TECHNICAL PHYSICS I.

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1 Introduction

„The most beautiful thing we can experience is the mysterious.
It is the source of all true art and science.“
Albert Einstein

Physics is fundamental natural science dealing with universal laws and principles of the universe. Discoveries in physics had a great impact on all natural sciences, and physics has been described as the fundamental science because other fields such as chemistry, biology, geology or astronomy investigate systems whose properties depend on the laws of physics. The emergence of physics as a science distinct from philosophy began with the dawn of modern science in the 16th and 17th centuries and continued through the revolution in physics at the beginning of the 20th century. The field has continued to expand, with a growing body of discoveries leading to the creation of five fundamental theories:

- **Classical mechanics** is often referred to as Newtonian mechanics after Issac Newton and his laws of motion. Mechanics is usually divided according to subject of its study into the particle mechanics, rigid body mechanics and continuum mechanics. Classical mechanics produces very accurate results within the domain of everyday experience. It is superseded by relativistic mechanics for objects moving at high velocities near the speed of light or quantum mechanics for behavior of elementary particles. Nevertheless, classical mechanics is still very useful, because it is much simpler and easier to apply than these other theories, and it has a very large range of approximate validity. Classical mechanics can be used to describe the motion of human-sized objects, many astronomical objects, and certain microscopic objects.

- **Classical electrodynamics** studies the phenomena associated with charged bodies and electric fields. Since moving charge produces a magnetic field, electrodynamics is concerned with effects such as magnetism, electromagnetic radiation, and electromagnetic induction. The area of classical electrodynamics, was first systematically explained by James Clerk Maxwell. Maxwell’s equations describe the phenomena of this area with great generality. Electromagnetism encompasses various real-world electromagnetic phenomena. For example, light is an oscillating electromagnetic field that is radiated from accelerating charged particles. Aside from gravity, most of the forces in everyday experience are ultimately results of electromagnetism.

- **Thermodynamics and statistical physics** study the effects of changes in temperature, pressure, and volume on physical systems on a macroscopic scale by analyzing the collective motion of their particles using statistics. Historically, thermodynamics developed out of the need to increase the efficiency of early steam engines. The starting point for most thermodynamic considerations are the laws of thermodynamics, which postulate that energy can be exchanged between physical systems as heat or work. They also postulate the existence of a quantity named entropy, which can be defined for any system. In thermodynamics, interactions between large ensembles of objects are studied and categorized. A system is composed of particles, whose average motions define its properties, which in turn are related to each other through equations of state. Statistical physics analyzes macroscopic systems by applying statistical principles to their microscopic constituents. It provides a framework for relating the microscopic properties of individual atoms and molecules to the macroscopic or bulk properties of materials that can be observed in everyday life.

- **Quantum theory** is the branch of physics dealing with atomic and subatomic systems and their interaction with radiation in terms of observable quantities. It is based on the observation that all forms of energy are released in discrete units called quanta. Quantum theory typically permits only statistical calculation of the observed features of subatomic particles in terms of wavefunctions. The Schrödinger equation plays the central role in quantum mechanics. The quantum theory proposes a dual nature for waves and particles, one aspect predominating in some situations, the other one predominating in other situations. The first contribution to the quantum theory was the explanation of black body radiation in 1900 by Max Planck. Niels Bohr used the quantum theory in 1913 to explain both atomic structure and atomic spectra, showing the connection between the electrons' energy levels and the frequencies of light given off and absorbed. A particularly important discovery of the quantum theory is the uncertainty principle, enunciated by Werner Heisenberg in 1927, which places an absolute theoretical limit on the accuracy of certain measurements. The development of quantum theory at the beginning 20th century revolutionized physics, and quantum mechanics. It is fundamental to most areas of current research.

- **Theory of relativity** is a generalization of classical mechanics that describes fast-moving or very massive systems. It includes special and general relativity. The special theory of relativity was proposed in 1905 by Albert Einstein. It is based on two postulates: The mathematical forms of the laws of physics are invariant in all inertial systems. The speed of light in a vacuum is constant and independent of the source or observer. Reconciling these two postulates requires a unification of space and time into the frame-dependent concept of spacetime. Theory of relativity also yields the equivalence of matter and energy, as expressed in the mass-energy equivalence formula. General relativity is the geometrical theory of gravitation published by Albert Einstein in 1915. It unifies special relativity, Newton's law of universal gravitation, and the insight that gravitational acceleration can be described by the curvature of space and time by extending special relativity to include transformations between non-inertial frames. In general relativity, the curvature of space-time is produced by the energy of matter and radiation. Many observations and experiments have confirmed many of the predictions of special and general theory of relativity, including gravitational time dilation, the gravitational redshift of light, signal delay, and gravitational radiation.

Each of these theories has been tested in numerous experiments and proven to be an accurate model of nature within its domain of validity. The fundamental theories form a foundation for the study and research of more specialized topics. Modern physics creates theoretical background for all modern technologies. The parts of physics applied in technical disciplines like mechanical, electrical or civil engineering are called technical physics. This textbook is based on the lectures on technical physics for students of mechanical engineering at the Slovak University of Technology. We hope that it can be useful also for students of other technical disciplines. The first volume, which you hold in your hands, starts with the introduction into the vector algebra. Main part of textbook deals with the mechanics of point particle and the description of gravitational field. Some useful tables and mathematical formulas are included in the appendices.

2 Vector algebra

Physical quantities can be divided into two following groups.

- **Scalars**, which are specified completely by a number and a unit and therefore have only magnitude. Some quantities that are scalars are e.g., mass, energy and time. Scalars can be handled by employing the rules of ordinary algebra.

- **Vectors**, which have both a magnitude and a direction. Examples of such quantities are e.g., velocity, acceleration and force. Vectors obey the special rules of manipulation, which will be explained in this chapter.

Accurate graphical representation of a vector on a diagram can be a simple arrow. The length of the arrow is proportional to the magnitude of the vector and the direction of the arrow is coincident with the direction of the vector. In printing the vector is usually represented by a boldfaced symbol, in handwriting we usually put an arrow above the symbol. If the vector quantity is written in absolute value or without vector notation it means only magnitude of this quantity.

2.1 Components of vector

For practical manipulation, it is very useful to resolve a vector into its components. The resolving of the vector $\boldsymbol{v}$ into its components v_x, v_y, v_z is represented by the equation

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}\,, \tag{2.1}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the unit vectors in the directions of axes x, y, z. The unit vector has the magnitude equal 1 unit, therefore the multiplications of components by unit vectors lead to the vector components, which have the same magnitudes. The graphical method of resolving the vector into its components is shown in Figure 2.1. The tail of the vector is placed at the origin of the coordinate system and the components of the vector are found by drawing perpendicular lines from the head of the vector to the axes x, y, z.

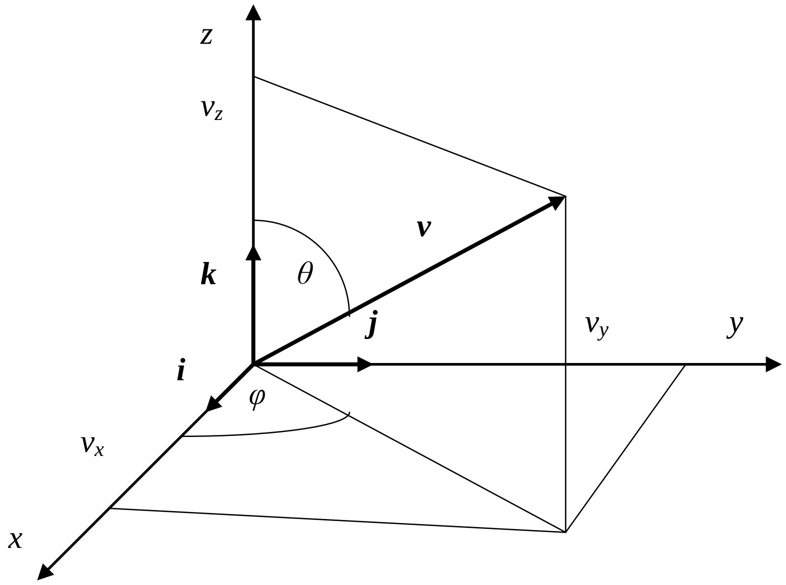

Figure 2.1: The resolving of the vector into its components.

The magnitude of the vector can be calculated using the Pythagoras theorem

$$v = \sqrt{v_x + v_y + v_z}\ . \tag{2.2}$$

If the direction of the vector ***v*** is given by the polar angle θ and the azimuthal angle φ, the components of this vector can by calculated using trigonometric functions

$$v_x = v\ \sin\theta \cos\varphi \tag{2.3}$$

$$v_y = v\ \sin\theta\ \sin\varphi \tag{2.4}$$

$$v_z = v\ \cos\theta\,. \tag{2.5}$$

Sometimes it is useful to represent the direction of the vector ***v*** by the direction cosines defined as:

$$l = \cos\ \alpha = \frac{v_x}{v} \tag{2.6}$$

$$m = \cos\ \beta = \frac{v_y}{v} \tag{2.7}$$

$$n = \cos\ \gamma = \frac{v_z}{v}, \tag{2.8}$$

which obey the relation

$$l^2 + m^2 + n^2 = 1\,. \tag{2.9}$$

2.2 Multiplication of vector by scalar

The multiplication of the vector $\boldsymbol{a}$ by the scalar s is represented by the equation

$$\boldsymbol{v} = s\boldsymbol{a} \,. \tag{2.10}$$

The vector $\boldsymbol{v}$ is the resultant vector, its magnitude is

$$v = sa \,. \tag{2.11}$$

The components of the resultant vector are

$$v_x = sa_x \tag{2.12}$$

$$v_y = sa_y \tag{2.13}$$

$$v_z = sa_z \,. \tag{2.14}$$

The direction of the resultant vector is given by the following rule: *„If the scalar is positive, the direction of the resultant vector is the same as the direction of the original vector. If the scalar is negative, the direction of the resultant vector is opposite to the direction of the original vector.“*

A graphical method of the multiplication of the vector by the scalar is shown in Figure 2.2 and Figure 2.3.

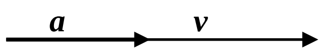

Figure 2.2: The multiplication of the vector by the positive scalar.

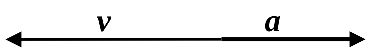

Figure 2.3: The multiplication of the vector by the negative scalar.

2.3 Addition and subtraction of two vectors

The addition of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is represented by the equation

$$\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}\,. \tag{2.15}$$

The vector $\boldsymbol{v}$ is the resultant vector and its components are

$$v_x = a_x + b_x \tag{2.16}$$

$$v_y = a_y + b_y \tag{2.17}$$

$$v_z = a_z + b_z\,. \tag{2.18}$$

The magnitude and the direction of the resultant vector are given on a diagram by the triangle rule: *„Append the vector **b** at the end of vector **a**, the resultant vector **v** is represented by an arrow connecting the tail of the vector **a** with the head of the vector **b**.”*

Graphical method of the addition of two vectors is shown in Figure 2.4.

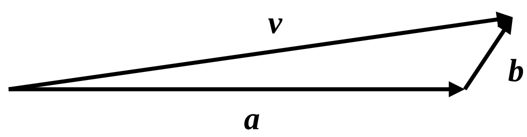

Figure 2.4: The addition of two vectors.

The addition of two vectors obeys the following laws:

- commutative law

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}\,; \tag{2.19}$$

- associative law

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = (\boldsymbol{a} + \boldsymbol{c}) + \boldsymbol{b} = (\boldsymbol{b} + \boldsymbol{c}) + \boldsymbol{a}\,; \tag{2.20}$$

- distributive law for multiplication by a scalar

$$s(\boldsymbol{a} + \boldsymbol{b}) = s\boldsymbol{a} + s\boldsymbol{b}\,. \tag{2.21}$$

The subtraction of two vectors is defined as the addition of the negative vector

$$\boldsymbol{v} = \boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b}) \,, \tag{2.22}$$

where the $-\boldsymbol{b}$ is the negative vector of the vector ***b***. The negative vector ***-b*** has the same magnitude as the vector ***b*** but the direction of the negative vector is opposite.

Graphical method of the subtraction of two vectors is illustrated in Figure 2.5.

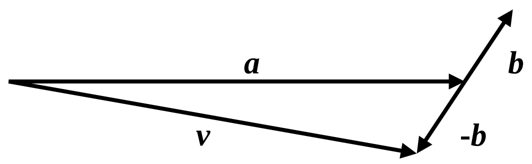

Figure 2.5: The subtraction of two vectors.

2.4 The scalar product of two vectors

The scalar product of two vectors ***a*** and ***b*** is defined as:

$$\boldsymbol{a} \cdot \boldsymbol{b} = s = ab\cos\varphi \,. \tag{2.23}$$

The scalar s is the resultant scalar and φ is the angle between vectors ***a*** and ***b***. This resultant scalar can be calculated using the components of vectors ***a*** and ***b*** by the formula

$$s = a_x b_x + a_y b_y + a_z b_z \,. \tag{2.24}$$

The meaning of the scalar product of two vectors is apparent from Figure 2.6. The scalar product represents the multiplication of magnitude of the vector **a** and the projection of the vector ***b*** into the direction of the vector ***a***.

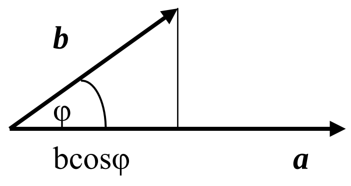

Figure 2.6: The scalar product of two vectors.

The scalar product obeys the following laws:

- commutative law

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}\,; \tag{2.25}$$

- distributive law for multiplication by a scalar

$$s(\boldsymbol{a} \cdot \boldsymbol{b}) = (s\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (s\boldsymbol{b})\,; \tag{2.26}$$

- distributive law for addiction of two vectors

$$\boldsymbol{c} \cdot (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \cdot \boldsymbol{a}) + (\boldsymbol{c} \cdot \boldsymbol{b})\,. \tag{2.27}$$

2.5 The vector product of two vectors

The vector product of two vectors ***a*** and ***b*** is defined as vector ***v*** perpendicular to the plane formed by vectors ***a*** and ***b*** with direction given by the right-hand rule: „*Swing the vector **a** into the vector **b** with the fingers of your right hand. Your thumb shows the direction of their vector product **v**.*“
The magnitude of the resultant vector is

$$\boldsymbol{v} = |\boldsymbol{a} \times \boldsymbol{b}| = ab\sin\varphi\,, \tag{2.28}$$

where φ is the angle between vectors ***a*** and ***b***. Using the components of the vectors ***a*** and ***b*** the resultant vector can be expressed as

$$\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}, \tag{2.29}$$

it means that the components of the resultant vector are

$$v_x = a_y b_z - a_z b_y \tag{2.30}$$

$$v_y = a_x b_z - a_z b_x \tag{2.31}$$

$$v_z = a_x b_y - a_y b_x\,. \tag{2.32}$$

The meaning of the vector product of two vectors is apparent from Figure 2.7. It is the vector with the magnitude equal to the area of a parallelogram created by the vectors ***a*** and ***b***.

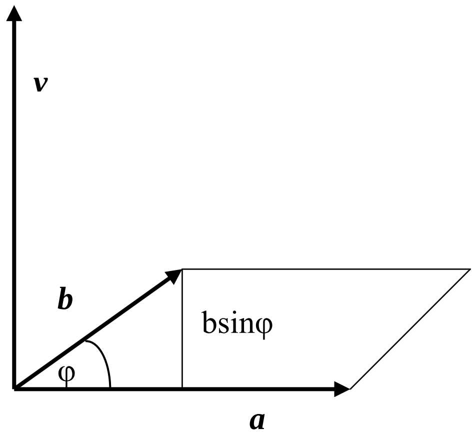

Figure 2.7: The vector product of two vectors.

The vector product obeys the following laws:

- anticommutative law

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \; ; \tag{2.34}$$

- distributive law for multiplication by a scalar

$$s(\boldsymbol{a} \times \boldsymbol{b}) = (s\boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (s\boldsymbol{b}) \; ; \tag{2.35}$$

- distributive law for addiction of two vectors

$$\boldsymbol{c} \times (\boldsymbol{a} + \boldsymbol{b}) = (\boldsymbol{c} \times \boldsymbol{a}) + (\boldsymbol{c} \times \boldsymbol{b}) \; . \tag{2.36}$$

3 Particle kinematics

Kinematics describes the motion using the concepts of space and time, regardless to the causes of the motion. The motion can be described either using mathematical equations or graphs. **Mathematical approach** is better for solving problems because it is more exact. **Graphical methods** provide insight that is more physical.

