int\limits_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_0^L E\mathrm{d}r = EL = 1.44 \cdot 10^{-2}\,\mathrm{V\,m^{-1}}\ .\ 1\,\mathrm{m} = 1.44 \cdot 10^{-2}\,\mathrm{V} = 14.4\,\mathrm{mV}$$

and the electric charge that will flow in time t^* will be

$$Q^* = \int\limits_0^{t^*} I\mathrm{d}t = It^* = 10\,\mathrm{A}\ .\ 10\,\mathrm{s} = 100\,\mathrm{C}\ .$$

6.8 *The parallel plate capacitor has electrodes with an area $S = 16\,\mathrm{cm}^2$, the distance between the electrodes $d = 0.2\,\mathrm{cm}$ and the space between the electrodes is filled with a dielectric with a relative permittivity $\epsilon_r = 6.9$. Calculate the capacitance of the capacitor, the charge on the electrodes, the electric field, the electric displacement field, the energy density and the energy of the electric field if the capacitor is connected to an electric voltage $U = 300\,\mathrm{V}$.*

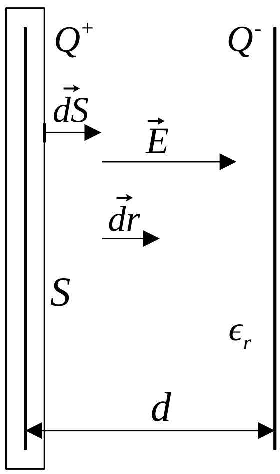

Fig. 32

The electric field between the electrodes of the capacitor can be expressed using the Gauss' law of electrostatics

$$\oint\limits_S \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_r \epsilon_0}\ ,$$

the closed area around the electrode (Fig. 32) has a total area $2S$, but the electric field is only on one side of it. The electric field is constant and has the direction of the surface element, therefore

$$\oint\limits_S \vec{E} \cdot \mathrm{d}\vec{S} = \oint\limits_S E\mathrm{d}S = E \oint\limits_S \mathrm{d}S = ES\ ,$$

which gives the equation

$$ES = \frac{Q}{\epsilon_r \epsilon_0} \ ,$$

from which it is possible to express the electric field

$$E = \frac{Q}{S\epsilon_r \epsilon_0}$$

and the voltage between the electrodes will be

$$U = \int\limits_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot \mathrm{d}\vec{r} = \int\limits_{r_1}^{r_2} E\mathrm{d}r = E \int\limits_{r_1}^{r_2} \mathrm{d}r = Ed = \frac{Q}{S\epsilon_r \epsilon_0} d \ .$$

The capacitance of the parallel plate capacitor will be

$$C = \frac{Q}{U} = \frac{S\epsilon_r \epsilon_0}{d} = \frac{16 \cdot 10^{-4}\,\mathrm{m}^2 \ . \ 6.9 \ . \ 8.854 \cdot 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}}{0.002\,\mathrm{m}} =$$

$$= 4.89 \cdot 10^{-11}\,\mathrm{F} = 48.9\,\mathrm{pF} \ .$$

The charge on the electrodes will be

$$Q = CU = 4.89 \cdot 10^{-11}\,\mathrm{F} \ . \ 300\,\mathrm{V} = 1.47 \cdot 10^{-8}\,\mathrm{C} = 14.7\,\mathrm{nC} \ .$$

The electric field will be

$$E = \frac{U}{d} = \frac{300\,\mathrm{V}}{0.002\,\mathrm{m}} = 1.5 \cdot 10^5\,\mathrm{V\,m}^{-1} = 150\,\mathrm{kV\,m}^{-1} \ .$$

The electric displacement field will be

$$D = \epsilon_r \epsilon_0 E = 6.9 \ . \ 8.854 \cdot 10^{-12}\,\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2} \ . \ 1.5 \cdot 10^5\,\mathrm{V\,m}^{-1} =$$

$$= 9.16 \cdot 10^{-6}\,\mathrm{C\,m}^{-2} = 9.16\,\mu\mathrm{C\,m}^{-2} \ .$$

The energy density of the electric field will be

$$w = \frac{1}{2} ED = \frac{1}{2} \ . \ 1.5 \cdot 10^5\,\mathrm{V\,m}^{-1} \ . \ 9.16 \cdot 10^{-6}\,\mathrm{C\,m}^{-2} = 0.687\,\mathrm{J\,m}^{-3}$$

The energy of the electric field will be

$$W = \frac{1}{2} CU^2 = \frac{1}{2} \ . \ 4.89 \cdot 10^{-11}\,\mathrm{F} \ . \ (300\,\mathrm{V})^2 = 2.2 \cdot 10^{-6}\,\mathrm{J} = 2.2\,\mu\mathrm{J} \ .$$

6.9 *Two resistors with resistances $R_1 = 4\,\Omega$ and $R_2 = 12\,\Omega$ are connected in parallel and connected to a source with electromotive voltage $U_e = 9\,\mathrm{V}$ and internal resistance $R_i = 1.5\,\Omega$. What electric currents are in the individual branches of the circuit?*

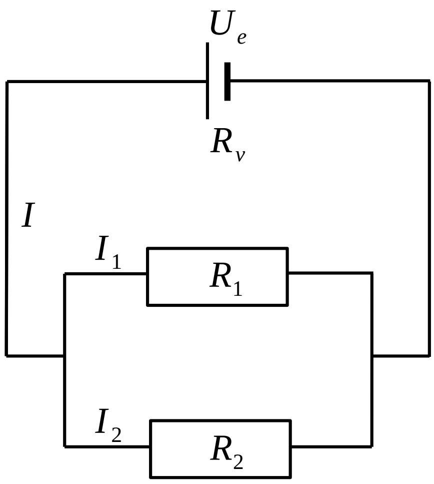

Fig. 33

For resistors connected in parallel (Fig. 33) with resistances R_1 and R_2, the following applies

$$U = U_1 = U_2 \; ,$$

$$I = I_1 + I_2 \; .$$

From Ohm's law

$$I = \frac{U}{R} \; ,$$

it follows

$$\frac{U}{R} = \frac{U}{R_1} + \frac{U}{R_2}$$

and the resulting resistance will be

$$R = \frac{R_1 R_2}{R_1 + R_2} \; .$$

The following applies to the electromotive voltage of the source

$$U_e = (R + R_i) I \; ,$$

which implies for electric current

$$I=\frac{U_e}{R+R_i}=\frac{U_e}{\frac{R_1R_2}{R_1+R_2}+R_i}=\frac{U_e(R_1+R_2)}{R_1R_2+R_i(R_1+R_2)}\ ,$$

after substitution

$$I=\frac{9\,\mathrm{V}\ .\ (4\,\Omega+12\,\Omega)}{4\,\Omega\ .\ 12\,\Omega+1.5\,\Omega\ .\ (4\,\Omega+12\,\Omega)}=2\,\mathrm{A}\ .$$

The electric current through the resistor with resistance R_1 will be

$$I_1=\frac{U}{R_1}=\frac{RI}{R_1}=\frac{\frac{R_1R_2}{R_1+R_2}I}{R_1}=\frac{R_2I}{R_1+R_2}\ ,$$

after substitution

$$I_1=\frac{12\,\Omega\ .\ 2\,\mathrm{A}}{4\,\Omega+12\,\Omega}=1.5\,\mathrm{A}\ .$$

The electric current through the resistor with resistance R_2 will be

$$I_2=\frac{U}{R_2}=\frac{RI}{R_2}=\frac{\frac{R_1R_2}{R_1+R_2}I}{R_2}=\frac{R_1I}{R_1+R_2}\ ,$$

after substitution

$$I_2=\frac{4\,\Omega\ .\ 2\,\mathrm{A}}{4\,\Omega+12\,\Omega}=0.5\,\mathrm{A}\ .$$

6.10 *The electric current in the conductor, whose electrical resistance is $R=10\,\Omega$, decreases linearly from the value $I_0=2\,\mathrm{A}$ to the zero value during the time $t_0=3\,\mathrm{s}$. What heat was generated in the conductor during this time, and what electric charge flowed through the conductor during this time?*

The electric current decreased linearly from the value I_0, that is

$$I=I_0-kt\ ,$$

to a zero value at time t_0, so the following applies

$$0=I_0-kt_0\ ,$$

from which it is possible to express the constant

$$k=\frac{I_0}{t_0}\ ,$$

the electric current will therefore vary with time as

$$I = I_0 - \frac{I_0}{t_0} t \,.$$

The following applies to heat

$$\mathrm{d}Q = P \mathrm{d}t \,,$$

where the power of the electric current will be

$$P = UI = RI^2 \,.$$

The heat generated in the conductor during time t_0 will be

$$Q = \int\limits_0^{t_0} P \mathrm{d}t = \int\limits_0^{t_0} RI^2 \mathrm{d}t = R \int\limits_0^{t_0} \left(I_0 - \frac{I_0}{t_0} t \right)^2 \mathrm{d}t = R \int\limits_0^{t_0} \left(I_0^2 - 2\frac{I_0^2}{t_0} t + \frac{I_0^2}{t_0^2} t^2 \right) \mathrm{d}t =$$

$$= R \left[I_0^2 t - \frac{I_0^2}{t_0} t^2 + \frac{I_0^2}{t_0^2} \frac{t^3}{3} \right]_0^{t_0} = \frac{R I_0^2 t_0}{3} \,,$$

after substitution

$$Q = \frac{10\,\Omega \,.\, (2\,\mathrm{A})^2 \,.\, 3\,\mathrm{s}}{3} = 40\,\mathrm{J} \,.$$

The following applies to electric charge

$$\mathrm{d}Q = I \mathrm{d}t \,.$$

The electric charge that flows through the conductor at time t_0, will be

$$q = \int\limits_0^{t_0} I \mathrm{d}t = \int\limits_0^{t_0} \left(I_0 - \frac{I_0}{t_0} t \right) \mathrm{d}t = \left[I_0 t - \frac{I_0}{t_0} \frac{t^2}{2} \right]_0^{t_0} = \frac{I_0 t_0}{2} \,,$$

after substitution

$$q = \frac{2\,\mathrm{A} \,.\, 3\,\mathrm{s}}{2} = 3\,\mathrm{C} \,.$$

7 Magnetic field

7.1 *An electric current* $I = 2\,\mathrm{A}$ *flows through a long, straight conductor. What is the magnetic induction of this conductor at a distance* $a = 0.5\,\mathrm{m}$ *from it?*