In this chapter, for the sake of simplification, we will further refer to a motion as being the motion of a single point-like mass particle. In reality, the objects have dimensions and other internal motions may be involved, which we will neglect here.

3.1 One dimensional motion

One-dimensional motion is the motion of the particle along the line, specialy along the *x*- axis. To describe this motion completely we need to know its position coordinate x relative to the chosen origin of the coordinate system at any time, i.e., $x(t)$.

There are several kinds of the motion along *x* with the function $x(t)$ and graphs that describe the motion.

a. ***No motion at all*** - the particle is not moving. Its position is fixed and constant in time x(t)=const . Graphically we can represent this case in so called **position-time graph** shown in Fig. 3.1.

a.

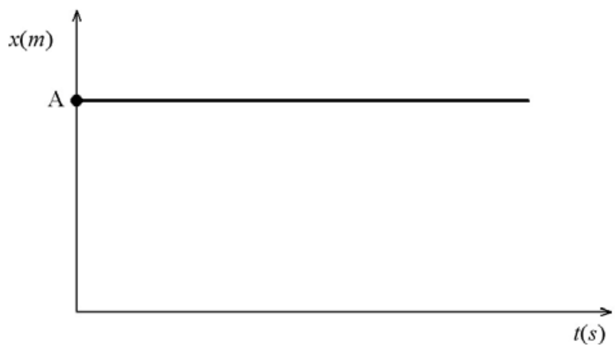

Figure 3.1: The position-time graph of no motion case.

The particle is located at the position given by the coordinate A, which does not change in time . We can express this fact by the equation

$$x(t) = \mathrm{A}\,. \tag{3.1}$$

b. ***Motion at a constant speed*** - we have to remember that speed and velocity are not the same in physics. The velocity is a vector defined by magnitude and direction. The speed is a scalar equal to the magnitude of the velocity vector. The position- time graph is in Fig. 3.2.

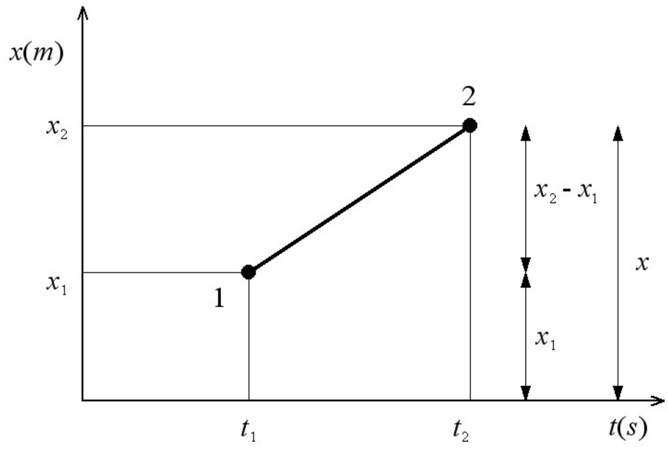

Figure 3.2: The position-time graph of motion at constant speed.

Let us consider the particle moving along x-axis from the position x_1 at the time t_1 to the position x_2 at the time t_2 as is shown in Fig. 3.2. Let us denote the values $x_1 = \mathrm{A}$ and $x_2 = \mathrm{B}t$ where $t = t_2 - t_1$. The expression $x = x_2 - x_1$ is called displacement. Then

$$x_1 = \mathrm{A}\,, \tag{3.2}$$

$$\mathrm{B} = \frac{x_2}{t}\,,$$

and

$$x_2 = \mathrm{B}t\,. \tag{3.3}$$

If

$$x = x_1 + x_2\,, \tag{3.4}$$

then we can rewrite it as:

$$x(t) = \mathrm{A} + \mathrm{B}t\,. \tag{3.5}$$

The last equation is the so called **analytical** expression for a linear one-dimensional motion at a constant speed.

c. ***Accelerated motion*** - is the motion at which the speed of a moving particle changes with time. The analytical expression of this motion is:

$$x(t) = \mathrm{A} + \mathrm{B}t + \mathrm{C}t^2 . \tag{3.6}$$

Physically the displacement, the velocity and the acceleration characterize the motion.

- **Displacement**

  If the particle moves along the *x*-axis from the position x_1 to the position x_2, its displacement is defined as $\Delta x = x_2 - x_1$, see Figure 3.3.

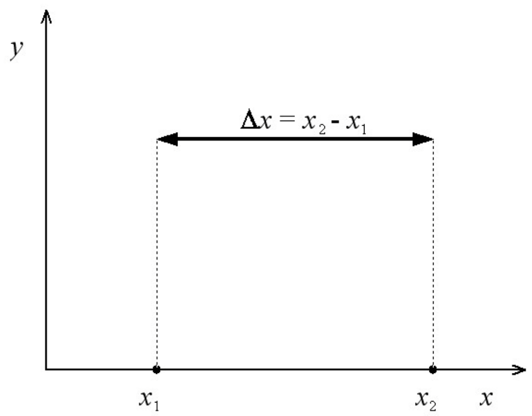

Figure 3.3: The displacement of the particle along the x-axis.

If $\Delta x > 0$ then $x_2 > x_1$ and the particle moves in the positive direction of the *x*-axis. If $\Delta x < 0$ then $x_2 < x_1$ and the particle moves in the opposite direction, i.e., along the axis to the left.

- **Average velocity**

  The ratio of the displacement Δx of the particle and the time interval Δt of the motion is average velocity. The position-time graph of the particle moving from the point A to the point B is shown in Fig. 3.4.

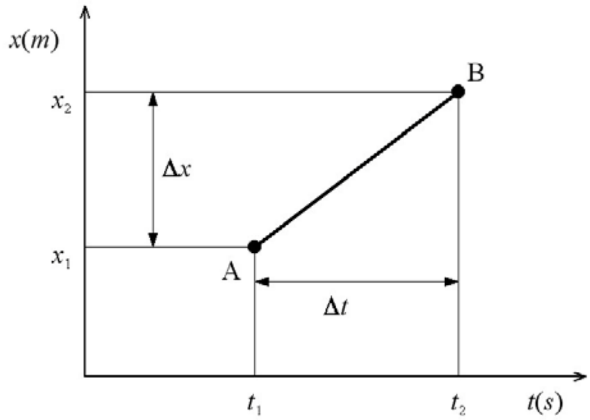

Figure 3.4: The position-time graph of the particle moving from the point A to the point B.

The average velocity of the particle v_{av} is defined as the ratio of its displacement, Δx, and the time interval Δt:

$$v_{\text{av}} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \tag{3.7}$$

The unit of average velocity is 1 meter per second : m s^{-1} in SI units. The abscissa $\overline{\text{AB}}$ is the hypotenuse of the rectangular triangle with the cathetuses Δt and Δx.

- **Instantaneous velocity**

It is defined as a limiting value of the average velocity as Δt approaches zero. Mathematically it is represented by the derivative of x with respect to t:

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t}. \tag{3.8}$$

Instantaneous velocity is the rate of change of position with time and it can be positive, negative or zero.

- **Average acceleration**

The ratio of the change of the velocity and the time interval is average acceleration. The velocity-time graph is shown in Fig. 3.5.

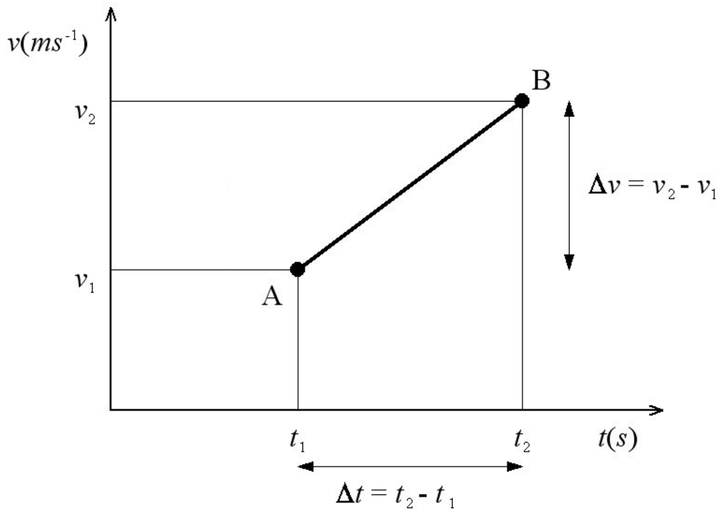

Figure 3.5: The velocity-time graph of motion of the particle moving from the point A to the point B.

The average acceleration is defined as:

$$a_{\mathrm{av}} = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}. \qquad (3.9)$$

The unit of average acceleration is 1 meter per second squared: ms^{-2} in SI units.

- **Instantaneous acceleration**

It is limiting value of the average acceleration as Δt approaches zero. Mathematically it is represented by the first derivative of velocity with respect to time or the second derivative of displacement with respect to time:

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}. \qquad (3.10)$$

Now, let‘s introduce so called **initial conditions**. Let t_1 be zero and the motion starts at the time $t_1 = t_0 = 0$ from the position $x_1 = x_0$ with the speed $v_1 = v_0$. Let in the time $t_2 = t$ the particle has the position $x_2 = x$ and the velocity $v_2 = v$:

$$\begin{array}{ll} t_1 = 0 & t_2 = t \\ x_1 = x_0 & x_2 = x \\ v_1 = v_0 & v_2 = v. \end{array} \qquad (3.11)$$

Graphically we can represent this situation in Figs. 3.6-a and 3.6-b, respectively.

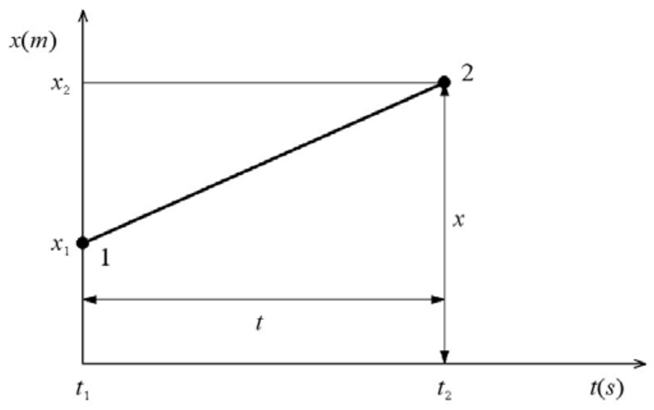

Figure 3.6.a: The position-time graph for motion of the particle from the point 1 to the point 2 taking into account the initial conditions.

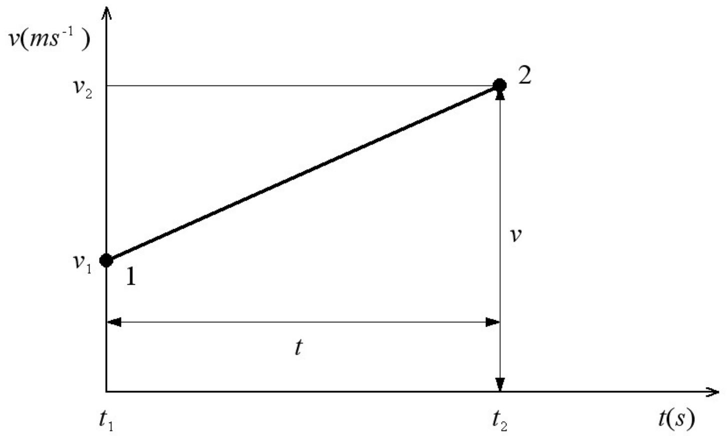

Figure 3.6.b: The velocity-time graph for motion of the particle from the point 1 to the point 2 taking into account the initial conditions.

From the figures it follows:

$$a = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v - v_0}{t}, \tag{3.12}$$

where we used the initial conditions. After some rearrangements we obtain:

$$at = v - v_0$$

$$v = v_0 + at\,. \tag{3.13}$$

To determine the position of the particle at any time t we have to start from the definition of the velocity of the particle:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \tag{3.14}$$

$$v\mathrm{d}t = \mathrm{d}x\,.$$

Therefore, the position x is given by

$$\int_{x_0}^{x} \mathrm{d}x = \int_{0}^{t} v\mathrm{d}t$$

$$\int_{x_0}^{x} \mathrm{d}x = \int_{0}^{t} \left(v_0 + at\right)\mathrm{d}t.$$

If we assume that acceleration of the moving particle is constant we get

$$\left[x\right]_{x_0}^{x} = \left[v_0 t\right]_0^t + \left[\frac{1}{2}at^2\right]_0^t$$

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2\,. \tag{3.15}$$

The last equation is valid only for uniformly accelerated motion.

Summary.

- no motion

$$x\left(t\right) = \mathrm{A} \tag{3.16}$$

$$v\left(t\right) = \frac{\mathrm{d}x}{\mathrm{d}t} = 0\,; \tag{3.17}$$

- motion at constant speed

$$x\left(t\right) = \mathrm{A} + \mathrm{B}t \tag{3.18}$$

$$v\left(t\right) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{A} + \mathrm{B}t\right) = \mathrm{B}\,; \tag{3.19}$$

- uniformly accelerated motion

$$x\left(t\right) = \mathrm{A} + \mathrm{B}t + \mathrm{C}t^2 \tag{3.20}$$

$$v\left(t\right) = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{A} + \mathrm{B}t + \mathrm{C}t^2\right) = \mathrm{B} + 2\mathrm{C}t \tag{3.21}$$

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = 2\mathrm{C}\,. \qquad (3.22)$$

3.2 Motion in two and three dimensions

In two dimensional motion the particle changes both x and y coordinates. Then the position of the particle is given by a **position vector** $\boldsymbol{r}$ drawn from the origin of some reference frame to the particle located in *xy* plane**.** It is lucidly represented in Fig. 3.7.

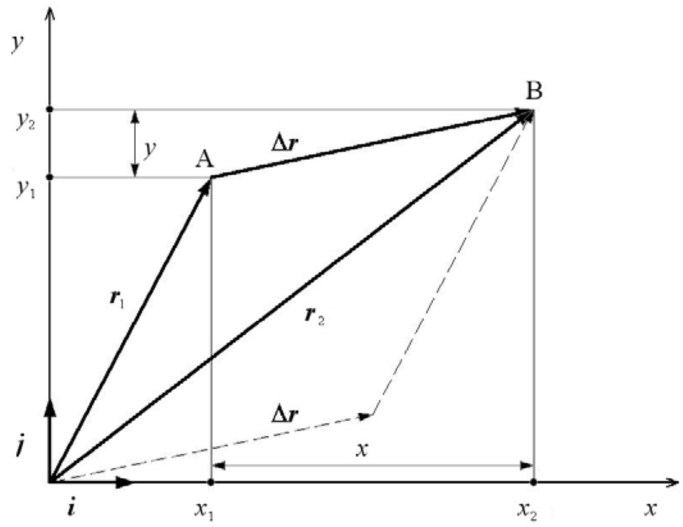

Figure 3.7: The displacement vector $\Delta\boldsymbol{r}$ of the particle moving from the point A to the point B.

Let us consider a particle moving from the point A to the point B. The coordinates of the point A are x_1, y_1 and its position vector is $\boldsymbol{r}_1$. Similarly, the coordinates of the point B are x_2, y_2 and the position vector is $\boldsymbol{r}_2$. The position vector of the motion changes from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ and the displacement of the particle is given by the vector $\Delta\boldsymbol{r}$. The vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ can be resolved to:

$$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} \qquad (3.23)$$

$$\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} \qquad (3.24)$$

for two dimensions and

$$\boldsymbol{r}_1 = x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1\boldsymbol{k} \qquad (3.25)$$

$$\boldsymbol{r}_2 = x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2\boldsymbol{k}$$

for three dimensions. Then the displacement vector $\Delta\boldsymbol{r}$ is:

$$\Delta\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = \left(x_2\boldsymbol{i} + y_2\boldsymbol{j}\right) - \left(x_1\boldsymbol{i} + y_1\boldsymbol{j}\right) = \left(x_2 - x_1\right)\boldsymbol{i} + \left(y_2 - y_1\right)\boldsymbol{j}\,. \tag{3.26}$$

Let us denote

$$x = x_2 - x_1 \tag{3.28}$$

$$y = y_2 - y_1. \tag{3.29}$$

After substitution of the equations (3.28) and (3.29) into (3.27) for $\Delta\boldsymbol{r}$ in two dimensions we obtain

$$\Delta\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} \tag{3.30}$$

and analogously in three dimensions

$$\Delta\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}\,. \tag{3.31}$$

The instantaneous velocity of the particle is defined as:

$$\boldsymbol{v} = \lim_{\Delta t\to 0}\frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}.$$

Substituting the equation 3.31 into the definition of the instantaneous velocity gives

$$\boldsymbol{v} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}\right) = \frac{\mathrm{d}x}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\boldsymbol{k}. \tag{3.32}$$

We know that

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}, \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t}, \qquad v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$$

are the components of the instantaneous velocity. Therefore, for the velocity vector we have

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}\,. \tag{3.33}$$

The velocity is tangent to the path of the particle at every point of the motion as it is evident from Fig. 3.8. As the particle moves from the point A to the point B along some path (see Fig. 3.8), its instantaneous velocity vector changes from v_1 at the time t_1 to v_2 at the time t_2.

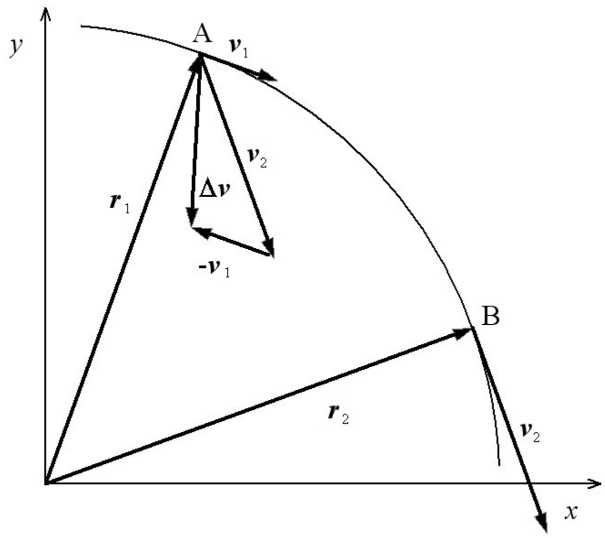

Figure 3.8: The sketch of the velocity vectors in circular motion.

The change of the velocity $\Delta \boldsymbol{v}$ is given by the expression:

$$\Delta \boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1 .$$

Then the average acceleration is:

$$\boldsymbol{a}_{\mathrm{av}} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v}_2 - \boldsymbol{v}_1}{t_2 - t_1} \tag{3.34}$$

and the instantaneous acceleration is the first derivative of the velocity with respect to time:

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}\right)$$

$$\boldsymbol{a} = \frac{\mathrm{d}v_x}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\boldsymbol{k}. \tag{3.35}$$

Because

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}, \qquad a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t}, \qquad a_z = \frac{\mathrm{d}v_z}{\mathrm{d}t}$$

are the components of the acceleration vector, then for the acceleration we can finally write:

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k} . \tag{3.36}$$

3.3 Circular motion

Uniform circular motion

Uniform circular motion is defined by the movement of an object in a circular path with constant linear speed. The magnitude of the velocity remains constant but its direction changes continuously. Examples of such a kind of the motion are the rotation of the satellites around the Earth, the circulation of the Moon around the Earth, a picked point on phonograph record etc. A circular motion of the particle is illustrated in Fig. 3.9.

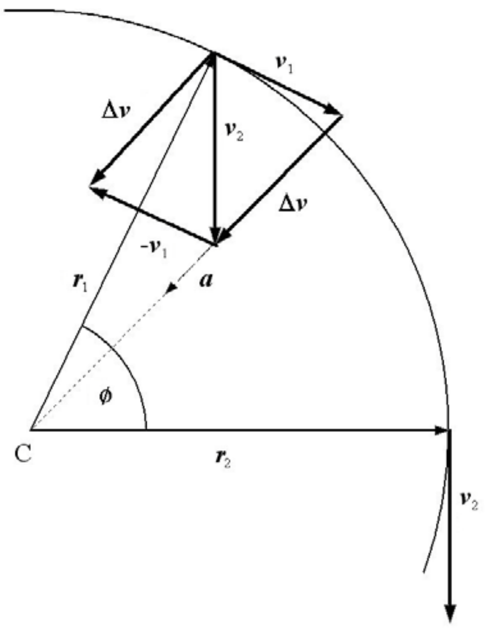

Figure 3.9: Vector analysis of the circular motion of the particle.

In Fig. 3.9 we can see circular motion of the particle. At the time t_1 the particle has the position vector $\boldsymbol{r}_1$ and its velocity $\boldsymbol{v}_1$, at the time t_2 its position vector is $\boldsymbol{r}_2$ and its velocity is $\boldsymbol{v}_2$. During the time interval $\Delta t = t_2 - t_1$ the position of the particle changes about the angle Φ and the velocity of the particle changes about $\Delta \boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$ as it is shown in Figs. 3.9 and 3.10.

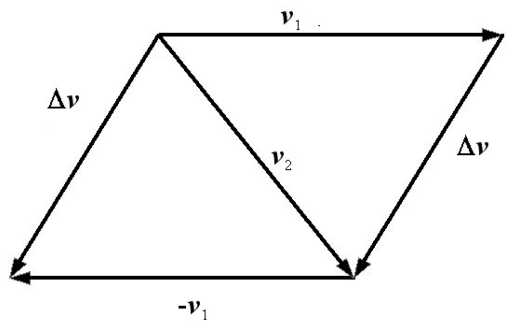

Figure 3.10: The vector analysis of the velocities in circular motion.

The vector $\Delta \boldsymbol{v}$ points toward the centre of the circle (the point C). The velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the tangents to the circle at the points A and B, respectively. Because the motion is uniform, both velocities have the same magnitudes. Then for the absolute values we can write:

$$|\boldsymbol{v}_1| = v \tag{3.37}$$

$$|\boldsymbol{v}_2| = v \tag{3.38}$$

$$|\Delta \boldsymbol{v}| = \Delta v. \tag{3.39}$$

The acceleration is pointed in the same direction as the vector $\Delta \boldsymbol{v}$ that means it is pointing to the centre of the circle:

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\,. \tag{3.40}$$

This acceleration is called the **radial or centripetal acceleration**. $\Delta \boldsymbol{v}$ is a change of the velocity as the particle moves from the point P_1 to the point P_2 as it is illustrated in Fig. 3.11.

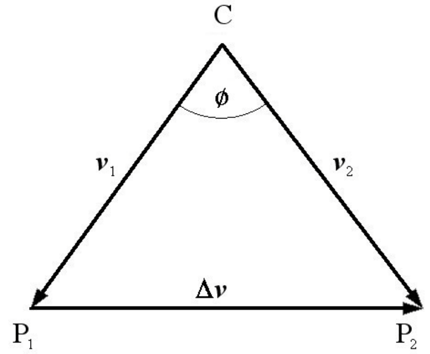

Figure 3.11: The change of the velocity as the particle moves from the point P_1 to the point P_2.

Now we can find the relation between the magnitude of the centripetal acceleration and the magnitude of the instantaneous velocity at any point. Let us denote Δl the path of the particle from the point A to the point B. If Δl approaches zero, the curve can be considered to a straight line. Then for the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ it holds:

$$|\boldsymbol{r}_1| = r \tag{3.41}$$

$$|\boldsymbol{r}_2| = r. \tag{3.42}$$

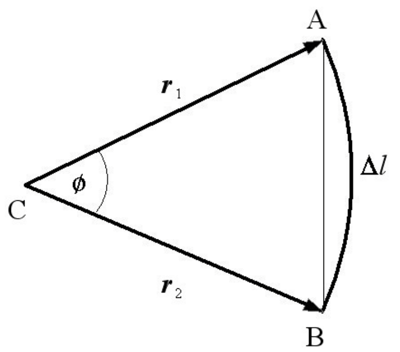

Figure 3.12: The path Δl gone by the particle
from the point A to the point B.

The triangles $\mathrm{P_1P_2C}$ from Fig. 3.11 and ABC from Fig. 3.12 are similar. For the similar triangles it holds:

$$\triangle \mathrm{P_1P_2C} \sim \triangle \mathrm{ABC}$$

$$\frac{\mathrm{P_1P_2}}{\mathrm{P_1C}} = \frac{\mathrm{AB}}{\mathrm{AC}}. \tag{3.43}$$

If we rewrite the last equation in the motion terminology, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta l}{r}.$$

Then Δv is:

$$\Delta v = \Delta l \frac{v}{r} \tag{3.44}$$

and subsequently using (3.44) the radial acceleration is:

$$a_r = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{v}{r} \frac{\Delta l}{\Delta t}. \tag{3.45}$$

From the definition of the velocity we have

$$v = \frac{\Delta l}{\Delta t}. \tag{3.46}$$

We can rewrite the radial acceleration into the form:

$$a_{\mathrm{r}} = \frac{v^2}{r}. \tag{3.47}$$

From this expression it follows that the radial (centripetal) acceleration is directly proportional to the square of the linear velocity and inversely proportional to the radius of the circle Its direction points toward the center of the circle.

Nonuniform circular motion

Nonuniform circular motion is circular motion, when the particle is moving along the circular path where the velocity changes in both the direction and the magnitude. The vectors of the velocity and the acceleration are illustrated in Fig. 3.13.

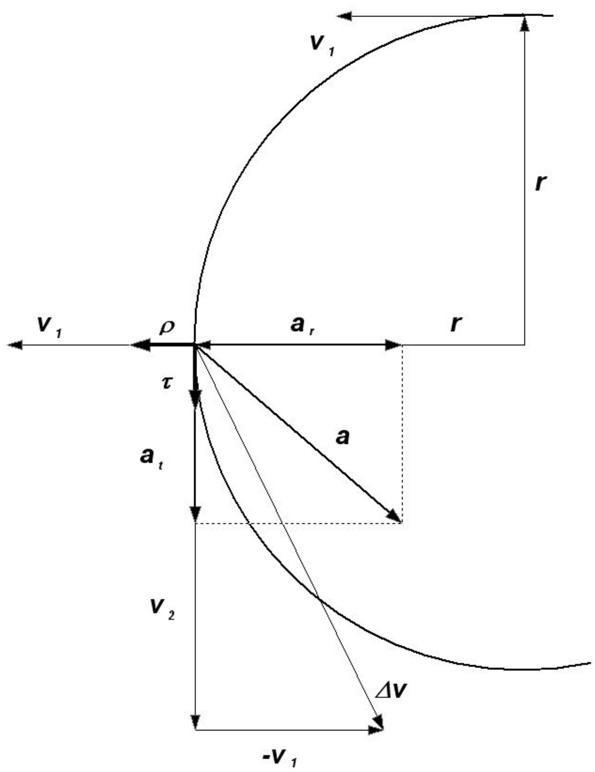

Figure 3.13: The vectors of the velocity and the acceleration
in nonuniform circular motion.

The velocity changes in both the direction and the magnitude so that $|\boldsymbol{v}_1| \neq |\boldsymbol{v}_2|$. The total acceleration vector $\boldsymbol{a}$ changes are both radial and tangential. The direction of the **radial acceleration** changes with the direction of the velocity. We introduce the unit vector $\boldsymbol{\rho}$ directed radially outward along the radius vector

$$\boldsymbol{a}_\mathrm{r} = \frac{v^2}{r}\boldsymbol{\rho} .$$

Now we introduce the unit vector $\boldsymbol{\tau}$ oriented along the tangent to the circle path. Therefore, the **tangential acceleration** can be expressed as:

$$\boldsymbol{a}_\mathrm{t} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} . \tag{3.48}$$

$\boldsymbol{a}_\mathrm{t}$ is perpendicular to $\boldsymbol{a}_\mathrm{r}$ and their directions change continuously in time.

The **total acceleration vector** is given by the sum of its radial and tangential vector components:

$$\boldsymbol{a} = \boldsymbol{a}_\mathrm{r} + \boldsymbol{a}_\mathrm{t} . \tag{3.49}$$

Then

$$\boldsymbol{a} = a_\mathrm{t}\boldsymbol{\tau} + a_\mathrm{r}\boldsymbol{\rho} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{\tau} - \frac{v^2}{r}\boldsymbol{\rho}. \tag{3.50}$$

The minus sign for $\boldsymbol{a}_\mathrm{r}$ indicates that it is always directed radially inward, opposite to the unit vector $\boldsymbol{\rho}$. The magnitude of the total acceleration is:

$$a = \sqrt{a_\mathrm{r}^2 + a_\mathrm{t}^2} . \tag{3.51}$$

Angular variables of circular motion

For description of circular motion it is often useful to use angular variables and not linear ones. In Fig. 3.14 we illustrate circular motion of the particle.

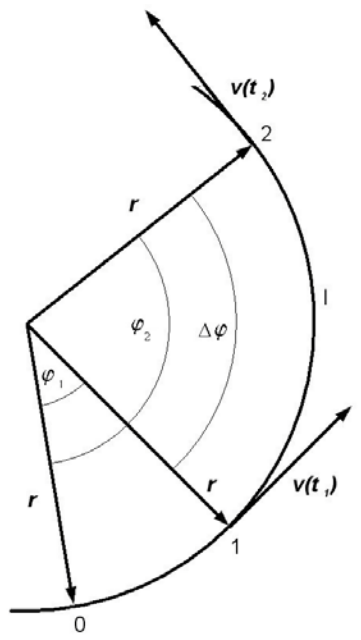

Fig. 3.14: Circular motion of the particle.

Let us take a reference point 0. At the time t_1 the particle is in the position 1 and its velocity is $\boldsymbol{v}_1$. At the time t_2 the particle moves to the point 2 and the velocity of the particle is $\boldsymbol{v}_2$. The angles φ_1 and φ_2 express the changes of the position vectors of the particle in the points 1 and 2 (relatively to any initial position 0), respectively. The angular displacement of the particle between the point 1 and the point 2 is:

$$\Delta\varphi = \varphi_2 - \varphi_1 . \tag{3.52}$$

Let l represents the length of the circle section, along which the particle passes during the time period $\Delta t = t_2 - t_1$. It is advantageous to express the angles in radians. **1 radian** is an angle subtended to the arc whose length l is equal to the radius:

$$\varphi = \frac{l}{r} \tag{3.53}$$

$$l = 2\pi r$$

$$\varphi = \frac{2\pi r}{r} = 2\pi . \tag{3.54}$$

The **average angular velocity** is the ratio of the change of the angular displacement $\Delta\varphi$ in the time interval Δt.

$$\omega_{\mathrm{av}} = \frac{\varphi_2 - \varphi_1}{t_2 - t_1} = \frac{\Delta\varphi}{\Delta t} . \tag{3.55}$$

The **instantaneous angular velocity** is a limiting value of the average angular velocity as the time interval Δt approaches zero or the first derivative of φ with respect to *t*:

$$\omega = \lim_{\Delta t \to 0} \omega_{av} = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt}. \tag{3.56}$$

The angular velocity has units of rad/s, or s^{-1}.

The **average angular acceleration** is defined as the ratio of the change in the angular velocity ω to the time interval Δt:

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}. \tag{3.57}$$

The **instantaneous angular acceleration** is defined as the limit of the average angular acceleration as Δt approaches zero or the derivative of the angular velocity ω with respect to time t:

$$\alpha = \lim_{\Delta t \to 0} \alpha_{av} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}. \tag{3.58}$$

The unit is rad/s^2 or s^{-2}.