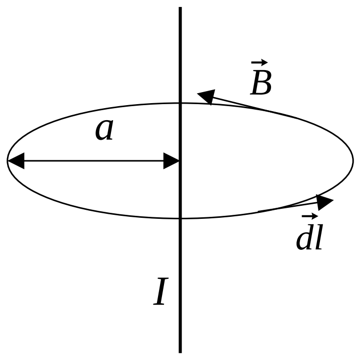

Fig. 34

The solution is possible using Ampère's circuital law, according to which the curve integral of the magnetic induction along any closed oriented curve is equal to the product of the magnetic constant and the electric current that flows through the area bounded by this curve

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I \ .$$

If a circle with a radius a around the conductor with the current I is chosen as the curve (Fig. 34), the vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ will have the same direction and the magnitude of the magnetic induction will be constant on this curve. Therefore, it will apply

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint B\mathrm{d}l = B \oint \mathrm{d}l = B2\pi a \ ,$$

from Ampère's circuital law then follows

$$B2\pi a = \mu_0 I \ ,$$

from which it is possible to express the magnetic induction as

$$B = \frac{\mu_0 I}{2\pi a} \, .$$

After substituting numerical values

$$B = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}\ .\ 2\,\mathrm{A}}{2\pi\ .\ 0.5\,\mathrm{m}} = 8 \cdot 10^{-7}\,\mathrm{T} = 0.8\,\mu\mathrm{T} \, .$$

7.2 *An electric current $I = 1.5\,\mathrm{A}$ flows through a circular conductor with radius $R = 10\,\mathrm{cm}$. Calculate the magnetic induction of the conductor on its axis at a distance $x = 20\,\mathrm{cm}$ from the centre of the circle and at the centre of the circle.*

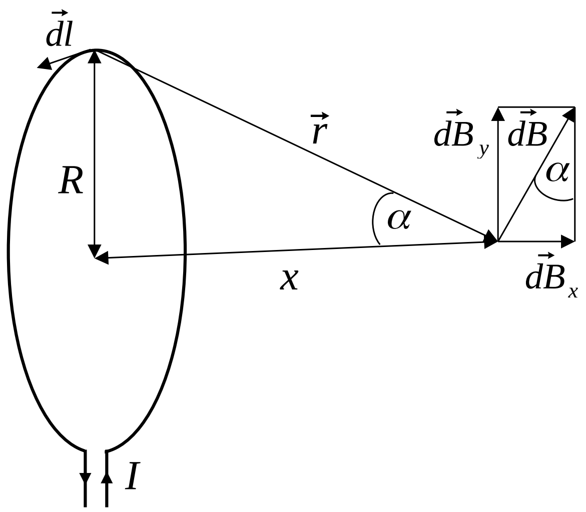

Fig. 35

The calculation of the magnetic induction of the conductor is possible using the Biot-Savart-Laplace law, which allows you to calculate the contribution

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}\vec{l} \times \vec{r}}{r^3} \, ,$$

to the magnetic induction from an element $\mathrm{d}\vec{l}$ at a location with position $\vec{r}$. Since the vectors $\mathrm{d}\vec{l}$ and $\vec{r}$ are perpendicular, the magnitude of this contribution can be expressed as

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l r \sin 90^\circ}{r^3} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} \, .$$

The vector $\mathrm{d}\vec{B}$ can be decomposed into components (Fig. 35)

$$\mathrm{d}\vec{B} = \mathrm{d}\vec{B}_x + \mathrm{d}\vec{B}_y \, .$$

The $\mathrm{d}\vec{B}_y$ components from the $\mathrm{d}\vec{l}$ elements, which lie on the opposite sides of the circle, are the same size and oppositely oriented, therefore, they will cancel each other and the resulting magnetic field will be given only by the sum of the $\mathrm{d}\vec{B}_x$ components, the magnitude of which is

$$\mathrm{d}B_x = \mathrm{d}B \sin\alpha = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} \sin\alpha \, .$$

The magnitude of the position vector can be expressed from the Pythagorean theorem

$$r = \sqrt{R^2 + x^2} \, ,$$

in a right triangle, also applies

$$\sin\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + x^2}} \, ,$$

from which it follows

$$\mathrm{d}B_x = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{R^2 + x^2} \frac{R}{\sqrt{R^2 + x^2}} = \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + x^2)^{\frac{3}{2}}} \mathrm{d}l \, .$$

The magnetic induction of the entire conductor can be calculated by integrating over the entire length of the conductor

$$B = \int \mathrm{d}B_x = \int_0^{2\pi R} \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + x^2)^{\frac{3}{2}}} \mathrm{d}l = \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + x^2)^{\frac{3}{2}}} \int_0^{2\pi R} \mathrm{d}l =$$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{(R^2 + x^2)^{\frac{3}{2}}} 2\pi R = \frac{\mu_0 I R^2}{2(R^2 + x^2)^{\frac{3}{2}}} \, ,$$

after substitution

$$B = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \,.\, 2\,\mathrm{A} \,.\, (0.1\,\mathrm{m})^2}{2\left[(0.1\,\mathrm{m})^2 + (0.2\,\mathrm{m})^2\right]^{\frac{3}{2}}} = 1.12 \cdot 10^{-6}\,\mathrm{T} = 1.12\,\mu\mathrm{T} \, .$$

The magnetic induction in the centre of the circle can be expressed from the resulting relation by substituting $x = 0$, which results in

$$B = \frac{\mu_0 I}{2R} = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \,.\, 2\,\mathrm{A}}{2 \,.\, 0.1\,\mathrm{m}} = 1.26 \cdot 10^{-5}\,\mathrm{T} = 12.6\,\mu\mathrm{T} \, .$$

7.3 *An electric current flows through a circular conductor with a radius of* $R = 5\,\mathrm{cm}$*, the magnetic induction in the centre of the circle is* $B = 5\,\mathrm{mT}$*. What is its magnetic moment?*

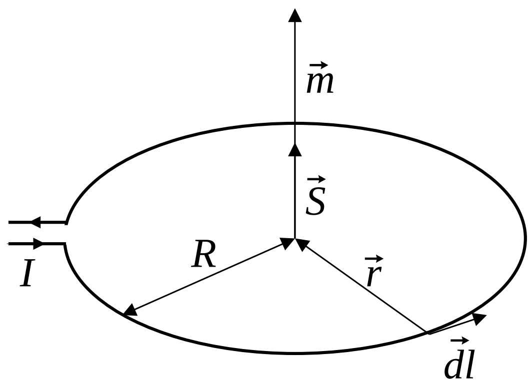

Fig. 36

The magnetic induction of the conductor is given by the Biot-Savart-Laplace law

$$\mathrm{d}\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}\vec{l} \times \vec{r}}{r^3} ,$$

where an element $\mathrm{d}\vec{l}$ (Fig. 36) of a conductor with an electric current I at a location with a position vector $\vec{r}$ contributes to the total magnetic induction by the contribution $\mathrm{d}\vec{B}$. Since the vectors $\mathrm{d}\vec{l}$ and $\vec{r}$ are perpendicular, it follows

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l r \sin 90°}{r^3} = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l}{r^2} .$$

Since the size of the position vector is equal to the radius of the circle, it will be

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi}\frac{\mathrm{d}l}{R^2} .$$

The magnetic induction of the entire conductor can be calculated by integrating over the entire length of the conductor

$$B = \int \mathrm{d}B = \int\limits_0^{2\pi R} \frac{\mu_0 I}{4\pi R^2}\mathrm{d}l = \frac{\mu_0 I}{4\pi R^2}\int\limits_0^{2\pi R}\mathrm{d}l = \frac{\mu_0 I}{4\pi R^2}2\pi R = \frac{\mu_0 I}{2R} ,$$

which can be used to express the electric current in a conductor

$$I = \frac{2BR}{\mu_0} .$$

The magnitude of the magnetic moment of the planar loop with surface S through which the electric current I flows is

$$m = IS = \frac{2BR}{\mu_0}\pi R^2 = \frac{2\pi B R^3}{\mu_0} ,$$

after substituting numerical values

$$m = \frac{2\pi \,.\, 5 \cdot 10^{-3}\,\mathrm{T} \,.\, (0.05\,\mathrm{m})^3}{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}} = 3.125\,\mathrm{A\,m^2} .$$

7.4 *Two long, straight conductors, parallel to each other, carry equal electric currents. The distance between them is* $a = 0.5\,\mathrm{m}$*, and the force exerted by one conductor per unit length of the other conductor is* $F_0 = 2 \cdot 10^{-7}\,\mathrm{N\,m^{-1}}$*. Calculate the electric currents in the conductors.*

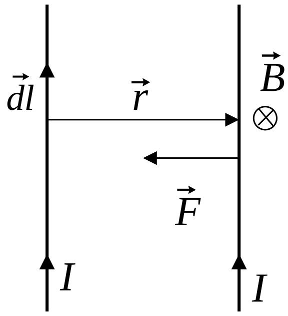

Fig. 37

The magnetic induction created by a straight conductor with an electric current I can be calculated using Ampère's circuital law

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I \ ,$$

if the integration loop is a circle with a radius a with one conductor in the centre, it follows

$$B 2\pi a = \mu_0 I \ ,$$

from which it is possible to express the magnetic induction:

$$B = \frac{\mu_0 I}{2\pi a} \ .$$

The force acting in the magnetic field $\vec{B}$ on the conductor element $\mathrm{d}\vec{l}$ through which the electric current I flows (Fig. 37), is expressed by Ampère's force law

$$\mathrm{d}\vec{F} = I \mathrm{d}\vec{l} \times \vec{B} \ ,$$

because the conductor element $\mathrm{d}\vec{l}$ and the magnetic induction $\vec{B}$ are perpendicular to each other, the magnitude of the force is

$$\mathrm{d}F = I \mathrm{d}l B \ ,$$

from which it is possible to express the magnitude of the force acting per unit length of the conductor

$$F_0 = \frac{\mathrm{d}F}{\mathrm{d}l} = IB \ .$$

After inserting the magnetic induction created by the second conductor

$$F_0 = \frac{\mu_0 I^2}{2\pi a} \ ,$$

from which it is possible to calculate the electric current

$$I = \sqrt{\frac{2\pi a F_0}{\mu_0}} \ ,$$

after inserting numerical values

$$I = \sqrt{\frac{2\pi \ . \ 0.5\,\mathrm{m} \ . \ 2\cdot 10^{-7}\,\mathrm{N\,m^{-1}}}{4\pi\cdot 10^{-7}\,\mathrm{N\,A^{-2}}}} = 0.707\,\mathrm{A} \ .$$

7.5 *Countercurrent electric currents $I_1 = I_2 = 5\,\mathrm{A}$ flow through two coaxial copper tubes in vacuum with radii $R_1 = 5\,\mathrm{mm}$ and $R_2 = 10\,\mathrm{mm}$. What is the magnetic induction at distances $r_2 = 3\,\mathrm{mm}$, $r_2 = 8\,\mathrm{mm}$ and $r_3 = 15\,\mathrm{mm}$ from of the common axis of the tubes?*