Relations between linear and angular variables in scalar form

It is evident that there exist relations between the linear and the angular variables. The angular equivalent of the position is the angle φ

Let us take the equation (3.8) for the linear velocity

$$v = \frac{dl}{dt},$$

and the equation (3.53) for the length of the arc

$$\varphi = \frac{l}{r}.$$

After the rearrangement we obtain

$$l = r\varphi. \tag{3.59}$$

Substituting the equation (3.59) in the equation (3.8) for the linear velocity we have

$$v = \frac{d}{dt}(r\varphi) = r\frac{d\varphi}{dt}.$$

From the equation (3.56) we know that

$$\omega = \frac{d\varphi}{dt}$$

and finally for the velocity and angular velocity relation we obtain:

$$v = r\omega. \tag{3.60}$$

From this equation it follows that the linear velocity is directly proportional to the angular velocity. Similarly for the linear tangential acceleration using (3.10) and (3.60) we can derive

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\omega) = r\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{3.61}$$

Taking into account equation (3.58)

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{3.62}$$

and substituting (3.62) into (3.61) gives us the expression for linear tangential acceleration with respect to the radial tangential acceleration:

$$a_t = r\alpha. \tag{3.63}$$

According to (3.47) and (3.60) for the linear radial acceleration we can write

$$a_\mathrm{r} = \frac{v^2}{r}$$

$$v = \omega r$$

$$a_\mathrm{r} = \frac{\omega^2 r^2}{r} = r\omega^2. \tag{3.64}$$

As you can see the tangential acceleration is proportional to the square of the angular velocity. Both the linear velocity and the linear radial acceleration are transformed into their angular equivalents through the linear variable radius r.

The equation (3.56) is related to the scalar description of the angular velocity. Generally, the velocity is the vector and has not only the magnitude but also the direction. The magnitude of vector $\boldsymbol{\omega}$ is ω and tells us how rapidly the particle is rotating. The direction of $\boldsymbol{\omega}$ is perpendicular to the plane of the circle and is determined by a right-hand rule. The vectors $\boldsymbol{r}$, $\boldsymbol{v}$ and $\boldsymbol{\omega}$ are related through cross product $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. The direction of the vector $\boldsymbol{v}$ is perpendicular to the plane created by the vectors $\boldsymbol{r}$ and $\boldsymbol{\omega}$.

The same rule is valid for the angular acceleration. We can write

$$\boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{\omega} \times \boldsymbol{r})}{\mathrm{d}t} = \boldsymbol{\omega} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r} = \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\alpha} \times \boldsymbol{r}.$$

For the description of the circular motion it is useful to introduce time variables, a period and a frequency. The **period** is time required for one revolution of the particle around the circle. The circumference of the circle is $2\pi r$ and the velocity is v. The time of one turn is

$$T = \frac{2\pi r}{v} \quad (3.65)$$

$$v = \omega r$$

$$T = \frac{2\pi r}{\omega r} = \frac{2\pi}{\omega}. \quad (3.66)$$

Now the unit is 1 second, s.

The **frequency** is defined by the number of revolutions of the particle per one second

$$f = \frac{1}{T} = \frac{\omega}{2\pi}. \quad (3.67)$$

More frequently used expression is

$$\omega = 2\pi f \, . \quad (3.68)$$

3.4 Classification of motions

The motion of a mass point (or a particle) in three-dimensional space can be classified as follows:

1. **Space motion** is the most general motion, which has three degrees of freedom.

2. **Surface motion** is planar motion in the plane or on the surface, e.g. projection at an angle, charge movement in homogeneous electrical field, the motion in gravitational field etc. It has two degrees of freedom.

3. **Motion along the curve** has one degree of freedom and we can briefly repeat the main equations of motion:

A ***Uniform linear motion*** - the particle has a constant velocity and no acceleration. The initial condition is $x = x_0$ at the time $t = 0$ and x_0 is equal to the constant c_1 (see Fig. 3.15). Under these conditions proceeding from general equations for the velocity it holds:

$$\boldsymbol{v} = v_x \boldsymbol{i} = \text{constant}$$

and for the acceleration

$$a_x = 0 \, .$$

We can determine the position as:

$$x = \int v_x \mathrm{d}t = v_x t + c$$
$$c = x_{/t=0} = x_0$$
$$x = x_0 + v_x t.$$

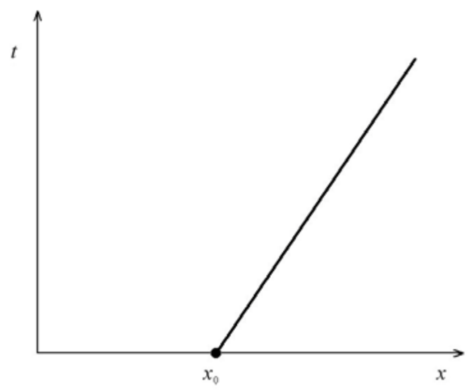

Figure 3.15: The position-time graph of the linear motion.

B ***Uniformly accelerated motion*** - the acceleration of the particle is constant and has the same direction as the velocity. The constant c_1 is x-coordinate of the velocity at the time $t = 0$ ($\boldsymbol{v}(0)$). The constant c_2 is x-coordinate of the particle at the time $t = 0$. It results from initial conditions. Therefore, the acceleration along the x-axis is given by

$$\boldsymbol{a} = a_x \boldsymbol{i} = \text{constant}$$

and the velocity

$$\boldsymbol{v}(0) = v_{x0} \boldsymbol{i}\ .$$

Its magnitude at any time t is

$$v_x = \int a_x dt = a_x t + c_1$$
$$c_1 = v_{/t=0} = v_{x0}$$
$$v_x = v_{x0} + a_x t.$$

Finally, the position of the particle at the time t is

$$x = \int v_x \mathrm{d}t = \int \left(v_{x_0} + a_x t \right) \mathrm{d}t$$
$$x = v_{x_0} t + \frac{a_x t^2}{2} + c_2$$
$$\mathrm{c}_2 = x_{/t=0} = x_0$$
$$x = x_0 + v_{x_0} t + \frac{1}{2} a_x t^2.$$

where x_0 is the position of the particle at the time $t = 0$, v_{x0} is the velocity of the particle at the time $t = 0$.

C ***Nonuniformly accelerated one dimensional motion*** is the motion when the acceleration is a continuous vector with the constant direction. The acceleration depends on the time

$$\boldsymbol{a} = a_x(t)\boldsymbol{i} ,$$

where $\boldsymbol{i}$ is the unit vector along x-axis.

The initial velocity $\boldsymbol{v}(0)$ has the same direction as the acceleration

$$\boldsymbol{v}(0) = v_{x0}\boldsymbol{i} .$$

Using the definition of the linear acceleration

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}$$

we have

$$\mathrm{d}v_x = a_x \mathrm{d}t .$$

Therefore, the linear velocity along x-axis at any time t is

$$v_x(t) = \int a_x(t)\mathrm{d}t + c_1.$$

Starting from the definition of the velocity

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t}$$

we can express the position of the particle at any time as

$$x(t) = \int v_x(t)\mathrm{d}t + c_2.$$

Circular motion is the motion with a constant radius of the curvature and the fixed center of the curvature S. Taking into account general equations for circular motion we can write the angular velocity as a function of t

$$\omega(t) = \int \alpha(t)\mathrm{d}t + c_1$$

and for the angle

$$\varphi(t) = \int \omega(t)\mathrm{d}t + c_2 .$$

D ***Uniform circular motion*** - the particle has a constant angular velocity and no angular acceleration. The initial condition for the angle is $\varphi = \varphi_0$ at the time $t = 0$ and φ_0 is equal to the constant c_1 (see Fig. 3.15). Due to these conditions for the angular velocity it holds:

$$\omega = \text{constant}.$$

Since

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

we have

$$\alpha = 0.$$

The angle at any time t is then

$$\varphi = \int \omega \mathrm{d}t = \omega t + c_1$$

$$c_1 = \varphi_{/t=0} = \varphi_0$$

$$\varphi = \varphi_0 + \omega t.$$

For the tangential acceleration we have

$$a_\mathrm{t} = \frac{\mathrm{d}v}{\mathrm{d}t} = r\frac{\mathrm{d}\omega}{\mathrm{d}t}$$

and if ω is constant, then $a_\mathrm{t} = 0$. The total acceleration equals radial acceleration

$$a = a_\mathrm{r}$$

or in the vector form

$$\boldsymbol{a} = a_r \boldsymbol{i}.$$

E ***Uniformly accelerated circular motion*** **–** the angular acceleration is constant

$$\alpha = \text{constant}.$$

From the definition of the angular velocity we can derive

$$\omega = \int \alpha \mathrm{d}t = \alpha t + c_1,$$

where the integration constant c_1 can be determined from the initial condition

$$c_1 = \omega_{/t=0} = \omega_0$$

$$\omega = \omega_0 + \alpha t.$$

Using the same procedure for the angular displacement we can derive

$$\varphi = \int \omega \mathrm{d}t = \int \left(\omega_0 + \alpha t\right) \mathrm{d}t = \omega_0 t + \frac{\alpha t^2}{2} + c_2$$

$$c_2 = \varphi_{/t=0} = \varphi_0$$

$$\varphi = \varphi_0 + \omega_0 t + \frac{1}{2}\alpha t^2 .$$

F ***Motion in homogeneous force field -*** Homogeneous force field gives the same acceleration to a particle in each point of the space, e.g. gravitational field. The illustrational example of such a motion is projectile motion in gravitational field of the Earth.

3.5 Projectile motion

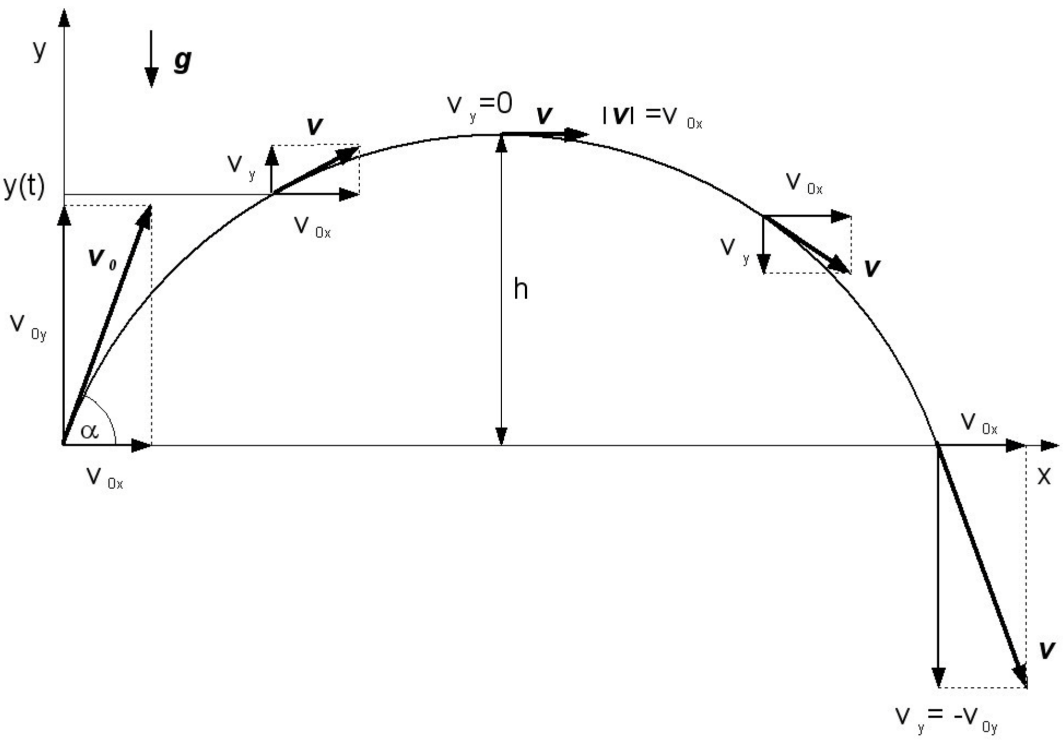

Figure 3.16: The projectile motion.

The projectile moves in two dimensions with constant acceleration $\boldsymbol{a} = \boldsymbol{g}$. Two rules hold:

1 $\boldsymbol{g}$ is a constant and directed downward;

2 the resistance of the air is negligible (for the sake of simplification).

Let us assume that the initial velocity of the projectile at the origin of the reference frame makes an angle α with the horizontal. The acceleration along x-axis is equal to zero and along y-axis is $-g$. Both x and y coordinates of the acceleration can be written as

$$a_x = 0$$
$$a_y = -g.$$

For the velocity at the time $t = 0$ we have:

$$v_{0x} = v_0 \cos\alpha \tag{3.69}$$

$$v_{0y} = v_0 \sin\alpha.. \tag{3.70}$$

From basic equations of the motion for x **components** of the position and the velocity vectors we can write:

$$x = x_0 + v_{0_x} t + \frac{1}{2} a_x t^2 \tag{3.71}$$

$$v_x = v_0 x + a_x t. \tag{3.72}$$

Taking into account the equation (3.69) and the fact that $x_0 = 0$, $a_x = 0$ we can rewrite the equations (3.71) and (3.72) as follows:

$$x = v_{0_x} t = v_0 t \cos\alpha \tag{3.73}$$

$$v_x = v_{0_x} = v_0 \cos\alpha. \tag{3.74}$$

A horizontal component of the velocity remains constant and equal to v_0.

In y **-direction** the acceleration a_y is equal $-g$. Substituting this one to the basic equations of motion

$$y = y_0 + v_{0_y} t + \frac{1}{2} a_y t^2 \tag{3.75}$$

$$v_y = v_{0_y} + a_y y \tag{3.76}$$

we obtain

$$y = v_{0_y} t - \frac{1}{2} g t^2 = v_0 t \sin\alpha - \frac{1}{2} g t^2 \tag{3.77}$$

$$v_y = v_0 \sin\alpha - gt. \tag{3.78}$$

Note that $y_0 = 0$ because the projectile starts from the origin.

Let us solve the equation (3.73) for the variable t and substitute this expression into the equation (3.77). We find

$$y = \tan\alpha\ x - \left(\frac{1}{2}\frac{g}{v_0^2\cos^2\alpha}\right)x^2, \tag{3.79}$$

which is valid for the angle in the range $0 < \alpha < \frac{\pi}{2}$. It is the equation of a **parabola.** The projectile follows this trajectory.

We can find the **maximal height** h of the projectile at the peak of its trajectory when $v_y = 0$ at the time t_1. Substituting the condition $v_y = 0$ into the equation (3.78) we have

$$0 = v_0\sin\alpha - gt$$

or

$$v_0\sin\alpha = gt\,. \tag{3.80}$$

From the equation (3.80) we express time and we take $t = t_1$:

$$t = \frac{v_0}{\mathrm{g}}\sin\alpha$$

$$t_1 = \frac{v_0}{\mathrm{g}}\sin\alpha\,. \tag{3.81}$$

Substituting the expression (3.81) into (3.75) we obtain the maximum height of the projectile trajectory, $y_{\max}$

$$y_{\max} = y_0 + v_{0_y}t_1 + \frac{1}{2}at_1^2\,.$$

Because

$$y_0 = 0\,,$$

then

$$y_{\max} = h = v_0\sin\alpha\ t_1 - \frac{1}{2}gt_1^2$$

$$h = v_0\sin\alpha\ \frac{v_0}{\mathrm{g}}\sin\alpha - \frac{1}{2}g\frac{v_0^2}{\mathrm{g}^2}\sin^2\alpha$$

$$h = \frac{v_0^2\sin^2\alpha}{g} - \frac{1}{2}\frac{v_0^2}{\mathrm{g}}\sin^2\alpha$$

$$h = \frac{1}{2}\frac{v_0^2\sin^2\alpha}{g}\,.$$

Time t_2 is duration time of the whole motion, i.e., from $y = 0$ through $y = h$ to $y = 0$. The horizontal distance $x_{\max}$ traveled by the projectile during the time t_2 is the double distance traveled up to the point of the peak during the time t_1

$$t_2 = 2t_1 . \tag{3.82}$$

Using the equations (3.69), (3.73) and (3.82) for a **maximum flying range** of the projectile we finally obtain:

$$x = t_2 v_0 \cos\alpha$$

$$x_{\max} = 2\frac{v_0}{g}\sin\alpha \; v_0\cos\alpha$$

$$x_{\max} = \frac{2v_0^2}{\mathrm{g}} \sin\alpha \; \cos\alpha \; .$$

4 Particle dynamics

In kinematics we describe the motion of particles based on the definition of displacement, velocity and acceleration. Now we shall describe the change in motion of particles using the concepts of force and mass. A particular body interacts with the surrounding bodies, e.g. an apple falls from the tree due to the gravity, a skier slides down the hill under effect of the gravity, friction with snow and air resistance, the billiard ball - gravity of another balls and the table, the Halley's comet round trip through the solar system due to the Sun gravity.