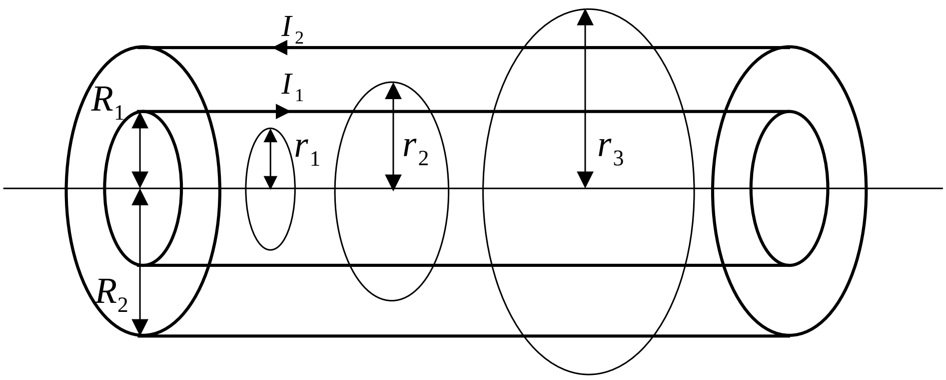

Fig. 38

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B}\cdot \mathrm{d}\vec{l} = \mu_0 I_{\text{net}} \ .$$

If the integration curve (Fig. 38) is a circle with radius r_1, then zero electric current flows through this curve, which implies

$$B_1 \oint_0^{2\pi r_1} \mathrm{d}l = 0 \ ,$$

$$B_1 2\pi r_1 = 0 \ .$$

Therefore, the magnetic induction at the distance r_1 will be

$$B_1 = 0\,\mathrm{T} \ .$$

If the integration loop is a circle with radius r_2, the electric current I_1 flows through this curve, which implies

$$B_2 \oint_0^{2\pi r_2} \mathrm{d}l = \mu_0 I_1 \ ,$$

$$B_2 2\pi r_2 = \mu_0 I_1 \ .$$

Therefore, the magnetic induction at the distance r_2 will be

$$B_2 = \frac{\mu_0 I_1}{2\pi r_2} = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \ . \ 5\,\mathrm{A}}{2\pi \ . \ 0.008\,\mathrm{m}} = 1.25 \cdot 10^{-3}\,\mathrm{T} = 0.125\,\mathrm{mT} \ .$$

If the integration loop is a circle with radius r_3, the net electric current flows through this curve is $I_1 - I_2 = 0$, which implies

$$B_3 \oint_0^{2\pi r_3} \mathrm{d}l = \mu_0 (I_1 - I_2) \ ,$$

$$B_3 2\pi r_3 = 0 \ .$$

Therefore, the magnetic induction at the distance r_3 will be

$$B_3 = 0\,\mathrm{T} \ .$$

7.6 *In a homogeneous magnetic field with magnetic induction $B = 0.5\,\mathrm{T}$, there is a rectangular conductor with sides $a = 5\,\mathrm{cm}$ and $b = 3\,\mathrm{cm}$, through which flows an electric current $I = 1\,\mathrm{A}$. The conductor can rotate around an axis that passes through the centres of the sides b and is perpendicular to the magnetic induction. What work will be done by the external forces that turn the conductor by an angle $\alpha = 90°$ from the stable position?*

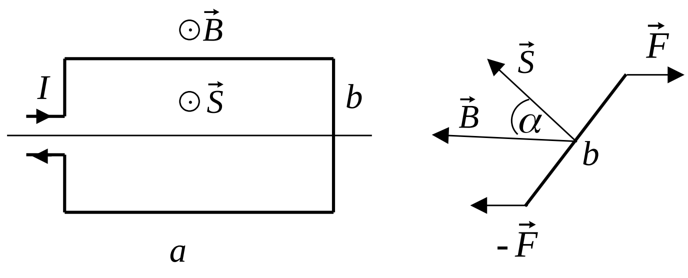

Fig. 39

A magnetic field exerts a torque on a closed loop with an electric current

$$\vec{M} = \vec{m} \times \vec{B} \ ,$$

where the magnetic moment of the loop is

$$\vec{m} = I\vec{S} \ .$$

The following applies

$$\vec{M} = I\vec{S} \times \vec{B} \ .$$

The magnetic field tries to rotate the loop so that the area vector of the loop and the magnetic induction vector have the same direction, this position is stable. For the loop to rotate about its axis, it must be acted upon by a couple of external forces (Fig. 39) that do the work when the loop rotates by $\mathrm{d}\alpha$

$$\mathrm{d}A = M\mathrm{d}\alpha \ .$$

The torque of the couple of external forces must be equal to the torque of the magnetic field

$$M = ISB\sin\alpha \ ,$$

therefore, the work of the couple of forces when turning the loop by $\mathrm{d}\alpha$ will be

$$\mathrm{d}A = ISB\sin\alpha\mathrm{d}\alpha$$

and the total work during the rotation by the angle α will be

$$A = \int\limits_0^\alpha ISB\sin\alpha\mathrm{d}\alpha = ISB\int\limits_0^\alpha \sin\alpha\mathrm{d}\alpha = ISB\left[-\cos\alpha\right]_0^\alpha = ISB(1-\cos\alpha) \ .$$

Because the area of the loop is

$$S = ab \ ,$$

it is possible to express the work as

$$A = IabB(1-\cos\alpha) \ ,$$

after substitution

$$A = 1\,\mathrm{A}\ .\ 0.05\,\mathrm{m}\ .\ 0.03\,\mathrm{m}\ .\ 0.5\,\mathrm{T}\ .\ (1-\cos 90^\circ) = 7.5\cdot 10^{-4}\,\mathrm{J} = 0.75\,\mathrm{mJ} \ .$$

7.7 *The toroid coil with a radius* $R = 10\,\mathrm{cm}$ *and the cross section radius* $r = 1\,\mathrm{cm}$ *has* $N = 10\,000$ *turns wound on a steel core through which an electric current* $I = 1\,\mathrm{A}$ *flows. The magnetic flux through the cross section of the core is* $\Phi = 7.5\,\mathrm{mWb}$. *Calculate the relative permeability of the core.*

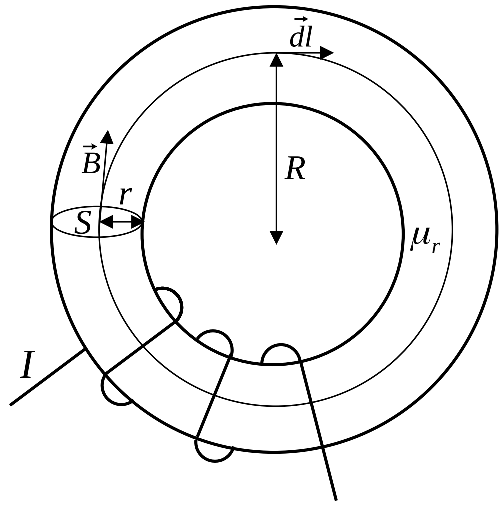

Fig. 40

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_r \mu_0 I_{\text{net}} \ .$$

If the integration curve (Fig. 40) is a circle with radius R centred at the centre of the toroid, the net electric current flowing through this curve is

$$I_{\text{net}} = NI \ .$$

The vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ have the same direction and the magnitude of the magnetic induction is constant, therefore

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint_0^{2\pi R} B \mathrm{d}l = B \oint_0^{2\pi R} \mathrm{d}l = B \left[l \right]_0^{2\pi R} = 2\pi R B \ ,$$

then from Ampère's circuital law follows

$$2\pi R B = \mu_r \mu_0 N I \ ,$$

from which the relative permeability can be expressed as

$$\mu_r = \frac{2\pi R B}{\mu_0 N I} \ .$$

From the magnetic induction flux

$$\Phi = BS ,$$

for the magnetic induction follows

$$B = \frac{\Phi}{S} = \frac{\Phi}{\pi r^2} ,$$

using which the relative permeability can be calculated as

$$\mu_r = \frac{2\pi R\Phi}{\mu_0 N I \pi r^2} = \frac{2R\Phi}{\mu_0 N I r^2} ,$$

after substitution

$$\mu_r = \frac{2 \,.\, 0.1\,\mathrm{m} \,.\, 7.5 \cdot 10^{-3}\,\mathrm{Wb}}{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \,.\, 10000 \,.\, 1\,\mathrm{A} \,.\, (0.01\,\mathrm{m})^2} = 1194 \,.$$

7.8 *What is the energy of the magnetic field of a toroid with radius* $R = 20\,\mathrm{cm}$ *on which* $N = 5000$ *turns with radius* $r = 1\,\mathrm{cm}$ *are wound, when an electric current* $I = 5\,\mathrm{mA}$ *flows through it?*

The magnetic induction can be calculated using Ampère's circuital law

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I_{\mathrm{net}} \,.$$

If the integration curve is a circle with radius R centred at the centre of the toroid, the net electric current flowing through this curve is

$$I_{\mathrm{net}} = NI ,$$

The vectors $\vec{B}$ and $\mathrm{d}\vec{l}$ have the same direction and the magnitude of the magnetic induction is constant, therefore

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \oint_0^{2\pi R} B \mathrm{d}l \cos 0^\circ = B \oint_0^{2\pi R} \mathrm{d}l = B 2\pi R ,$$

then from Ampère's circuital law follows

$$B 2\pi R = \mu_0 N I ,$$

from which the magnetic induction can be expressed as

$$B = \frac{\mu_0 N I}{2\pi R} \,.$$

The magnetic flux through one turn is

$$\Phi = \int_S \vec{B} \cdot \mathrm{d}\vec{S} = B \int_S \mathrm{d}S = BS = B\pi r^2$$

and the magnetic flux through N turns of the toroid is

$$\Phi_{\text{total}} = N\Phi = NB\pi r^2 \ ,$$

After substituting the magnetic induction, the magnetic flux is

$$\Phi_{\text{total}} = \frac{\mu_0 N^2 I r^2}{2R} \ ,$$

by which the inductance of the toroid can be calculated as

$$L = \frac{\Phi_{\text{total}}}{I} = \frac{\mu_0 N^2 r^2}{2R} \ .$$

The energy of the magnetic field in the toroid is

$$W_m = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 r^2 I^2}{4R} \ ,$$

after substitution

$$W_m = \frac{4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}} \ . \ (5000)^2 \ . \ (0.01\,\mathrm{m})^2 \ . \ (0.005\,\mathrm{A})^2}{4 \ . \ 0.2\,\mathrm{m}} =$$

$$= 9.81 \cdot 10^{-8}\,\mathrm{J} = 98.1\,\mathrm{nJ} \ .$$

7.9 *A rectangular conductor with sides $a = 20\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$ s located in the Earth's magnetic field with the magnetic induction $B = 45\,\mu\mathrm{T}$. The conductor can rotate about an axis that passes through the centre of side b and is perpendicular to the magnetic induction. What is the waveform and the amplitude of the induced voltage in the conductor if the conductor rotates in the magnetic field with a frequency of $f = 50 s^{-1}$.*

According to Faraday's law of induction, the induced electromotive voltage is equal to the negative time change of the magnetic flux

$$U_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ ,$$

where the magnetic flux is defined as

$$\Phi = \int_S \vec{B} \cdot \mathrm{d}\vec{S} = \int_S B \mathrm{d}S \cos\alpha \ .$$

Since the magnetic field induction is constant

$$\Phi = B \int_S \mathrm{d}S \cos\alpha = BS\cos\alpha = Bab\cos\alpha \ ,$$

and the angle between the loop surface vector and the magnetic induction vector is

$$\alpha = \omega t = 2\pi f t \ ,$$

thus, the magnetic induction flux will be

$$\Phi = Bab\cos(2\pi f t) \ .$$

Faraday's law of electromagnetic induction, therefore, for the induced voltage follows

$$U_i = -\frac{\mathrm{d}\left[Bab\cos(2\pi f t)\right]}{\mathrm{d}t} = Bab2\pi f \sin(2\pi f t) = U_0 \sin(2\pi f t) \ ,$$

where the amplitude of the induced voltage is

$$U_0 = Bab2\pi f \ ,$$

after substituting numerical values

$$U_0 = 45 \cdot 10^{-6}\,\mathrm{T} \ .\ 0.2\,\mathrm{m} \ .\ 0.1\,\mathrm{m} \ .\ 2\pi \ .\ 50\,\mathrm{s}^{-1} = 2.83 \cdot 10^{-4}\,\mathrm{V} = 0.283\,\mathrm{mV} \ .$$

7.10 *Calculate the amplitude and the waveform of the induced electric current in a rectangular copper conductor with sides* $a = 10\,\mathrm{cm}$ *and* $b = 4\,\mathrm{cm}$ *cross section* $S = 2\,\mathrm{mm}^2$ *and resistivity* $\rho = 1.7 \cdot 10^{-8}\,\Omega\,\mathrm{m}$, *which in a homogeneous magnetic field with by induction* $B = 5\,\mathrm{mT}$ *rotates with a frequency* $f = 100\,\mathrm{s}^{-1}$.

According to Faraday's law of induction, the induced electromotive voltage is equal to the negative time change of the magnetic flux through the loop

$$U_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} \ .$$

where the magnetic flux through the loop is

$$\Phi = \int_S \vec{B} \cdot \mathrm{d}\vec{S} = \int_S B\mathrm{d}S\cos\alpha = B\int_S \mathrm{d}S\cos\alpha = BS\cos\alpha = Bab\cos\alpha \ ,$$

because the angle between the loop surface vector and the magnetic induction vector is

$$\alpha = \omega t = 2\pi f t \ ,$$

the magnetic flux will be

$$\Phi = Bab \cos{(2\pi f t)} \ .$$

The induced voltage then follows

$$U_i = -\frac{\mathrm{d}\left[Bab \cos{(2\pi f t)}\right]}{\mathrm{d}t} = Bab 2\pi f \sin(2\pi f t) \ .$$

The relationship between induced voltage and current can be expressed using Ohm's law

$$I_i = \frac{U_i}{R} \ ,$$

where the electrical resistance of the conductor can be calculated as

$$R = \rho \frac{l}{S} = \rho \frac{2(a+b)}{S} \ .$$

The waveform of the induced electric current will be

$$I_i = \frac{BabS\pi f}{\rho(a+b)} \sin(2\pi f t) = I_0 \sin(2\pi f t) \ ,$$

and its amplitude will be

$$I_i = \frac{BabS\pi f}{\rho(a+b)} \ ,$$

after substituting numerical values

$$I_0 = \frac{5 \cdot 10^{-3}\,\mathrm{T}\ .\ 0.1\,\mathrm{m}\ .\ 0.04\,\mathrm{m}\ .\ 2 \cdot 10^{-6}\,\mathrm{m}^2\ .\ \pi\ .\ 100\,\mathrm{s}^{-1}}{1.7 \cdot 10^{-8}\,\Omega\,\mathrm{m}\ .\ (0.1\,\mathrm{m} + 0.04\,\mathrm{m})} = 5.28\,\mathrm{A} \ .$$

8 Oscillations and waves

8.1 *Imagine a straight shaft between Europe and Australia passing through the centre of the Earth. If a body enters the shaft, it will be acted upon by a force that is directed towards the centre of the Earth and is directly proportional to the distance from the centre of the Earth. Calculate how long it would take a body that was dropped into the shaft to travel from Europe to Australia and back, and what speed the body would have when passing through the centre of the Earth. Gravitational acceleration on the surface of the Earth is* $g = 9.81\,\mathrm{m\,s^{-2}}$ *and the radius of the Earth is* $R_E = 6370\,\mathrm{km}$.

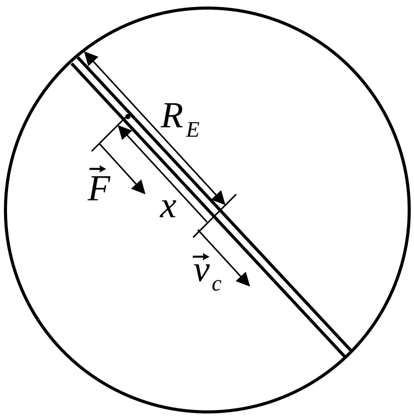

Fig. 41

The body in the shaft will be acted upon by a force (Fig. 41) whose magnitude is

$$F = -kx \ ,$$

from Newton's second law, it follows

$$ma = F \ ,$$

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \ .$$

On the surface of the Earth, the force is equal to the weight of the body

$$kR_E = mg \ ,$$

from which the constant is

$$k = m\frac{g}{R_E} \ ,$$

using which it is possible to write the equation of motion in the form

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{g}{R_E}x \ ,$$

which is the equation of harmonic motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x \ ,$$

whose angular frequency is

$$\omega = \sqrt{\frac{g}{R_E}} \ ,$$

and its solution has the form

$$x = x_0 \cos(\omega t + \alpha) \ .$$

Since at time $t = 0\,\mathrm{s}$ s the position of the body was $x = R_E$, the following applies

$$x_0 = R_E \ ,$$

$$\alpha = 0 \ ,$$

so the equation describing the movement of the body will be

$$x = R_E \cos\left(\sqrt{\frac{g}{R_E}}t\right) \ .$$

The journey of a body from Europe to Australia and back is a period of motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E}{g}} \ ,$$

after substituting numerical values

$$T = 2\pi\sqrt{\frac{6.37 \cdot 10^6\,\mathrm{m}}{9.81\,\mathrm{m\,s^{-2}}}} = 5063\,\mathrm{s} = 84\,\mathrm{min} \ .$$

The velocity of the body can be expressed as

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\left[R_E \cos\left(\sqrt{\frac{g}{R_E}}t\right)\right]}{\mathrm{d}t} = -\sqrt{R_E g}\sin\left(\sqrt{\frac{g}{R_E}}t\right) \ .$$

For a body in the centre of the Earth, the following applies

$$R_E \cos\left(\sqrt{\frac{g}{R_E}}t_c\right) = 0 \ ,$$

hence, the time is

$$t_c = \frac{\pi}{2}\sqrt{\frac{R_Z}{g}} \ .$$

After substituting into the velocity of the body, the velocity of the body at the centre of the Earth will be

$$v_c = -\sqrt{R_Z g}\sin\left(\sqrt{\frac{g}{R_E}}t_c\right) = -\sqrt{R_E g}\sin\left(\sqrt{\frac{g}{R_E}}\,\frac{\pi}{2}\sqrt{\frac{R_E}{g}}\right) = -\sqrt{R_E g} \ ,$$

where the negative symbol means that the v_s direction is opposite to the x direction, after substituting the numerical values, the velocity of the body in the centre of the Earth will be

$$v_c = \sqrt{6.37 \cdot 10^6\,\mathrm{m} \ . \ 9.81\,\mathrm{m\,s^{-}1}} = 7905\,\mathrm{m\,s^{-1}} \ .$$

8.2 *Two bodies with masses* $m_1 = 5\,\mathrm{kg}$ *and* $m_2 = 3\,\mathrm{kg}$ *are connected by a spring whose spring constant is* $k = 100\,\mathrm{N\,m^{-1}}$*. We bring the bodies closer to each other, thereby compressing the spring and then releasing the bodies. Calculate the period of oscillation of the bodies.*

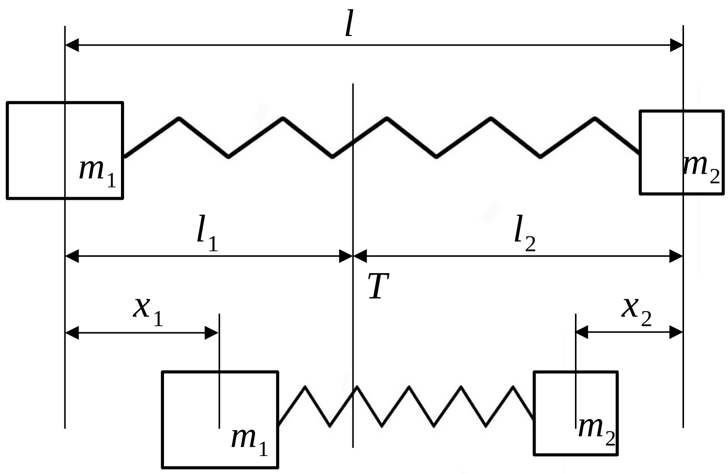

Fig. 42

If the origin of the coordinate system is in the common centre of gravity of the bodies (Fig. 42), the following will apply

$$0 = \frac{-m_1 l_1 + m_2 l_2}{m_1 + m_2} ,$$

where l_1 and l_2 are the distances of the centres of gravity of individual bodies from their common centre of gravity. It follows that

$$m_1 l_1 = m_2 l_2 .$$

Since it is an isolated system, after compressing the spring, the position of the centre of gravity will not change and will apply

$$0 = \frac{-m_1(l_1 - x_1) + m_2(l_2 - x_2)}{m_1 + m_2} ,$$

where x_1 and x_2 are the deviations of the bodies from their equilibrium positions. It follows that

$$m_1 x_1 = m_2 x_2 .$$

The force acting on the first body will be

$$F_1 = -k(x_1 + x_2) .$$

Because

$$x_2 = x_1 \frac{m_1}{m_2} ,$$

the force acting on the first body can be expressed as

$$F_1 = -k \frac{m_1 + m_2}{m_2} x_1 = -k_1 x_1 ,$$

where

$$k_1 = k \frac{m_1 + m_2}{m_2} .$$

The angular frequency of the first body will be

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}$$

and the period of the first body will be

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi \sqrt{\frac{m_1 m_2}{k(m_1 + m_2)}} ,$$

after substitution

$$T_1 = 2\pi\sqrt{\frac{5\,\mathrm{kg}\,.\,3\,\mathrm{kg}}{100\,\mathrm{N\,m^{-1}}\,.\,(5\,\mathrm{kg}+3\,\mathrm{kg})}} = 0.86\,\mathrm{s}\ .$$

The force acting on the other body will be

$$F_2 = -k(x_1 + x_2)\ .$$

Because

$$x_1 = x_2\frac{m_2}{m_1}\ ,$$

it is possible to express the force acting on the second body as

$$F_2 = -k\frac{m_1+m_2}{m_1}x_2 = -k_2x_2\ ,$$

where

$$k_2 = k\frac{m_1+m_2}{m_1}\ .$$

The angular frequency of the second body's motion will be

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k(m_1+m_2)}{m_1m_2}}$$

and the period of motion of the second body will be

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi\sqrt{\frac{m_1m_2}{k(m_1+m_2)}}\ .$$

The periods of movement of the first and second bodies are therefore the same $T_1 = T_2 = 0.86\,\mathrm{s}$.