At the beginning let us consider a body as a particle. **Dynamics** is the science, which describes the change of the particle under the effect of the force and the mass. The procedure to solve the dynamics systems is as follows:

a. We introduce the force $\boldsymbol{F}$ and we define it in the terms of the acceleration.
b. We develop a procedure for assigning a mass m to a body so that we could understand the fact that different bodies have different accelerations in the same environment.
c. We try to find the ways of calculating the forces that act on the bodies from the properties of the body and of its environment - the **force laws**.

The **force** gives us the relations of an environment to body motion. The laws of the mechanics taken together with the force laws constitute the laws of dynamics.

4.1 Newton's laws

Newton's first law

Problem of motion and its reasons was a central topic of "Natural philosophy" that was a basis of today's physics. Sir Isaacs Newton was the principal architect of classical mechanics. His three laws of motion were presented in "Philosophiae Naturalis Principia Mathematica" called simply "Principia" published in 1686.

We must find a way to free a body from all influences of its environment or from all forces. The motion would be like if its external forces were truly zero. No external force is needed to keep a body moving with the constant velocity. It is difficult to find a situation when no external force acts on a body. The force of gravity will act on an object on or near the Earth and resistive process as friction or air resistance oppose the motion on the ground or in the air.

There is no distinction between a body on which no external forces act and the body on which the sum or the resultant of all the resultant forces is zero. The resultant of all forces acting on a body is called the **net force**.

Let us consider that the net force acting on a body is zero. If the body is at rest it will remain at rest. If the body is moving with the constant velocity it will continue to do so:

$$\sum_{i=1}^{n} \boldsymbol{F}_i = 0 . \tag{4.1}$$

This law is called Newton's first law. If the net force acting on a body is zero, then it is possible to find a set of reference frames in which the body has no acceleration.

The tendency of a body to remain at rest or in an uniform linear motion is called **inertia**. The first Newton's law is called the **law of inertia**. The reference frames, to which this law is applied, are the **inertial frames**. Observers in different reference frames moving with the constant velocity relative to each other have the same values of the acceleration. There is not just one frame in which the acceleration happens to be zero. We have a body at rest in the frame and no net forces acting on it. If the body does not remain at rest, the frame is not the inertial frame. Similarly, we have a body in motion at the constant velocity (subject under no net force). If its velocity changes, the frame is not the inertial frame.

Obviously we apply the laws of classical mechanics from the point of view of an observer in the inertial frame. There exist the exceptions like, e.g. the accelerating car, the orbiting satellite, the merry-go-round etc.

There is no distinction in the first law between a body at rest and one moving with the constant velocity. Both are natural if the net force acting on the body is zero. It is **natural state of body's motion.**

Newton's second law

Many experiments showed that the greater the mass of an object, the smaller its acceleration will be for a given force. The acceleration produced by a given force is inversely proportional to the mass being accelerated. We can express this fact as follows:

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m} \tag{4.2}$$

or

$$\boldsymbol{F} = m\boldsymbol{a} . \tag{4.3}$$

The law given by equation (4.3) is called Newton's second law. We can express it in component form as:

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \\ F_z &= ma_z \ , \end{aligned} \tag{4.4}$$

where F_x, F_y, F_z and a_x, a_y, a_z are the components of acting force and acceleration, respectively. Unit of the force in SI system is 1 Newton - $1\ \mathrm{N} = \mathrm{kg}\cdot\mathrm{m}\cdot\mathrm{s}^{-2}$. If $\boldsymbol{F}$ is the resultant force or the net force then the equations (4.3) and (4.4) can be transformed to:

$$\sum_{\mathrm{i}=1}^{\mathrm{n}} \boldsymbol{F}_{\mathrm{i}} = m\boldsymbol{a} \tag{4.5}$$

or

$$\begin{aligned} \sum F_x &= ma_x \\ \sum F_y &= ma_y \\ \sum F_z &= ma_z \ . \end{aligned} \tag{4.6}$$

Newton's third law

Let us consider two bodies, A and B in the fig. 4.1. The body A exerts a force $\boldsymbol{F}_{\mathrm{BA}}$ on the body B and the body B exerts a force $\boldsymbol{F}_{\mathrm{AB}}$ on the body A . These forces are called **attractive forces**.

Figure 4.1: The forces of interaction between two particles.

This situation can be expressed by the **third Newton's law**. To every action there is an equal and opposite oriented reaction. In other words, when two bodies exert the mutual forces on each other, two forces are always equal in the magnitude and opposite in the direction. A mathematical expression of the third Newton's law is:

$$\boldsymbol{F}_{\mathrm{AB}} = -\boldsymbol{F}_{\mathrm{BA}} \ . \tag{4.7}$$

We can see that the forces always occur in pairs. The isolated force cannot exist.

Some applications of Newton’s laws

In what follows we give several examples of the application of the Newton’s laws:

1. The satellite is orbiting around the Earth. The force of the action of the Earth on the satellite is $\boldsymbol{F}_{\mathrm{ES}}$. It has the same magnitude but the opposite direction relative to the force acting on the Earth by the satellite is $\boldsymbol{F}_{\mathrm{SE}}$.

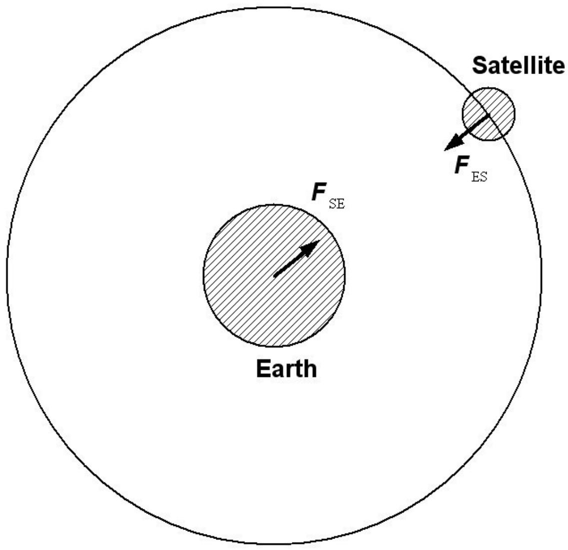

Figure 4.2: The forces of interaction between the Earth and the satellite.

2. Fig. 4.3.a shows a block of mass m hung by three strings. We consider the knot at the junction of the three strings to be “the body”.

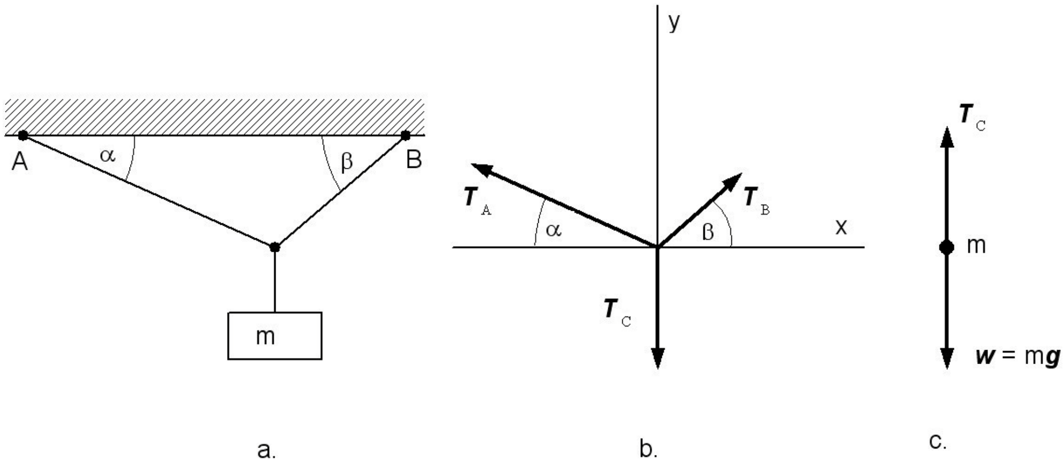

Figure 4.3: a) A block of mass hangs from three strings A, B and C, b) free-body diagram of the knot that joins the strings, c) the free-body diagram of the block.

Fig. 4.3.b shows the free-body diagram of the knot, which remains at rest under the action of three forces $\boldsymbol{T}_{\mathrm{A}}$, $\boldsymbol{T}_{\mathrm{B}}$ and $\boldsymbol{T}_{\mathrm{C}}$ called the tensions of the strings. We assume that, like the string, the knot is massless, so its weight does not appear in the diagram. Choosing the x and y axes as shown in Fig. 4.3.b we can resolve the forces into their x and y components. The acceleration components are zero because the system is at rest. We can write for

x component: $$\sum F_x = T_{\mathrm{A}x} + T_{\mathrm{B}x} = ma_x = 0 \tag{4.8}$$

y component: $$\sum F_y = T_{\mathrm{A}y} + T_{\mathrm{B}y} + T_{\mathrm{C}y} = ma_y = 0\,. \tag{4.9}$$

From Fig. 4.3.b we can see that:

$$\begin{aligned} T_{\mathrm{A}x} &= -T_{\mathrm{A}}\cos\alpha \\ T_{\mathrm{A}y} &= T_{\mathrm{A}}\sin\alpha \\ T_{\mathrm{B}x} &= T_{\mathrm{B}}\cos\beta \\ T_{\mathrm{B}y} &= T_{\mathrm{B}}\sin\beta \\ T_{\mathrm{C}x} &= 0 \\ T_{\mathrm{C}y} &= -T_{\mathrm{C}}\ . \end{aligned} \tag{4.10}$$

Now we examine the free-body diagram of the mass in Fig. 4.3.c:

$$\begin{aligned} T_{\mathrm{C}y} - mg &= ma_y = 0 \\ T_{\mathrm{C}} &= mg\ . \end{aligned} \tag{4.11}$$

3. A block of mass m is held in place by a string on a frictionless plane inclined at angle θ (see Fig. 4.4.a). We have to find the tension on the string and the normal force exerted on the block by the plane.

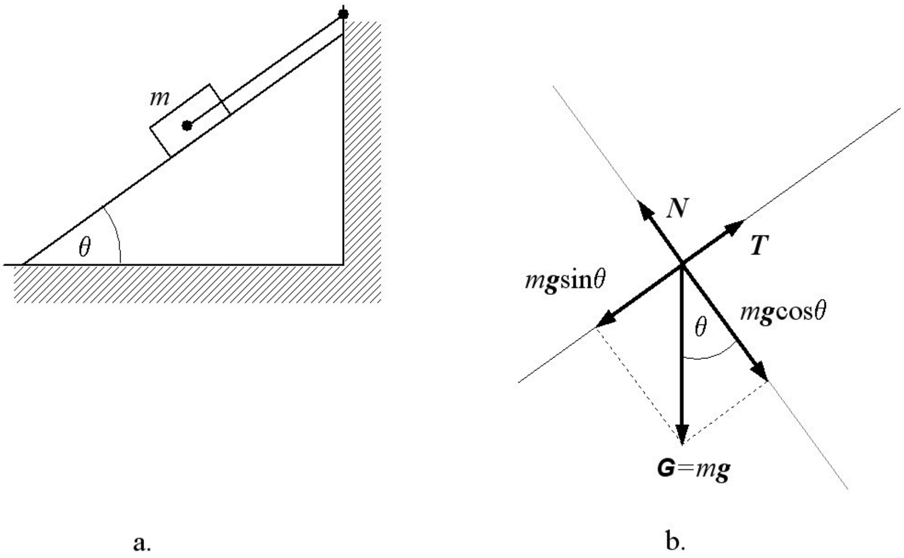

Figure 4.4: a) The block of mass on the inclined plane and b) the free-body diagram of the block of mass.

The free-body diagram of the block is shown in Fig. 4.4.b. The block is acted on by the normal force $\boldsymbol{N}$, its weight $\boldsymbol{W} = m\boldsymbol{g}$ and the tension $\boldsymbol{T}$ of the string. We choose a coordinate system with the x axis along the plane and the y axis perpendicular to it. With this choice, two of the forces ($\boldsymbol{T}$ and $\boldsymbol{N}$) are already resolved into their components. In this case there is no acceleration and the sum of these forces must be zero. The weight is resolved into its x ($-mg\ \sin\theta$) and y ($-mg\ \cos\theta$) components. The force equations are as follows:

x component: $$\sum F_x = T - mg\ \sin\theta = ma_x = 0\,, \tag{4.12}$$

y component: $$\sum F_y = N - mg\ \cos\theta = ma_y = 0\,. \tag{4.13}$$

Solving these equations we obtain:

$$T = mg\ \sin\theta, \tag{4.14}$$

$$N = mg\ \cos\theta\ . \tag{4.15}$$

4. Consider two unequal masses connected by a string that passes over an ideal pulley (whose mass is negligible and whose axle rotates with negligible friction), as shown in Fig. 4.5.a. This arrangement is known as **Atwood's machine**. Let m_1 be less than m_2. We want to find the tension in the string and the acceleration of the masses.

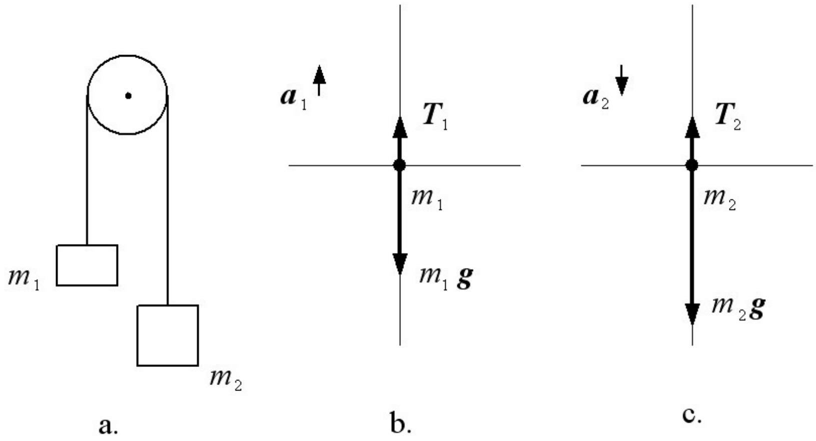

Figure 4.5: a) The Atwood's machine b) the free-body diagram of m_1 and c) m_2.

Because we anticipate the masses to have only the vertical accelerations, we choose the positive y direction to be the direction of the motion for each mass. Only y components need to be considered. The free-body diagrams are shown in Figs. 4.5.b and c. Then the equations of the motion are as follows:

block 1: $$\sum F_y = T_1 - m_1 g = m_1 a_1 \ , \tag{4.16}$$

block 2: $$\sum F_y = m_2 g - T_2 = m_2 a_2 \ , \tag{4.17}$$

where a_1 and a_2 are the accelerations of m_1 and m_2, respectively. Because the string is massless and does not stretch, and the pulley is massless and frictionless, then $T_1 = T_2 = T$ ($|\boldsymbol{T}_1| = |\boldsymbol{T}_2| = T$) and $a_1 = a_2 = a$ ($|\boldsymbol{a}_1| = |\boldsymbol{a}_2| = a$). Substituting these relations and solving (4.16) and (4.17) simultaneously, we find:

$$a = \frac{m_2 - m_1}{m_2 + m_1} g \ , \tag{4.18}$$

$$T = \frac{2m_1 m_2}{m_1 + m_2} g \ . \tag{4.19}$$

In special case when $m_1 = m_2 = m$, the acceleration $a = 0$ and the tension $T = m_1 g = m_2 g = mg$ what is typical for balanced situation. In another special case when $m_2 >> m_1$, then $a = g$ and $T = 2m_1 g$.

4.2 The center of mass

The center of mass of the system of the particles

Above we have mentioned the motion of a single particle approximated to an ideal one. The system of the particles or real bodies is exposed to a complex motion, not only a linear or circular one. Generally, the motion of the real body can be replaced by the motion of its **center of mass**.

In Fig. 4.6 we can see the system of two bodies with masses m_1 and m_2 at the positions x_1 and x_2. The mass of the system is concentrated in the fictional point CM with coordinate x_{CM} .

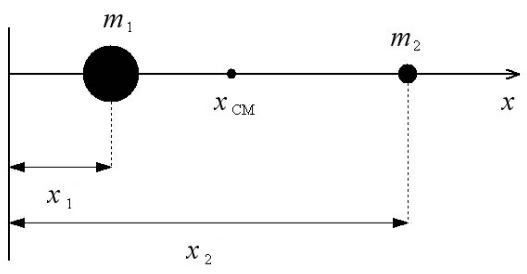

Figure 4.6: The center of mass of the system consisting of two particles.