8.3 *Calculate the period of harmonic motion of a body of mass* $m = 100\,\mathrm{g}$ *suspended on a spring. A force* $F_1 = 0.2\,\mathrm{N}$ *is needed to extend the spring by* $x_1 = 10\,\mathrm{cm}$.

The equation of motion of a body with mass m performing harmonic motion has the form

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx\ ,$$

where x is the displacement of the body from the equilibrium position, and k is the spring constant. The equation has a solution in the form

$$x = x_0\cos(\omega t + \alpha)\ ,$$

where x_0 is the amplitude and α is the phase constant of the motion. For the angular frequency applies

$$\omega = \sqrt{\frac{k}{m}} \ .$$

A force F_1 is required to extend the spring by x_1, therefore

$$F_1 = kx_1 \ ,$$

for the spring constant, it follows

$$k = \frac{F_1}{x_1} \ .$$

The period of harmonic motion can be calculated as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{mx_1}{F_1}} \ ,$$

after substitution

$$T = 2\pi \ . \sqrt{\frac{0.1\,\mathrm{kg}\ .\ 0.1\,\mathrm{m}}{0.2\,\mathrm{N}}} = 1.4\,\mathrm{s} \ .$$

8.4 *Mechanical work* $A_1 = 0.25\,\mathrm{J}$ *is required to extend the spring by* $x_1 = 5\,\mathrm{cm}$. *What will be the angular frequency of a body with mass* $m = 0.5\,\mathrm{kg}$, *that will oscillate on this spring?*

The force required to extend the spring by x is

$$F = kx \ .$$

The mechanical work when stretched by x_1 will therefore be

$$A_1 = \int\limits_0^{x_1} F\mathrm{d}x = \int\limits_0^{x_1} kx\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_0^{x_1} = \frac{1}{2}kx_1^2 \ ,$$

from which it is possible to express the spring constant

$$k = \frac{2A_1}{x_1^2} \ .$$

The harmonic motion of a body is described by the equation of motion

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \ .$$

Its solution has the form

$$x = x_0 \cos(\omega t + \alpha) \ ,$$

where x_0 is the amplitude and α is the phase constant of the motion. For the angular frequency applies

$$\omega = \sqrt{\frac{k}{m}} \ .$$

After substituting for the spring constant, the angular frequency will be

$$\omega = \sqrt{\frac{2A_1}{mx_1^2}}$$

and after substituting numerical values

$$\omega = \sqrt{\frac{2 \ . \ 0.25\,\mathrm{J}}{0.5\,\mathrm{kg} \ . \ (0.05\,\mathrm{m})^2}} = 20\,\mathrm{s}^{-1} \ .$$

8.5 *A horizontal board performs a harmonic motion in the horizontal direction with a period of $T = 3\,\mathrm{s}$. The body lying on the board starts to slide when the amplitude of oscillations reaches the value $x_0 = 0.5\,\mathrm{m}$. What is the coefficient of friction between the body and the board?*

A frictional force acts on the body on the board, the magnitude of which is given by the multiplication of the coefficient of friction and the normal force

$$F_t = \mu N \ .$$

The magnitude of the normal force is equal to the product of the mass of the body and the acceleration of gravity

$$N = mg \ ,$$

therefore, the magnitude of the frictional force will be

$$F_t = \mu mg \ .$$

Because the body moves together with the board, it is in a non-inertial frame of reference, and in addition to the frictional force, the body also has an inertial force

$$F_z = -ma \ .$$

The motion of the board is described by the function

$$x = x_0 \cos(\omega t + \alpha) \ ,$$

from which the speed of the plate follows

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}[x_0 \cos(\omega t + \alpha)]}{\mathrm{d}t} = -x_0 \omega \sin(\omega t + \alpha) \ ,$$

from which the acceleration of the board follows

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}[-x_0 \omega \sin(\omega t + \alpha)]}{\mathrm{d}t} = -x_0 \omega^2 \cos(\omega t + \alpha) \ ,$$

which can be used to express the magnitude of the inertial force

$$F_z = m x_0 \omega^2 \cos(\omega t + \alpha) = F_{z0} \cos(\omega t + \alpha) \ ,$$

where the amplitude of the inertial force is

$$F_{z0} = m x_0 \omega^2 \ .$$

The body starts to slide when the amplitude of the inertial force equals the frictional force

$$F_{z0} = F_t \ ,$$

that is, when it will be valid

$$m x_0 \omega^2 = \mu m g \ ,$$

where the angular frequency can be expressed using the period

$$\omega = \frac{2\pi}{T} \ ,$$

from which the coefficient of friction follows

$$\mu = \frac{4\pi^2 x_0}{T^2 g} \ ,$$

after substitution

$$\mu = \frac{4\pi^2 \ . \ 0.5\,\mathrm{m}}{(3\,\mathrm{s})^2 \ . \ 9.81\,\mathrm{m\,s^{-2}}} = 0.22 \ .$$

8.6 *The particle performs damped harmonic motion, the dependence of the particle's position on time is given by the function* $x = 5\,\mathrm{cm}\, e^{-1.4\,\mathrm{s}^{-1}t}\cos(1.6\pi\ \mathrm{s}^{-1}t)$. *Calculate the damping coefficient, the logarithmic decrement of the damping, the time it takes for the amplitude of the oscillations to drop to one-hundredth of the original value, and the angular frequency at which the particle would oscillate if the damping force stopped acting.*

The position of a particle in damped harmonic motion is described by a function

$$x = x_0 e^{-bt}\cos(\omega t + \alpha)\ ,$$

from which it follows that the damping coefficient is

$$b = 1.4\,\mathrm{s}^{-1}\ ,$$

and the logarithmic decrement of the damping is

$$\delta = bT = b\frac{2\pi}{\omega} = 1.4\,\mathrm{s}^{-1}\frac{2\pi}{1.6\pi\ \mathrm{s}^{-1}} = 0.875\ .$$

Time for the amplitude to drop to one hundredth

$$x_0 e^{-bt_1} = \frac{x_0}{100}\ ,$$

will be

$$t_1 = \frac{\ln 100}{b} = \frac{\ln 100}{1.4\,\mathrm{s}^{-1}} = 3.29\,\mathrm{s}\ .$$

For the angular frequency of the damped harmonic oscillator applies

$$\omega = \sqrt{\omega_0^2 - b^2}\ ,$$

from which it is possible to express the angular frequency of motion without damping

$$\omega_0 = \sqrt{\omega^2 + b^2} = \sqrt{(1.6\pi\ \mathrm{s}^{-1})^2 + (1.4\,\mathrm{s}^{-1})^2} = 5.22\,\mathrm{s}^{-1}\ .$$

8.7 *The result of adding two harmonic motions on a line is the motion described by the equation* $x = x_0\cos(2\,\mathrm{s}^{-1}\ .\ t)\cos(50\,\mathrm{s}^{-1}\ .\ t)$. *Calculate the angular frequencies of the original harmonic motions and the angular frequency of the shocks of the resulting motion.*

When adding two harmonic motions

$$x = x_0 \cos(\omega_1 t) \ ,$$

$$x = x_0 \cos(\omega_2 t) \ ,$$

based on the principle of superposition, the resulting motion is

$$x = x_0 \cos(\omega_1 t) + x_0 \cos(\omega_2 t) \ .$$

Using the relationship

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right) \ ,$$

it is possible to write the resulting motion as

$$x = 2x_0 \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)\cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \ .$$

The angular frequencies of the original motions can be obtained by solving the system of equations

$$\frac{\omega_1 - \omega_2}{2} = 2\,\mathrm{s}^{-1} \ ,$$

$$\frac{\omega_1 + \omega_2}{2} = 50\,\mathrm{s}^{-1} \ ,$$

which implies

$$\omega_1 = 52\,\mathrm{s}^{-1} \ ,$$

$$\omega_2 = 48\,\mathrm{s}^{-1} \ .$$

The resulting motion can be written as a harmonic motion

$$x = A\cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \ ,$$

whose amplitude changes slowly

$$A = \left|2x_0 \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)\right| \ ,$$

the period of this amplitude is

$$T_A = \frac{2\pi}{\frac{|\omega_1 - \omega_2|}{2}} = \frac{4\pi}{|\omega_1 - \omega_2|} \ .$$

Because two amplifications and two attenuations occur in one period of the amplitude, that is, two shocks, for their period applies

$$T_s = \frac{T_A}{2} = \frac{2\pi}{|\omega_1 - \omega_2|}$$

and the angular frequency of the shocks is

$$\omega_s = \frac{2\pi}{T_s} = |\omega_1 - \omega_2| = \left|52\,\mathrm{s}^{-1} - 48\,\mathrm{s}^{-1}\right| = 4\,\mathrm{s}^{-1} \ .$$

8.8 *The wave travels through a medium, the displacement of medium particles is described by the function* $u = A\cos 2\pi(bt - hx)$, *where* $A = 2\cdot 10^{-6}\,\mathrm{m}$, $b = 5000\,\mathrm{s}^{-1}$ *a* $h = 1\,\mathrm{m}^{-1}$. *Calculate the wavelength, frequency, period, amplitude, velocity of the wave, the maximum value of the velocity and acceleration of particle oscillations and write a function for the same wave travelling in the opposite direction.*

A function describing the wave in the direction of the x-axis

$$u = u_0 \cos(\omega t - kx) \ ,$$

where k is the wave number

$$k = \frac{2\pi}{\lambda} \ ,$$

can be rewritten to the form

$$u = u_0 \cos 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right) \ ,$$

which implies that the wavelength of the wave is

$$\lambda = \frac{1}{h} = \frac{1}{1\,\mathrm{m}^{-1}} = 1\,\mathrm{m} \ .$$

The frequency ot the wave is

$$f = b = 5000\,\mathrm{s}^{-1} \ ,$$

the period of the wave is

$$T = \frac{1}{f} = \frac{1}{b} = 2\cdot 10^{-4}\,\mathrm{s} \ ,$$

the amplitude of the wave is

$$u_0 = A = 2\cdot 10^{-6}\,\mathrm{m} \ ,$$

and the speed of the wave is

$$c = \frac{\lambda}{T} = \frac{b}{h} = \frac{1\,\mathrm{m}}{2\cdot 10^{-4}\,\mathrm{s}} = 5000\,\mathrm{m\,s}^{-1} \ .$$

The speed of the oscillation of the particles is

$$v = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}\left[A\cos 2\pi(bt - hx)\right]}{\mathrm{d}t} = -2\pi b A \sin 2\pi(bt - hx) \ .$$

The maximum value of the speed of the oscillations of the particles is

$$v_{max.} = 2\pi bA = 2\pi \; . \; 5000\,\mathrm{s}^{-1} \; . \; 2 \cdot 10^{-6}\,\mathrm{m} = 0.0628\,\mathrm{m\,s}^{-1} \; .$$

The acceleration of the oscillation of the particles is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\left[-2\pi bA \sin 2\pi(bt - hx)\right]}{\mathrm{d}t} = -(2\pi b)^2 A \cos 2\pi(bt - hx) \; .$$

The maximum value of the acceleration of the oscillations of the particles is

$$a_{max.} = (2\pi b)^2 A = (2\pi \; . \; 5000\,\mathrm{s}^{-1})^2 \; . \; 2 \cdot 10^{-6}\,\mathrm{m} = 1973.9\,\mathrm{m\,s}^{-2} \; .$$

For a wave travelling in the opposite direction, the following applies

$$x \to -x \; ,$$

therefore, the equation describing the wave traveling in the opposite direction will have the form

$$u = A \cos 2\pi(bt + hx) \; .$$

8.9 *The speed of sound in steel can be determined by creating a wave in a steel rod fixed in the middle, which vibrates the air in a Kundt's tube, in which a standing wave is created. Calculate the speed of sound in steel and the tensile modulus of steel if the distance between two standing wave nodes in air is* $x = 8\,\mathrm{cm}$, *the length of the rod is* $l = 1.2\,\mathrm{m}$, *the speed of sound in air is* $v = 340\,\mathrm{m\,s}^{-1}$ *and the density of steel is* $\rho = 7800\,\mathrm{kg\,m}^{-3}$.

The condition for the formation of standing waves in the air and the rod, as well as the formation of resonance, is

$$f = f' \; .$$

The frequency of the wave in the air applies

$$f = \frac{v}{\lambda} \; ,$$

where v and λ are the speed and the wavelength of the wave in air. For the frequency of the wave in the rod applies

$$f' = \frac{v'}{\lambda'} \; ,$$

where v' and λ' are the speed and the wavelength of the wave in the rod. Therefore, it is possible to rewrite the condition for the formation of standing waves in the form

$$\frac{v}{\lambda} = \frac{v'}{\lambda'} ,$$

from which, for the speed of the wave in the rod, it follows

$$v' = v\frac{\lambda'}{\lambda} ,$$

where the wavelength in air can be determined as twice the distance between two nodes

$$\lambda = 2x$$

and the wavelength in the rod can be determined as twice the length of the rod

$$\lambda' = 2l ,$$

using which, for the speed of sound in steel, follows

$$v' = v\frac{2l}{2x} = v\frac{l}{x} ,$$

after substitution

$$v' = 340\,\mathrm{m\,s^{-1}} \,.\, \frac{1.2\,\mathrm{m}}{0.08\,\mathrm{m}} = 5100\,\mathrm{m\,s^{-1}} \,.$$

The wave equation has the general form

$$\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2} \,.$$

Using Hooke's law, it is possible to derive the equation for waves in steel

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} ,$$

therefore, the speed of sound in steel will be

$$v'^2 = \frac{E}{\rho} ,$$

from which the tensile modulus of steel follows

$$E = v'^2\rho ,$$

after substitution

$$E = (5100\,\mathrm{m\,s^{-1}})^2 \,.\, 7800\,\mathrm{kg\,m^{-3}} = 2.03 \cdot 10^{11}\,\mathrm{Pa} \,.$$

8.10 *The whistle with closed end produces a tone of fundamental frequency* $f = 130.5\,\mathrm{Hz}$. *Calculate the length of the whistle and the fundamental frequency, if the end of the whistle is open. The speed of sound in air* $v = 340\,\mathrm{m\,s^{-1}}$.

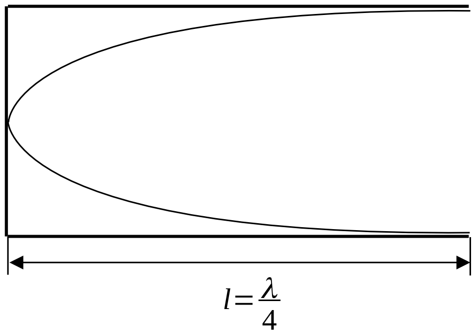

Fig. 43

If one end of the whistle is closed and the other is open (Fig. 43), there will be an antinode at the closed end and a node at the open end. The length of the whistle and the wavelength of the fundamental frequency follow

$$l = \frac{\lambda}{4} \,.$$

Because the wavelength and the frequency apply

$$f = \frac{v}{\lambda} \,,$$

it is possible to express the length of the whistle as

$$l = \frac{v}{4f} \,,$$

after substitution

$$l = \frac{340\,\mathrm{m\,s^{-1}}}{4\,.\,130.5\,\mathrm{Hz}} = 0.65\,\mathrm{m} \,.$$

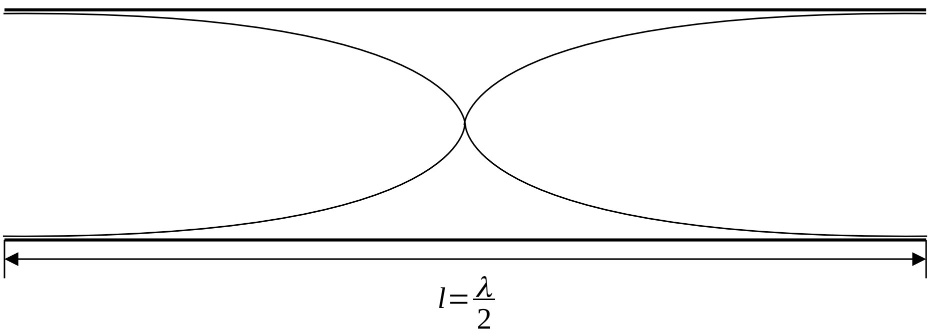

Fig. 44

If both ends of the whistle are open (Fig. 44), there will be antinodes at both ends of the whistle. The length of the whistle and the wavelength of the fundamental frequency follow

$$l = \frac{\lambda}{2} \, .$$

Because the wavelength and the frequency apply

$$f = \frac{v}{\lambda} \, ,$$

it is possible to express the length of the whistle as

$$f = \frac{v}{2l} \, ,$$

after substitution

$$f = \frac{340\,\mathrm{m\,s^{-1}}}{2\;.\;0.65\,\mathrm{m}} = 261.5\,\mathrm{Hz} \, .$$

9 Optics

9.1 *Two monochromatic plane electromagnetic waves of the same frequency, polarised in the same plane, with amplitudes $E_{01} = 5\,\mathrm{V\,m^{-1}}$ and $E_{02} = 7\,\mathrm{V\,m^{-1}}$ propagate in the same direction in vacuum. Calculate the resulting wave intensity if the waves are a) incoherent b) coherent and the phase shift between them is $\delta = \frac{\pi}{3}$.*

a) In the superposition of two incoherent waves, the resulting wave intensity is equal to the sum of the wave intensities

$$I = I_1 + I_2 \,,$$

which implies

$$I = \frac{1}{2}c\epsilon_0 E_{01}^2 + \frac{1}{2}c\epsilon_0 E_{02}^2 = \frac{1}{2}c\epsilon_0 (E_{01}^2 + E_{02}^2) \,,$$

after insertion

$$I = \frac{1}{2} \,.\, 3 \cdot 10^8\,\mathrm{m\,s^{-1}} \,.\, 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \,.\, [(5\,\mathrm{V\,m^{-1}})^2 + (7\,\mathrm{V\,m^{-1}})^2] =$$

$$= 0.098\,\mathrm{W\,m^{-2}} \,.$$

b) In the superposition of two coherent waves, the resulting wave intensity is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta \,,$$

which implies

$$I = \frac{1}{2}c\epsilon_0 E_{01}^2 + \frac{1}{2}c\epsilon_0 E_{02}^2 + c\epsilon_0 E_{01}E_{02}\cos\delta = \frac{1}{2}c\epsilon_0\left(E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos\delta\right) \,,$$

after insertion

$$I = \frac{1}{2} \,.\, 3 \cdot 10^8\,\mathrm{m\,s^{-1}} \,.\, 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}} \,.\, [(5\,\mathrm{V\,m^{-1}})^2 + (7\,\mathrm{V\,m^{-1}})^2 +$$

$$+2 \,.\, 5\,\mathrm{V\,m^{-1}} \,.\, 7\,\mathrm{V\,m^{-1}} \,.\, \cos\frac{\pi}{3}] = 0.145\,\mathrm{W\,m^{-2}} \,.$$

9.2 *A plano-convex lens is laid on a planar plate. Light of wavelength* $\lambda = 598\,\mathrm{nm}$ *is incident perpendicularly on the flat side of the lens. In the reflected light, the Newton's rings are observed. The radius of the fifth dark ring is* $r_5 = 5\,\mathrm{mm}$. *Calculate the radius of the convex surface of the lens and the radius of the fourth dark ring.*