Then the x coordinate of the center of mass of the system of two particles in one dimension is:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{\sum_{i=1}^{2} x_i m_i}{\sum_{i=1}^{2} m_i} \ .$$

Similarly, we can derive the coordinate of the center of mass of the system of n particles in three dimensions. We must take into account that the total mass of the system of n particles is the sum of masses $m_1, m_2, \ldots, m_n$ of all particles

$$M = m_1 + m_2 + \ldots + m_n = \sum_{i}^{n} m_i \ . \qquad (4.20)$$

Then

$$x_{CM} = \frac{x_1 m_1 + x_2 m_2 + \ldots + x_n m_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} x_i m_i . \tag{4.21}$$

For the velocity of the center of the mass in one dimension it holds:

$$v_{CM,x} = \frac{1}{M}\left(m_1 v_{1,x} + m_2 v_{2,x} + \ldots + m_n v_{n,x}\right) = \frac{1}{M}\sum_{i=1}^{n} m_i v_{i,x} \; . \tag{4.22}$$

We can extend the center of mass conception to three dimensions:

$$\begin{aligned}
x_{CM} &= \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} x_i m_i}{M} = \frac{1}{M}\sum_{i=1}^{n} x_i m_i \; , \\
y_{CM} &= \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} y_i m_i}{M} = \frac{1}{M}\sum_{i=1}^{n} y_i m_i \; , \\
z_{CM} &= \frac{\sum_{i=1}^{n} z_i m_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} z_i m_i}{M} = \frac{1}{M}\sum_{i=1}^{n} z_i m_i \; .
\end{aligned} \tag{4.23}$$

Now we will use the general form of the position vector of the center of mass ($\boldsymbol{r}_{CM} = x_{CM}\boldsymbol{i} + y_{CM}\boldsymbol{j} + z_{CM}\boldsymbol{k}$). Hence these three equations can be written as a single expression giving the position vector of the center of mass:

$$\boldsymbol{r}_{CM} = \frac{1}{M}\left(m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2 + \ldots + m_n \boldsymbol{r}_n\right) = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{r}_i \; . \tag{4.24}$$

To determine the velocity of the center of mass we can take the first derivative of the previous equation:

$$\boldsymbol{v}_{CM} = \frac{d\boldsymbol{r}_{CM}}{dt} = \frac{1}{M}\left(m_1 \frac{d\boldsymbol{r}_1}{dt} + m_2 \frac{d\boldsymbol{r}_2}{dt} + \ldots + m_n \frac{d\boldsymbol{r}_n}{dt}\right) \tag{4.25}$$

or

$$\boldsymbol{v}_{CM} = \frac{1}{M}\left(m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 + \ldots + m_n \boldsymbol{v}_n\right) = \frac{1}{M}\sum_{i=1}^{n} m_i \boldsymbol{v}_i \; . \tag{4.26}$$

Differentiating (4.26) once more gives a relation for the acceleration of the center of mass:

$$\boldsymbol{a}_{CM} = \frac{d\boldsymbol{v}_{CM}}{dt} = \frac{1}{M}\left(m_1 \frac{d\boldsymbol{v}_1}{dt} + m_2 \frac{d\boldsymbol{v}_2}{dt} + \ldots + m_n \frac{d\boldsymbol{v}_n}{dt}\right) \tag{4.27}$$

or

$$\boldsymbol{a}_{\mathrm{CM}} = \frac{1}{M}\left(m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + \dots m_{\mathrm{n}}\boldsymbol{a}_{\mathrm{n}}\right) = \frac{1}{M}\sum_{\mathrm{i}=1}^{\mathrm{n}} m_{\mathrm{i}}\boldsymbol{a}_{\mathrm{i}} \ . \tag{4.28}$$

We can rewrite the equation (4.28) as:

$$M\boldsymbol{a}_{\mathrm{CM}} = m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + \dots + m_{\mathrm{n}}\boldsymbol{a}_{\mathrm{n}} \ . \tag{4.29}$$

Using the second Newton's law we obtain:

$$M\boldsymbol{a}_{\mathrm{CM}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \dots + \boldsymbol{F}_{\mathrm{n}} \, .$$

From this equation it follows that the total force acting on a system of n particles is equal to the total mass M of this system times the acceleration $\boldsymbol{a}_{\mathrm{CM}}$ of the center of mass of the system.

The particles can interact among themselves. The forces of the interaction are called the **internal forces**. The forces that originate beyond the system under consideration are the **external forces.** We can rewrite the upper equation in the form:

$$\sum \boldsymbol{F}_{\mathrm{ext}} = M\boldsymbol{a}_{\mathrm{CM}} \tag{4.30}$$

or in the component form:

$$\begin{aligned} \sum F_{\mathrm{ext},\,x} &= Ma_{\mathrm{CM},\,x} \\ \sum F_{\mathrm{ext},\,y} &= Ma_{\mathrm{CM},\,y} \\ \sum F_{\mathrm{ext},\,z} &= Ma_{\mathrm{CM},\,z} \ . \end{aligned} \tag{4.31}$$

The center of mass of a system moves like an imaginary particle, which is acted upon by the net external force.

4.3 Linear momentum

Linear momentum of the single particle is a vector $\boldsymbol{p}$ defined as the product of its mass and its velocity $\boldsymbol{v}$:

$$\boldsymbol{p} = m\boldsymbol{v} \, . \tag{4.32}$$

These two vectors have the same direction. The momentum has unit of $\mathrm{kg\ m\ s^{-1}}$. Newton expressed the second law of motion in the terms of momentum:

The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force.

In symbolic form we can write:

$$\sum \boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \ , \tag{4.33}$$

where $\sum \boldsymbol{F}$ represents the resultant force acting on the particle. The proof of equivalence of $\boldsymbol{F} = m\boldsymbol{a}$ and $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$ is as follows:

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{v}\right) = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a}\ . \tag{4.34}$$

Suppose we have the system of n particles with masses $m_1, m_2, \ldots, m_n$ and constant total mass M $\left(M = \sum_{i=1}^{n} m_i\right)$. The particles may interact with each other. Each particle has a certain value of the velocity and the linear momentum. The total momentum of the system $\boldsymbol{P}$ is the sum of the momenta of all particles:

$$\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \ldots + \boldsymbol{p}_n = m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 + \ldots + m_n\boldsymbol{v}_n = \sum_{i=1}^{n} m_i\boldsymbol{v}_i = M\boldsymbol{v}_{CM}\ . \tag{4.35}$$

Comparing this equation with (4.29) we obtain:

$$\boldsymbol{P} = M\boldsymbol{v}_{CM}\ . \tag{4.36}$$

The total linear momentum of a system of the particles is equal to the product of the system and the total mass of the system and the velocity.

After differentiation of the last equation and taking into consideration the constant total mass M we obtain:

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = M\frac{\mathrm{d}\boldsymbol{v}_{CM}}{\mathrm{d}t} = M\boldsymbol{a}_{CM}\ . \tag{4.37}$$

Comparing this equation with (4.30) we can rewrite the second Newton's law:

$$\sum \boldsymbol{F}_{ext} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\,. \tag{4.38}$$

If the net external force is zero we can say about **isolated system**. Then

$$\sum \boldsymbol{F}_{ext} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = 0 \tag{4.39}$$

and hence

$$\boldsymbol{P} = \text{constant}\ . \tag{4.40}$$

This expression is called the **law of conservation of linear momentum.**

When the net external force acting on a system is zero, the total linear momentum of the system remains constant.

4.4 Work

Work done by a constant force

Consider a block of mass that undergoes a displacement along a straight line under the action of a constant force $\boldsymbol{F}$. Let us analyse some possible situations that can occur. For the sake of simplification we shall assume that the block is homogeneous and its mass m is concentrated in its center of mass. In this case we can replace the block by a particle.

A . Motion takes place in straight line in the direction of the force. The **work** W done by the constant force on the particle of mass is defined as the product of the magnitude F of the force $\boldsymbol{F}$ and the magnitude s of the displacement along the acting force:

$$W = Fs . \tag{4.41}$$

B . Generally, a constant force makes an angle φ with the displacement. The work done by the force acting on the particle is the product of the **component** of the force in the direction of the displacement $(F \cos\varphi)$ and the magnitude of the displacement $\boldsymbol{s}$ as is shown in Fig. 4.7.

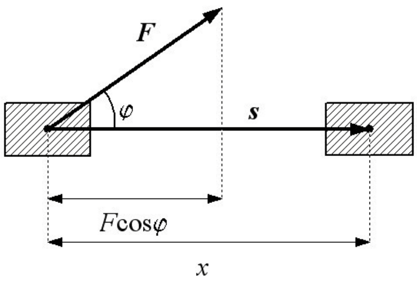

Figure 4.7: The projection of the force $\boldsymbol{F}$ acting on the body into the direction of the displacement $\boldsymbol{s}$.

Therefore, the work done by this force is given by $W = F\cos\varphi \, s$.

C . Other forces may also act on the particle. The total work done on the particle is the sum of all values of the work done by all separate forces. The net force can be found and then the total work, the **net work**, done by this net force can be calculated.

D . Another possible case is if the angle between the force acting on the particle and the displacement is zero $\left(\varphi = 0^o\right)$. This situation was mentioned in point A. and it is represented in Fig. 4.8.

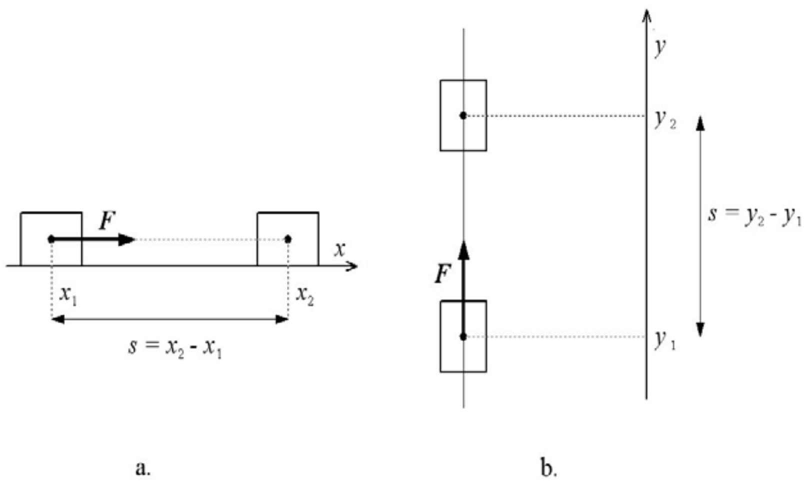

Figure 4.8: The displacement of the body under the force with the reciprocal zero angle θ.

The force acting on the particle has component only in the direction of motion. This direction can be, for instance, in the direction of x-axis (Fig. 4.8-a) or y-axis (Fig. 4.8-b). The work done by the force is in this case

$$W = Fs ,$$

because $\cos 0^o = 1$.

E . If the force has no component in the direction of motion, ***F*** does **no work** on the body. The representation of this situation is in Fig. 4.9.

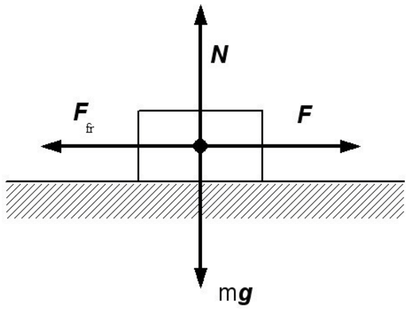

Figure 4.9: The displacement and the force analysis of the body under the force with the reciprocal right angle θ.

The force $\boldsymbol{N}$ is the reaction of surface on the body and it is called the **normal force**. The force $m\boldsymbol{g}$ is the **gravity-weight force.** The force $\boldsymbol{F}_{\mathrm{fr}}$ is the **friction force**. Neither $\boldsymbol{N}$ nor $m\boldsymbol{g}$ do work, because they are perpendicular to the force $\boldsymbol{F}$ and $\cos 90^o = 0$. The only force that does the work is the force of friction $\boldsymbol{F}_{\mathrm{fr}}$ and it acts against $\boldsymbol{F}$ and the velocity of body motion.

Another example is shown in Fig. 4.10. A body is attached to a cord and revolves in a horizontal circle. The tension $\boldsymbol{T}$ on the cord does not work on the body because it has no component in the direction of the displacement. The velocity is tangent to the circle and therefore it is perpendicular to the tension $(\boldsymbol{T} \perp \boldsymbol{v})$.

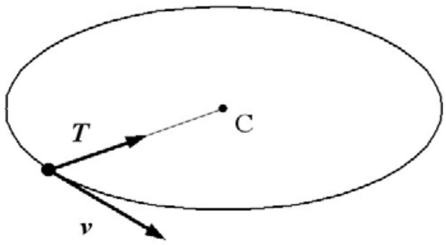

Figure 4.10: The body is attached to a cord and revolving in the horizontal circle.

F . For the scalar equation we can write:

$$W = F\cos\varphi\, s. \tag{4.42}$$

There must be a component of $\boldsymbol{F}$ in the direction of $\boldsymbol{s}$. The projection of it is shown in Fig. 4.11.

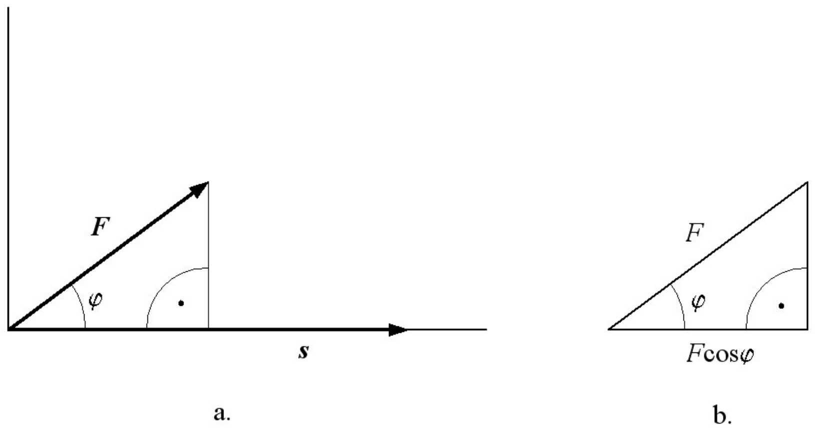

Figure 4.11: The projection of the force $\boldsymbol{F}$ acting on the body in the direction of the displacement $\boldsymbol{s}$.

But because of scalar multiplication properties, we can rewrite this equation also as:

$$W = F\ s\cos\varphi. \tag{4.43}$$

This equation represents the projection of the displacement $\boldsymbol{s}$ in the direction of the force $\boldsymbol{F}$ (Fig. 4.12).

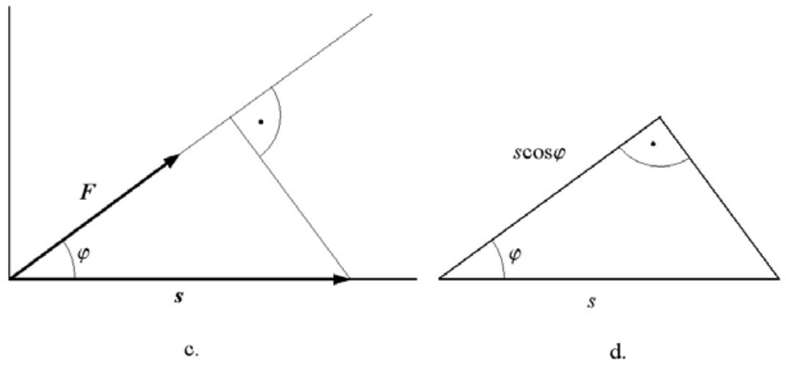

Figure 4.12: The projection of the displacement $\boldsymbol{s}$ of the body in the direction of the force $\boldsymbol{F}$.