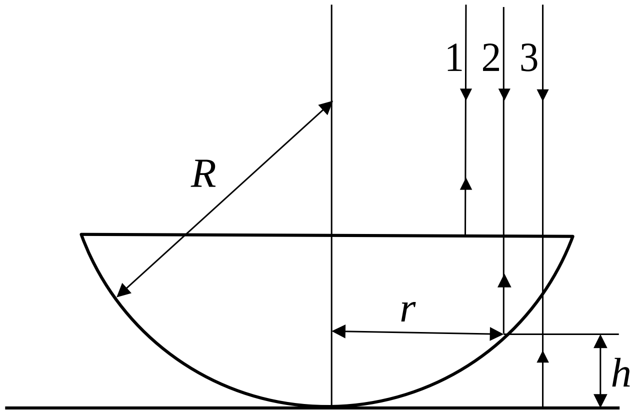

Fig. 45

Part of the light (Fig. 45) incident on the flat surface of the lens is reflected from this surface (ray 1), and part penetrates the lens. In the penetrating part of the light, some light is reflected from the convex surface of the lens (ray 2), and some penetrates further and is reflected up to the plane plate (ray 3). Ray 1 does not interfere with either ray 2 or ray 3 because their path difference is greater than the coherent length of the light. Interference occurs between rays 2 and 3. Their path difference is

$$\Delta = 2h + \frac{\lambda}{2}\ ,$$

because ray 3 has twice passed through a layer of air with refractive index $n = 1$ and on reflection from the optically denser medium on the bottom plate, there is a phase change of π, which corresponds to a path difference of $\frac{\lambda}{2}$. The interference minimum condition is

$$2h + \frac{\lambda}{2} = (2m+1)\frac{\lambda}{2} \qquad m = 0, 1, 2, \dots\ .$$

The Pythagorean Theorem implies

$$(R-h)^2 + r^2 = R^2\ ,$$

from which it is possible to express

$$r^2 = h(2R - h) \ .$$

Since $h << R$, the following holds

$$r^2 = h(2R - h) \approx h2R \ ,$$

which implies

$$h = \frac{r^2}{2R} \ .$$

When inserted into the condition for the interference minimum, then

$$r_m = \sqrt{mR\lambda} \ ,$$

from which it is possible to express the radius of the lens as

$$R = \frac{r_m^2}{m\lambda} \ ,$$

from the values for the fifth interference minimum

$$R = \frac{r_5^2}{5\lambda} \ .$$

the lens radius can be calculated

$$R = \frac{(5 \cdot 10^{-3}\,\mathrm{m})^2}{5 \ . \ 598 \cdot 10^{-9}\,\mathrm{m}} = 8.36\,\mathrm{m} \ .$$

The radius of the fourth interference minimum is

$$r_4 = \sqrt{4\lambda R} \ ,$$

after inserting

$$r_4 = \sqrt{4 \ . \ 598 \cdot 10^{-9}\,\mathrm{m} \ . \ 8.36\,\mathrm{m}} = 4.47\,\mathrm{mm} \ .$$

9.3 *A plano-convex lens is laid on a planar plate. Light is incident perpendicularly on the flat side of the lens. In the reflected light, we observe Newton's rings. If the space between the lens and the plate is filled with liquid, the radius of the fourth dark ring will be the same as the radius of the third dark ring when there was air in the space between the lens and the plate. Calculate the index of refraction of the liquid.*

Since the space between the lens and the planar plate is filled by a liquid with refractive index n, the difference in the optical paths of the rays reflected from the convex part of the lens and the planar plate is

$$\Delta = 2nh + \frac{\lambda}{2} .$$

The condition of the interference minimum is

$$2nh + \frac{\lambda}{2} = (2m+1)\frac{\lambda}{2} \qquad m = 0, 1, 2, \dots ,$$

because

$$h = \frac{r^2}{2R} ,$$

the radius of the m-th dark ring is

$$r_m = \sqrt{\frac{mR\lambda}{n}} ,$$

so the radius of the fourth dark ring, if there is a liquid in the space, is

$$r_4 = \sqrt{\frac{mR\lambda}{n}} .$$

and the radius of the third dark ring, if there is air in the space, is

$$r_3 = \sqrt{mR\lambda} ,$$

because for air $n = 1$. The condition of equality of radii

$$r_4 = r_3 ,$$

implies

$$\sqrt{\frac{mR\lambda}{n}} = \sqrt{mR\lambda} ,$$

from which the refractive index of the liquid can be expressed

$$n = \frac{4}{3} \approx 1.33 .$$

9.4 *To prevent light loss by reflection, the glass plate, whose refractive index is* $n_1 = 1.66$*, is covered on both sides with a thin covering of transparent material. What must be the index of refraction of the cover layer, and for what minimum thickness of the cover layer will light of wavelength* $\lambda = 520\,\mathrm{nm}$ *pass through the plate without loss? Assume that the light is incident perpendicularly on the wafer, and the losses due to absorption of light are negligible.*

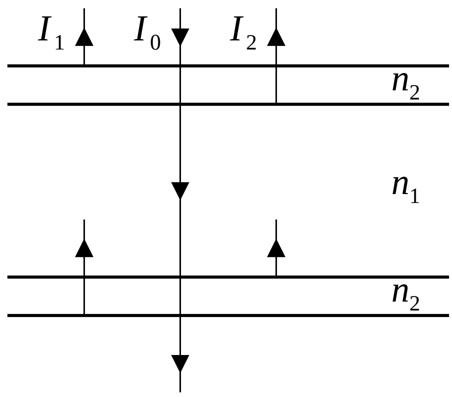

Fig. 46

The light rays (Fig. 46) that are reflected from the cover layer and the glass plate are coherent because they are produced by splitting a single ray, and the cover layer is thinner than the coherent length of light. Light will pass through without loss if the intensity of the reflected light is zero, that is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(\frac{2\pi}{\lambda}\Delta\right) = 0 \; ,$$

where the intensity of light reflected at the interface between the air and the cover layer is

$$I_1 = \left(\frac{n_2 - 1}{n_2 + 1}\right)^2 I_0 \; ,$$

and the intensity of light reflected at the interface between the cover layer and the glass plate is

$$I_2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 (I_0 - I_1) \approx \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_0 \; .$$

The intensity of the reflected light will be zero if the conditions are simultaneously satisfied

$$I_1 = I_2 \qquad \text{and} \qquad \cos\left(\frac{2\pi}{\lambda}\Delta\right) = -1 \; .$$

The first condition implies

$$\left(\frac{n_2 - 1}{n_2 + 1}\right)^2 I_0 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 I_0 \; .$$

For the refractive index of the cover layer, the following must therefore hold

$$n_2 = \sqrt{n_1} = \sqrt{1.66} = 1.30 \; .$$

The second condition for the difference of optical paths must hold

$$\Delta = (2m-1)\frac{\lambda}{2} \ .$$

Since the difference in the optical paths is

$$\Delta = 2hn_2 \ ,$$

for the thickness of the cover layer, it follows

$$h = \frac{(2m-1)\lambda}{4n_1} \qquad m = 1, 2, 3, \dots \ .$$

The smallest thickness of the cover layer will be at $m = 1$, so it is equal to

$$h_{min} = \frac{\lambda}{4n_1} = \frac{520 \cdot 10^{-9}\,\mathrm{m}}{4\ .\ 1.30} = 100\,\mathrm{nm} \ .$$

9.5 *When a perpendicularly parallel beam of violet light with wavelength $\lambda_1 = 420\,\mathrm{nm}$ is incident on the slit, the center of the second dark band can be seen on the screen at an angle $\alpha_1 = 4°53'$ from the normal to the plane of the slit. At what angle will the center of the third dark band be seen if we illuminate the slit with green light of wavelength $\lambda_2 = 550\,\mathrm{nm}$?*

If the light bends at the slit, minima are formed on the shade at the points for which

$$d \sin\alpha = k\lambda \qquad k = 1, 2, 3, \dots \ ,$$

therefore, for the second minimum of light with wavelength λ_1 it is valid

$$d \sin\alpha_1 = 2\lambda_1$$

and for the third minimum of light with wavelength λ_2 it is valid

$$d \sin\alpha_2 = 3\lambda_2 \ .$$

Dividing the equations by each other produces a new equation

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{2\lambda_1}{3\lambda_2} \ ,$$

from which it is possible to express

$$\sin\alpha_2 = \frac{3\lambda_2}{2\lambda_1}\sin\alpha_1 \ ,$$

after inserting values

$$\sin\alpha_2 = \frac{3\ .\ 550 \cdot 10^{-9}\,\mathrm{m}}{2\ .\ 420 \cdot 10^{-9}\,\mathrm{m}}\ .\ \sin 4°53' = 0.167 \ ,$$

which corresponds to the angle

$$\alpha_2 = 9°37' \ .$$

9.6 *Calculate the illuminance of a small circular area of radius* $r << R$ *located at a distance* $R = 2\,\mathrm{m}$ *from a point source with luminous intensity* $I = 20\,\mathrm{cd}$*, if the normal to the area points to the point source.*

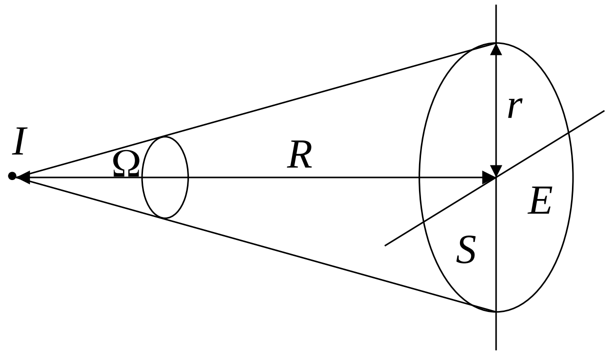

Fig. 47

Illuminance is defined as the ratio of the luminous flux to the size of the area on which the luminous flux falls

$$E = \frac{\Phi}{S} \,.$$

Because a light source whose luminous intensity is I emits a luminous flux to a solid angle Ω (Fig. 47)

$$\Phi = I\Omega \,,$$

the illuminance is

$$E = \frac{I\Omega}{S} \,.$$

The condition $r <<< R$ allows the use of the relationship between the spherical surface S, the radius of the sphere R and the solid angle Ω

$$S = \Omega R^2 \,,$$

using which the illuminance can be expressed as

$$E = \frac{I\Omega}{\Omega R^2} = \frac{I}{R^2} \,,$$

after inserting values

$$E = \frac{20\,\mathrm{cd}}{(2\,\mathrm{m})^2} = 5\,\mathrm{lx} \,.$$

9.7 *A light source whose luminance is* $L = 100\,\mathrm{cd\,m^{-2}}$ *has the shape of a disc with radius* $R = 0.5\,\mathrm{m}$. *Calculate the illuminance at a point located at a distance* $a = 5\,\mathrm{m}$ *from the center of the light source.*