G. The work is the scalar quantity and can be positive or negative. If the force has the opposite direction as the displacement $\left(\varphi = 180^o\right)$, the work is negative $\left(\cos 180^o = -1\right)$:

$$W = Fs\cos\alpha = Fs\cos 180^o = Fs(-1) = -Fs. \tag{4.44}$$

If the component of the force is in the same direction as the displacement, the work done by the applied force is positive ($\varphi = 0$)

Our definition of the work may not be the same as a physical point of view. For illustration we can mention a person holding heavy weight in the air. He may be working hard in physiological sense, but as far as physics, his work is zero because no displacement occurs.

Unit of the work is 1 **Joule** (abbreviation J)**.**

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = m\boldsymbol{a} \cdot \boldsymbol{s} \,,$$

and hence for the unit we obtain:

$$\mathrm{J} = \mathrm{Nm} = \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{2} \cdot \mathrm{m} = \mathrm{kg} \cdot \mathrm{m}^{2} \cdot \mathrm{s}^{-2} \,.$$

Work done by a variable force

Let us consider the work done by the force that is not constant. For simplification we will consider one-dimensional case. The force $\boldsymbol{F}$ acts on a body in the direction x and varies in the magnitude with x according to a function $F(x)$. The body moves under the action of this force from an initial position x_{i} to a final position x_{f}. The dependence of the function $F(x)$ on x is illustrated on the plot $\boldsymbol{F}$ versus x in Fig. 4.13 .

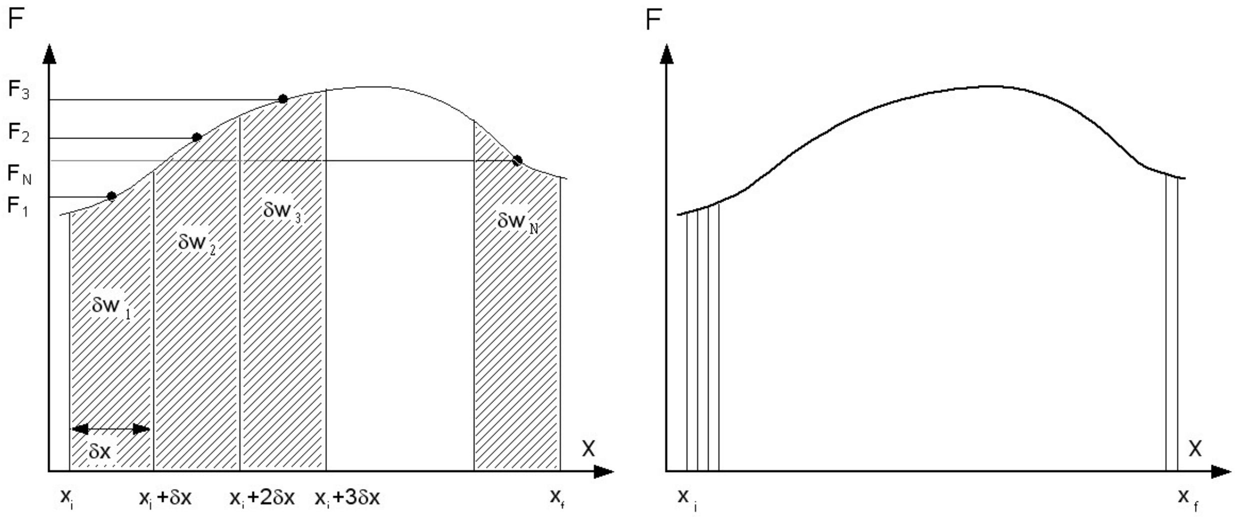

Figure 4.13: The dependence of the $F(x)$ on x with large width of δx a.) and with δx approaching zero b.).

We divide the interval from x_{i} to x_{f} into N small intervals of equal widths δx (Fig. 4.13.a). The first interval can be approximated to a rectangle with sides δx and F_1, where F_1 is almost constant force and δx is the displacement from x_{i} to $x_{\mathrm{i}} + \delta x$. The area of this rectangle is the small work δW_1 for a small displacement δx as:

$$\delta W_1 = F_1 \delta x . \tag{4.45}$$

In the second rectangle there is the displacement δx from $x_{\mathrm{i}} + \delta x$ to $x_{\mathrm{i}} + 2\delta x$ and $F(x)$ is nearly constant value F_2. Similarly we can write all other work elements up to $\boldsymbol{N}$ -th one:

$$\begin{aligned}
\delta W_1 &= F_1 \delta x \\
\delta W_2 &= F_2 \delta x \\
&\vdots \\
\delta W_{\mathrm{N}} &= F_{\mathrm{N}} \delta x \ .
\end{aligned} \tag{4.46}$$

In the interval from x_{i} to x_{f} the total work W done by $F(x)$ can be expressed as a sum of all small work elements:

$$\begin{aligned}
W &= \delta W_1 + \delta W_2 + \ldots + \delta W_{\mathrm{N}} = \\
&= F_1 \delta x + F_2 \delta x + \ldots + F_{\mathrm{N}} \delta x
\end{aligned} \tag{4.47}$$

or

$$W = \sum_{\mathrm{n}=1}^{\mathrm{N}} F_{\mathrm{n}} \delta x \, . \tag{4.48}$$

If the width of the intervals δx is infinitesimally small the force F_{n} in each interval is equal to constant. We obtain exact value of the work done by the force F :

$$W = \lim_{\delta x \to 0} \sum_{\mathrm{n}=1}^{\mathrm{N}} F_{\mathrm{n}} \delta x \, . \tag{4.49}$$

The limit expression can be replaced by the definite integral within the limits from x_{i} to x_{f} :

$$W = \int_{x_{\mathrm{i}}}^{x_{\mathrm{f}}} F(x) \mathrm{d}x \, . \tag{4.50}$$

This definite integral is equal to the area under the $F(x)$ versus x curve between the values x_{i} and x_{f} .

4.5 The power

Now we are interested in the fact how rapidly the work is done. The rate at which the work is done is called **power**. The **average power** $\overline{P}$ delivered by an agent that exerts a particular force on a body is the total work done by that force on the body divided by the total time interval:

$$\overline{P} = \frac{\Delta W}{\Delta t} \, . \tag{4.51}$$

The **instantaneous power** P delivered by an agent is the small amount of work done in the infinitesimal time interval $\mathrm{d}t$:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \, . \tag{4.52}$$

Using the definition of $\mathrm{d}W$ we can rewrite the power in the form:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v} \,. \tag{4.53}$$

If the force acting on the body and the velocity of the body are parallel, the equation (4.53) can be arranged as follows:

$$P = \boldsymbol{F} \cdot \boldsymbol{v} = Fv \cos 0^{o} = Fv \,. \tag{4.54}$$

From this equation it follows that the power is positive. If the force acting on the body and the velocity of the body are antiparallel then P is:

$$P = \boldsymbol{F} \cdot \boldsymbol{v} = Fv \cos 180^{o} = -Fv \tag{4.55}$$

and the power is negative.

If the power is constant in time then $P = \overline{P}$. Then we can write:

$$P = \frac{W}{t} \tag{4.56}$$

or

$$W = Pt \,. \tag{4.57}$$

The SI unit is 1 **watt** with abbreviation 1 W . Then

$$P = \frac{W}{t} = \frac{\boldsymbol{F} \cdot \boldsymbol{s}}{t} = \frac{m\boldsymbol{a} \cdot \boldsymbol{s}}{t} \tag{4.58}$$

and hence in units we get:

$$\mathrm{W = J\,s^{-1} = N\,ms^{-1} = kg\,m\,s^{-2}m\,s^{-1} = kg\,m^{2}s^{-3}} \,.$$

4.6 Kinetic energy

We shall consider how the work acts on the motion of a particle. Let us replace the influence of all acting forces by the net force $\boldsymbol{F}_{\mathrm{net}}$. The net work W_{net} is the work done by all forces that act on the particle. Two approaches to find the total work W are possible.

A . We can find the net force as a sum of all acting forces:

$$\boldsymbol{F}_{\mathrm{net}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_{\mathrm{N}} \tag{4.59}$$

and then treat this force as a single net force. Then we use the formula for the work in one dimensional case as:

$$W = \int_{x_\mathrm{i}}^{x_\mathrm{f}} F\left(x\right)\mathrm{d}x . \tag{4.60}$$

For more than one dimension we have

$$W = \int_\mathrm{i}^\mathrm{f} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_\mathrm{i}^\mathrm{f} F \cos\varphi \, \mathrm{d}s . \tag{4.61}$$

B. We can calculate the work done by each of all forces that act on the particle, where i is the beginning and f is the end of the path of the particle under the action of the force $\boldsymbol{F}$. The angle φ is the angle between the force $\boldsymbol{F}$ and the displacement $\boldsymbol{s}$ of the particle. Therefore,

$$\begin{aligned} W_1 &= \int \boldsymbol{F}_1 \cdot \mathrm{d}\boldsymbol{s} \\ W_2 &= \int \boldsymbol{F}_2 \cdot \mathrm{d}\boldsymbol{s} \\ &\vdots \\ W_\mathrm{N} &= \int \boldsymbol{F}_\mathrm{N} \cdot \mathrm{d}\boldsymbol{s} . \end{aligned} \tag{4.62}$$

The total work is given by:

$$W = W_1 + W_2 + \ldots + W_\mathrm{N} . \tag{4.63}$$

Let us examine the influence of the constant force on the particle moving in one dimension from x_i to x_f:

$$W_\mathrm{net} = F_\mathrm{net}\left(x_\mathrm{f} - x_\mathrm{i}\right) = ma\left(x_\mathrm{f} - x_\mathrm{i}\right). \tag{4.64}$$

To determine the final velocity of the particle we must solve the quadratic equation for the variable t:

$$x_\mathrm{f} = x_\mathrm{i} + v_\mathrm{i}t + \frac{1}{2}at^2 . \tag{4.65}$$

Solving this quadratic equation for the value of t we have:

$$t = \frac{-v_\mathrm{i} + \sqrt{v_\mathrm{i}^2 + 2a\left(x_\mathrm{f} - x_\mathrm{i}\right)}}{a} . \tag{4.66}$$

For the final velocity we use the following equation (derived in Chapter 3):

$$v_\mathrm{f} = v_\mathrm{i} + at . \tag{4.67}$$

From this equation the time is

$$t = \frac{v_\mathrm{f} - v_\mathrm{i}}{a} . \tag{4.68}$$

We can set the times t from equations (4.66) and (4.68) to be equal and then express the final velocity:

$$v_\mathrm{f}^2 = v_\mathrm{i}^2 + 2a\left(x_\mathrm{f} - x_\mathrm{i}\right). \tag{4.69}$$

Hence

$$\begin{aligned} \frac{v_\mathrm{f}^2 - v_\mathrm{i}^2}{2} &= a\left(x_\mathrm{f} - x_\mathrm{i}\right) \quad / \cdot m \\ \frac{mv_\mathrm{f}^2}{2} - \frac{mv_\mathrm{i}^2}{2} &= ma\left(x_\mathrm{f} - x_\mathrm{i}\right) = W_\mathrm{net}. \end{aligned} \tag{4.70}$$

Generally, the expression for the kinetic energy is:

$$K = \frac{1}{2}mv^2. \tag{4.71}$$

Using the equation (4.70) for the net work we obtain:

$$W_\mathrm{net} = K_\mathrm{f} - K_\mathrm{i} = \Delta K. \tag{4.72}$$

The last equation is called the **work-energy theorem.**

The net work done by the forces acting on a particle is equal to the change in the kinetic energy of the particle.

The general proof of the work-energy theorem is as follows:

$$F_\mathrm{net} = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{4.73}$$

$$F_\mathrm{net} = mv\frac{\mathrm{d}v}{\mathrm{d}x} \tag{4.74}$$

$$W_\mathrm{net} = \int F_\mathrm{net}\mathrm{d}x = \int mv\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = \int mv\ \mathrm{d}v \tag{4.75}$$

$$W_\mathrm{net} = \int_{v_\mathrm{i}}^{v_\mathrm{f}} mv\ \mathrm{d}v = m\int_{v_\mathrm{i}}^{v_\mathrm{f}} v\ \mathrm{d}v \tag{4.76}$$

$$W_\mathrm{net} = \frac{1}{2}mv_\mathrm{f}^2 - \frac{1}{2}mv_\mathrm{i}^2. \tag{4.77}$$

4.7 Conservative and nonconservative forces

It is useful to divide the forces into two kinds: conservative and nonconservative ones. The **conservative force** depends only on the position. The work acting on a particle moving between any two points depends only on the initial and final position of these points. If the work done by the force depends on the path length, this force is called the **nonconservative force**.

To illustrate behavior of the conservative systems we consider the one-dimensional motion of a particle acted on by three separate forces.

A . The spring force

In Fig. 4.14 we can see a block of mass attached to a spring of the force constant k. For the force proportional to the displacement acting on the spring it holds:

$$F = -\mathrm{k}x\,. \tag{4.78}$$

Minus sign in this expression means that the force is always directed toward the equilibrium position. The work done by this force is:

$$W = \int F\left(x\right)\mathrm{d}x = \int\left(-\mathrm{k}x\right)\mathrm{d}x = -\frac{1}{2}\mathrm{k}x^2\,. \tag{4.79}$$

Assume that the block slides without friction across a horizontal surface. Five situations can occur.

1. In the situation 1 an external agent compresses the spring and the block is displaced to the distance $x = +d$ from its relaxation position at $x = 0$.
2. The external agent is suddenly removed at time $t = 0$. The spring begins to do work on the block. The block moves from the position $x = +d$ to $x = 0$. The work of the transition from the situation 1 to the situation 2 is then:
$$W = \left[-\frac{1}{2}\mathrm{k}x^2\right]_d^0 = \frac{1}{2}\mathrm{k}d^2\,. \tag{4.80}$$
3. Passing through the position $x = 0$ the sign of the force reverses. The spring acts to slow down the block doing the negative work on it. The position of the block changes from the position $x = 0$ to $x = -d$. During the transition from the situation 2 to 3 the force does work:
$$W = \left[-\frac{\mathrm{k}x^2}{2}\right]_0^{-d} = -\frac{1}{2}\mathrm{k}d^2\,. \tag{4.81}$$
4. When the block returns to its relaxation position from $x = -d$ to $x = 0$ the spring force does work:
$$W = \left[-\frac{1}{2}\mathrm{k}x^2\right]_{-d}^0 = \frac{1}{2}\mathrm{k}d^2\,. \tag{4.82}$$
5. The last situation shows the block being pushed back to its initial position from $x = 0$ to $x = +d$. The work done by the force is:
$$W = \left[-\frac{\mathrm{k}x^2}{2}\right]_0^{d} = -\frac{1}{2}\mathrm{k}d^2\,. \tag{4.83}$$

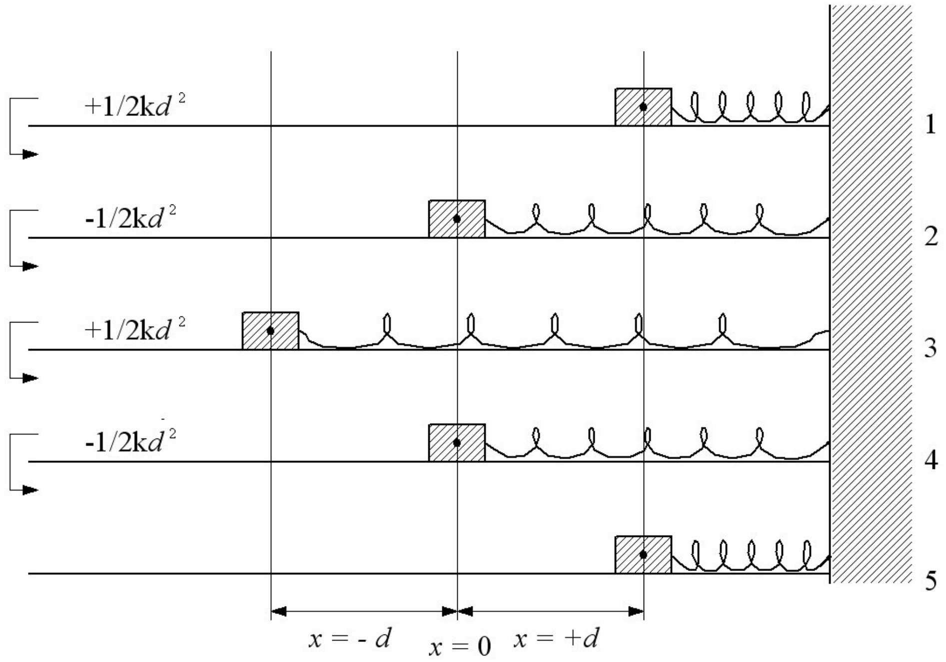

Figure 4.14:Movement of the block of mass attached to the spring.