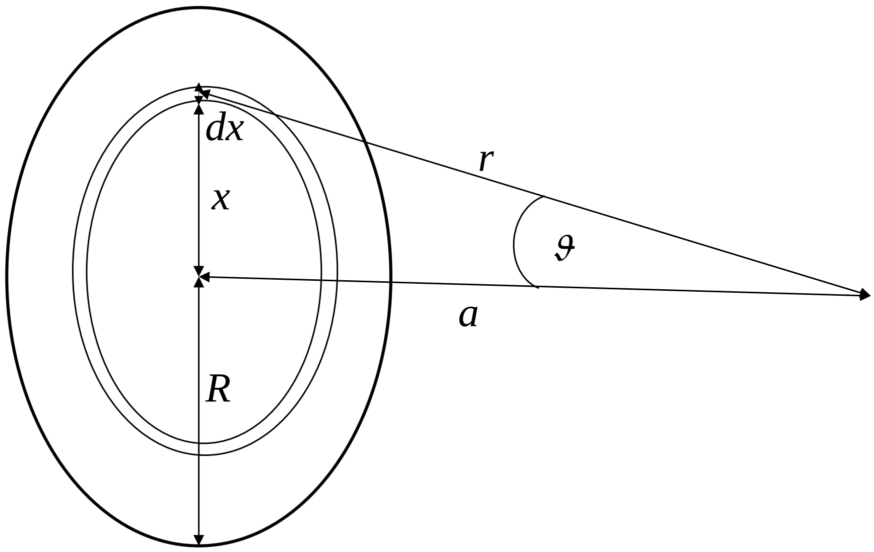

Fig. 48

The light source can be divided into concentric circular rings of radius x and thickness $\mathrm{d}x$, whose area (Fig. 48) is

$$\mathrm{d}S = 2\pi x \mathrm{d}x \ .$$

The luminous intensity of the ring in the direction of the given point is

$$\mathrm{d}L_{\vartheta} = L \mathrm{d}S \cos\vartheta = L 2\pi x \mathrm{d}x \cos\vartheta$$

and the illuminance produced by the ring at a given point is

$$\mathrm{d}E = \frac{\cos\vartheta}{r^2} L 2\pi x \mathrm{d}x \cos\vartheta = \frac{\cos^2\vartheta}{r^2} L 2\pi x \mathrm{d}x \ .$$

The distance of the ring from the point is

$$r = \sqrt{x^2 + a^2} \ ,$$

and is also valid

$$\cos\vartheta = \frac{a}{r} = \frac{a}{\sqrt{x^2 + a^2}} \ ,$$

which can be used to express the illuminance produced by the ring as

$$E = 2\pi L a^2 \frac{x}{(x^2 + a^2)^2} \mathrm{d}x$$

and the illuminance produced by the whole light source can be calculated by integrating

$$E = 2\pi L a^2 \int_0^R \frac{x}{(x^2 + a^2)^2} \mathrm{d}x = \pi L a^2 \left[-\frac{1}{x^2 + a^2} \right]_0^R = \frac{\pi L R^2}{R^2 + a^2} ,$$

after inserting values

$$E = \frac{\pi \ . \ 100\,\mathrm{cd\,m^{-2}} \ . \ (0.5\,\mathrm{m})^2}{(5\,\mathrm{m})^2 + (0.5\,\mathrm{m})^2} = 3.11\,\mathrm{lx} \ .$$

9.8 *A wall is illuminated by two identical bulbs side by side at a distance $d = 2\,\mathrm{m}$ from the wall. When one bulb is switched off, calculate how the second bulb must move to keep the illuminance of the wall the same as before.*

For the illuminance of an area by a point source whose luminous intensity is I, the following holds

$$E = \frac{I}{r^2} \cos\alpha \ .$$

Since the distance of the bulb from the wall is d, and the light rays fall perpendicular to the wall $\alpha = 0°$, the illuminance of the wall from a single bulb is

$$E = \frac{I}{d^2} \ .$$

Since both bulbs have the same luminous intensity and are at the same distance from the wall, the illuminance of the wall from the two bulbs is

$$E_2 = 2E = \frac{2I}{d^2} \ .$$

After switching off one bulb, the second bulb must be moved to a distance x from the wall so that the illuminance of the wall from one bulb

$$E_1 = \frac{I}{x^2}$$

remains the same as when illuminated by two bulbs

$$E_1 = E_2 \ .$$

Thus it must be valid

$$\frac{I}{x^2} = 2\frac{I}{d^2} ,$$

which, for the distance of the bulb from the wall, implies

$$x = \frac{d}{\sqrt{2}} ,$$

after inserting

$$x = \frac{2\,\mathrm{m}}{\sqrt{2}} = 1.41\,\mathrm{m} .$$

9.9 *The table is illuminated by two bulbs, which are placed on the ceiling at a distance* $d = 1\,\mathrm{m}$ *from each other and height* $h = 1.5\,\mathrm{m}$ *above the table. The luminous intensity of each bulb is* $I = 100\,\mathrm{cd}$. *Calculate the illuminance a) on the table centred between the bulbs b) on the table directly under one of the bulbs.*

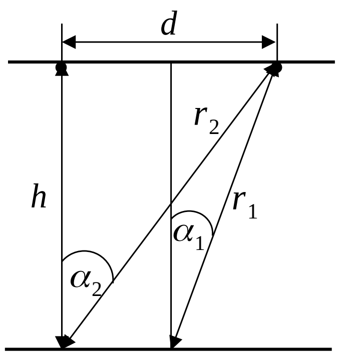

Fig. 49

a) On the table in the middle between the bulbs (Fig. 49), the illuminance of the light from each bulb is the same

$$E_1 = E_2 = \frac{I}{r_1^2}\cos\alpha_1 .$$

Because

$$r_1 = \sqrt{h^2 + \left(\frac{d}{2}\right)^2}$$

and also

$$\cos\alpha_1 = \frac{h}{r_1} ,$$

the illuminance from the individual bulbs is

$$E_1 = E_2 = \frac{Ih}{\left[h^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}} \; .$$

The illuminance on the table centred between the bulbs will be the sum of the illuminances from the two bulbs

$$E = E_1 + E_2 = \frac{2Ih}{\left[h^2 + \left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}} \; ,$$

after inserting values

$$E = \frac{2 \,.\, 100\,\mathrm{cd} \,.\, 1.5\,\mathrm{m}}{\left[(1.5\,\mathrm{m})^2 + (\frac{1\,\mathrm{m}}{2})^2\right]^{\frac{3}{2}}} = 76\,\mathrm{lx} \; .$$

b) On the table below the bulb, the illuminance will be from the bulb that is directly above it

$$E_1 = \frac{I}{h^2}$$

and the illuminance from the bulb next to it

$$E_2 = \frac{I}{r_2^2} \cos\alpha_2 \; .$$

Because

$$r_2 = \sqrt{d^2 + h^2}$$

and also

$$\cos\alpha_2 = \frac{h}{r_2} \; ,$$

the illuminance from the side bulb can be expressed as

$$E_2 = \frac{Ih}{(d^2 + h^2)^{\frac{3}{2}}} \; .$$

The illuminance on the table under one of the bulbs will be the sum of the illuminances from both bulbs

$$E = E_1 + E_2 = \frac{I}{h^2} + \frac{Ih}{(d^2 + h^2)^{\frac{3}{2}}} \; ,$$

after inserting values

$$E = \frac{100\,\mathrm{cd}}{(1.5\,\mathrm{m})^2} + \frac{100\,\mathrm{cd} \,.\, 1.5\,\mathrm{m}}{[(1\,\mathrm{m})^2 + (1.5\,\mathrm{m})^2]^{\frac{3}{2}}} = 70\,\mathrm{lx} \; .$$

9.10 *In the centre above the circular table top with radius* $R = 80\,\mathrm{cm}$ *is a light source with luminous intensity* $I = 100\,\mathrm{cd}$. *At what height above the table should the light source be placed so that the illuminance of the edge of the table is maximised? What is the maximum illuminance of the edge of the table?*

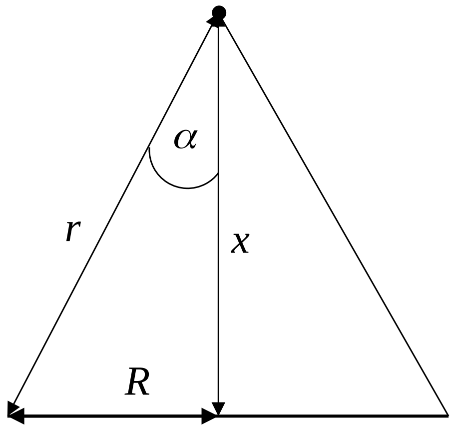

Fig. 50

The illuminance of the table edge is

$$E = \frac{I}{r^2} \cos\alpha \ .$$

Because (Fig. 50)

$$r^2 = R^2 + x^2 \ ,$$

and also

$$\cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{R^2 + x^2}} \ ,$$

the illuminance of the table edge can be expressed as

$$E = \frac{Ix}{(R^2 + x^2)^{\frac{3}{2}}} \ .$$

The extremum of this function must satisfy the condition

$$\frac{\mathrm{d}E}{\mathrm{d}x} = 0 \ ,$$

which implies

$$I\frac{(R^2 + x^2)^{\frac{3}{2}} - x\frac{3}{2}(R^2 + x^2)^{\frac{1}{2}}2x}{(R^2 + x^2)^3} = 0 \ .$$

This condition is satisfied for the distance of the light source from the table

$$x = \frac{R}{\sqrt{2}} \; ,$$

after insertion

$$x = \frac{0.8\,\mathrm{m}}{\sqrt{2}} = 0.57\,\mathrm{m} \; ,$$

at this distance from the light source, the illuminance at the edge of the table is

$$E = \frac{100\,\mathrm{cd}\;.\;0.57\,\mathrm{m}}{[(0.8\,\mathrm{m})^2 + (0.57\,\mathrm{m})^2]^{\frac{3}{2}}} = 60\,\mathrm{lx} \; .$$

Physical constants

- acceleration of gravity: $g = 9.81\,\mathrm{m\,s^{-2}}$
- gravitation constant: $\kappa = 6.67 \cdot 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$
- Avogadro constant: $N_A = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$
- Boltzmann constant: $k = 1.38 \cdot 10^{-23}\,\mathrm{J\,K^{-1}}$
- elementary charge: $e = 1.602 \cdot 10^{-19}\,\mathrm{C}$
- electric constant: $\epsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}$
- magnetic constant: $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N\,A^{-2}}$
- speed of light in vacuum: $c = 3 \cdot 10^{8}\,\mathrm{m\,s^{-1}}$
- radius of the Earth: $R_Z = 6.371 \cdot 10^{6}\,\mathrm{m}$
- mass of the Earth: $M_Z = 5.972 \cdot 10^{24}\,\mathrm{kg}$

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