Individual contributions of work in all stages summed together give us the **total work**. This total work that gets the block into its beginning position in **complete cycle is zero**.

B . The force of gravity.

In Fig. 4.15 we have a ball acted on by the Earth’s gravity. We have three situations:

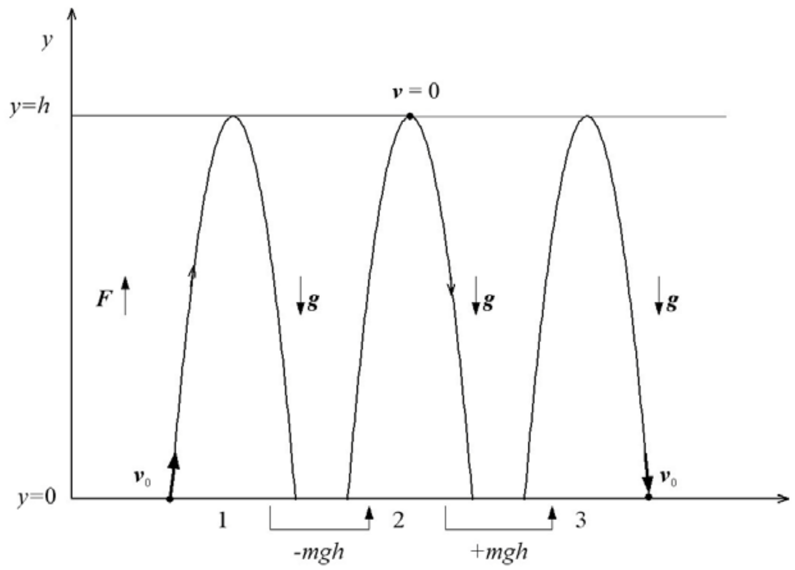

Figure 4.15: The ball is thrown upward (1), stopping in the air (2)
and falling down (3).

1. The ball is ejected upward by an external agent that gives initial speed $\boldsymbol{v}_0$ and an initial kinetic energy $\frac{1}{2}m\boldsymbol{v}_0^2$. The first trajectory of the ball is from $y=0$ up to the top at $y=h$, where h is the height of the throw. The ball is rising and the Earth does work on it:

$$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \int_0^h m(-g)\,\mathrm{d}y = -mg\left[y\right]_0^h = -mgh. \tag{4.84}$$

The change of the kinetic energy from the position 1 to 2 is:

$$\Delta K = W = K_2 - K_1 = \frac{1}{2}mv^2{}_{/v=0} - \frac{1}{2}mv_0^2 = -\frac{1}{2}mv_0^2. \tag{4.85}$$

2. At the top the ball stops its motion and its velocity is zero. The position of the ball changes from $y=h$ to $y=0$. The force of gravity begins to do work:

$$W = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = mg\int_h^0 \mathrm{d}y = \left[-mgh\right]_h^0 = mgh. \tag{4.86}$$

For the change of the kinetic energy it holds:

$$\Delta K = W = K_2 - K_1 = -\left(-\frac{1}{2}mv^2 - \frac{1}{2}mv^2{}_{/v=0}\right) = \frac{1}{2}mv^2. \tag{4.87}$$

The **total work** done by the force of gravity during the **round trip is zero.**

If a body moves under the action of a force that does no total work during any round trip, then the force is **conservative**. Otherwise it is **noncoservative** one. In other words, if the work done by a force moving a body from an initial location to a final location is independent on the path taken between these two points, then the force is **conservative**, otherwise it is **nonconservative** one.

In Fig. 4.16 we can see the path of the particle moving from the point 1 to the point 2.

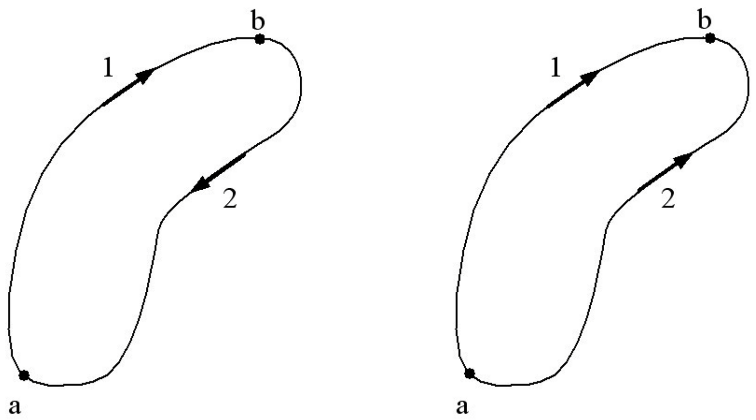

Figure 4.16: The path of the particle from the point a to b.

We have the particle moving from the point a to the point b and back. Let us consider that the force $\boldsymbol{F}$ acting on the particle is the conservative one. The **total work** done on the particle by that force during complete cycle **must be zero**

$$\begin{aligned} &W_{\mathrm{ab},1} + W_{\mathrm{ba},2} = 0 \\ &\int_{\mathrm{a}}^{\mathrm{b}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}_{/\text{path }1} + \int_{\mathrm{b}}^{\mathrm{a}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}_{/\text{path }2} = 0\,, \end{aligned} \tag{4.88}$$

where $W_{\mathrm{ab},1}$ is the work done by the force when particle moves from a to b along the path 1 and $W_{\mathrm{ba},2}$ is the work done by the force when the particle moves from b to a along the path 2. The last equation is the first criterion for a conservative force - the mathematical statement equivalent.

4.8 Potential energy

Potential energy, represented by the symbol U, is an energy of configuration of a system. It is an energy stored by the system because of relative position or orientation of the parts of the system.

Let us consider a system on which only one force acts. When we change the configuration of the system (e.g. moving one of its parts), the work W is done by the conservative force. We define the change in potential energy ΔU corresponding to a particular change in configuration:

$$\Delta U = -W \,. \tag{4.89}$$

It is the negative value of the work done by the conservative force (for instance the force acting on the spring). Let us look at the configuration of the block in the spring, (Fig. 4.14), when the spring is in the relaxed state. The work done by the spring force on the block, with respect to the state when the block is momentarily at rest, is:

$$W = -\frac{1}{2}kd^2 \,. \tag{4.90}$$

Therefore the change of potential energy of the system is:

$$\Delta U = -W = +\frac{1}{2}kd^2 \,. \tag{4.91}$$

From the work energy theorem the change in the kinetic energy of the block is:

$$\Delta U + \Delta K = 0 \tag{4.92}$$

or

$$\Delta\left(U + K\right) = 0 \,. \tag{4.93}$$

The change of the total energy $\left(U + K\right)$ is zero. If the change of the sum $\left(U + K\right)$ is zero, then the value of this sum must be constant:

$$U + K = E = \text{constant.} \tag{4.94}$$

The equation (4.94) represents the **mechanical energy conservation law**. The quantity E is called the **total mechanical energy.**

5 Gravitational field

Gravitational field is a model used in physics to explain the gravity in the universe. Still today, three fundamental physical theories explain gravity in different ways.

- **The gravitational field in classical mechanics** is not an actual entity, but merely a model used to describe the effects of gravity. The gravitational field can be determined using Newton's law of universal gravitation. Determined in this way, the gravitational field around a single particle is a vector field consisting at every point of a vector pointing directly towards the particle. The magnitude of the field at every point is calculated with the Newton’s law, and represents the force per unit mass of any object at that point in the space. The field around multiple particles is merely the vector sum of the fields around each individual particle.

- **The gravitational field in theory of relativity** is determined as the solution of Einstein's field equations. These equations are dependent on the distribution of matter and energy in a space, unlike Newtonian gravity, which is dependent only on the distribution of matter. The fields themselves in general relativity represent the curvature of spacetime. General relativity states that being in a region of curved space is equivalent to accelerating up the gradient of the field. By Newton's second law, this will cause an object to experience a fictitious force if it is held still with respect to the field. This is why a person will feel himself pulled down by the force of gravity while standing still on the Earth's surface. In general the gravitational fields predicted by general relativity differ in their effects only slightly from those predicted by classical mechanics, but there are a number of easily verifiable differences, one of the most well known being the bending of light in such fields.

- **The gravitational field in quantum theory** is mediated by the hypothetical elementary particle called graviton. If it exists, the graviton must be massless particle and it must have a spin of 2. Gravitons are postulated because of the great success of the standard model of elementary particles at modeling the behavior of all other forces of nature with similar particles: electromagnetism with the photon, the strong interaction with the gluons, and the weak interaction with the W and Z bosons.

This chapter deals with the theory of gravitational field in the framework of the classical mechanics. Next two approaches are subjects of more advanced textbooks on theory of relativity or quantum theory. In the classical limit, both approaches give identical results, including Newton's law of universal gravitation.

5.1 The gravitational force

In 1687 Sir Issac Newton formulated the law which has the fundamental role in classical mechanics for description of gravitational force. Newton’s law of universal gravitation says: *“Every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of this force is along the line joining the particles.”*

It is usually written as the vector equation

$$\boldsymbol{F} = -G\frac{m_1 m_2}{r^3}\boldsymbol{r} \,. \tag{5.1}$$

The force $\boldsymbol{F}$ is the gravitational force exerted on the particle of mass m_2 by the particle of mass m_1. The vector $\boldsymbol{r}$ is the position vector of the particle of mass m_2 with respect to the particle of mass m_1. The minus sign indicates that the gravitational force is opposite direction as the direction of position vector, it means that the gravitational force has the direction towards the particle of mass m_1. The magnitude of the position vector in the numerator can be divided with the cube of the position vector in the denominator. The multiplication by the displacement vector thus gives the direction to the gravitational force but the dependence remains inversely proportional to the square of the distance between the particles. The notation is drawn in Figure 5.1.

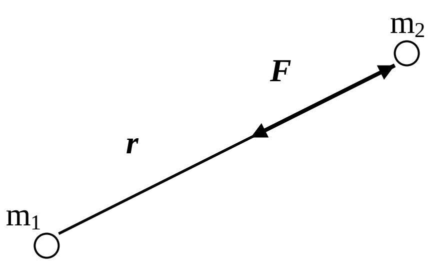

Figure 5.1: The Newton’s law of universal gravitation.

The constant G is called the universal gravitational constant and its value is

$$G = 6.67428(67)\cdot 10^{-11}\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}\,. \tag{5.2}$$

If the particle of mass m is placed inside the gravitational field created by system of n particles, the resulting gravitational force can by calculated as the sum of all gravitational forces acting on it

$$\boldsymbol{F} = \sum_{\mathrm{i}}^{\mathrm{n}} \boldsymbol{F}_{\mathrm{i}} = -Gm\sum_{\mathrm{i}}^{\mathrm{n}} \frac{M}{r_{\mathrm{i}}^3}\boldsymbol{r}_{\mathrm{i}}\,, \tag{5.3}$$

where $\boldsymbol{r_i}$ is the position vector of the particle of mass M_{i}. If the particle is placed inside the gravitational field created by an object with continuously distributed mass M, the resulting gravitational force can be obtained by replacing the sum in (5.3) by integral

$$\boldsymbol{F} = \int \mathrm{d}\boldsymbol{F} = -Gm\int \frac{\mathrm{d}M}{r^3}\boldsymbol{r}\,, \tag{5.4}$$

where $\boldsymbol{r}$ is the position vector of the mass element dM.

5.2 The gravitational field

If we want to study the properties of the gravitational field we have to introduce the quantity which characterizes the gravitational field itself. Such physical quantity is called gravitational field and it is defined as

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{m}\,. \tag{5.5}$$

The $\boldsymbol{F}$ is the gravitational force acting on the particle of mass m. Using the Newton's law of universal gravitation, we can express the gravitational field created by the particle of mass M in a particular point in space with the position vector $\boldsymbol{r}$ as:

$$\boldsymbol{E} = -Gm\frac{M}{r^3}\boldsymbol{r}\,. \tag{5.6}$$

Analogously as for gravitational force our result for gravitational field can by generalized for the system of particles

$$\boldsymbol{E} = \sum_{\mathrm{i}}^{\mathrm{n}} \boldsymbol{E}_{\mathrm{i}} = -G\sum_{\mathrm{i}}^{\mathrm{n}} \frac{M}{r_{\mathrm{i}}^3}\boldsymbol{r}_{\mathrm{i}} \tag{5.7}$$

as well as for the object with continuously distributed mass

$$\boldsymbol{E} = \int \mathrm{d}\boldsymbol{E} = -G\int \frac{\mathrm{d}M}{r^3}\boldsymbol{r}\,. \tag{5.8}$$

5.3 The gravitational potential energy

Now we will describe the motion of a particle of mass m in gravitational field created by the particle of mass M from the energetic point of view. The work done by the gravitational force $\boldsymbol{F}$ when the particle moves from the point $\boldsymbol{r}_1$ to the point $\boldsymbol{r}_2$ is defined as:

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \,. \tag{5.9}$$

Using Newton's law of universal gravitation we can express this work as:

$$W_{12} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \left(-G \frac{mM}{r^3} \boldsymbol{r} \right) \cdot \mathrm{d}\boldsymbol{r} = -GmM \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r^3} = -GmM \left[-\frac{1}{r} \right]_{r_1}^{r_2} = GmM \left(\frac{1}{r_2} - \frac{1}{r_1} \right). \tag{5.10}$$

The gravitational potential energy at the point $\boldsymbol{r}_2$ with respect to the point vector $\boldsymbol{r}_1$ is defined as:

$$U_{21} = -W_{12} \,, \tag{5.11}$$

and thus we can write

$$U_{21} = GmM \left(\frac{1}{r_1} - \frac{1}{r_2} \right). \tag{5.12}$$

The absolute gravitational potential energy at the point $\boldsymbol{r}$ is defined as a potential energy with respect to infinity, which can be expressed as:

$$U = \lim_{r_1 \to \infty} \left[GmM \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right] = -\frac{GmM}{r} \,. \tag{5.13}$$

The minus sign indicates that the potential energy is negative at any finite distance, that is, the potential energy is zero at infinity and decreases as the separation distance decrease. This corresponds to the fact that the gravitational force is attractive.

5.4 The gravitational potential

If we want to characterize the properties of gravitational field without dependence on the mass of moving particle, we have to introduce gravitational potential:

$$V = \frac{U}{m} \,. \tag{5.14}$$

Now, we can express the gravitational potential as:

$$V = -G\frac{M}{r}. \tag{5.15}$$

Analogously as for gravitational force and gravitational field our result for the gravitational potential can by generalized for the system of particles

$$V = \sum_{\mathrm{i=1}}^{\mathrm{n}} V_{\mathrm{i}} = -G\sum_{\mathrm{i}}^{\mathrm{n}}\frac{M_{\mathrm{i}}}{r_{\mathrm{i}}} \tag{5.16}$$

as well as for the object with continuously distributed mass

$$V = \int \mathrm{d}V = -G\int\frac{\mathrm{d}M}{r}. \tag{5.17}$$

5.5 The relation between gravitational field and gravitational potential

The relation between the gravitational field and gravitational potential will be clear if we express the gravitational potential in terms of gravitational force

$$V = \frac{U}{m} = -\frac{W}{m} = -\frac{1}{m}\int_{\infty}^{r}\boldsymbol{F}\cdot d\boldsymbol{r}. \tag{5.18}$$

Using the definition of gravitational field we get

$$V = -\int_{\infty}^{r}\boldsymbol{E}\cdot d\boldsymbol{r}. \tag{5.19}$$

It means that

$$dV = -\boldsymbol{E}\cdot d\boldsymbol{r}. \tag{5.20}$$

The gravitational field and the gravitational potential are functions of x, y, z, therefore

$$\frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -\left(E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z\right). \tag{5.21}$$

If we compare the left and the right sides of this equation we can see that

$$E_x = -\frac{\partial V}{\partial x} \tag{5.22}$$

$$E_y = -\frac{\partial V}{\partial y} \tag{5.23}$$

$$E_z = -\frac{\partial V}{\partial z} \tag{5.24}$$

From these equations follows that the gravitational field is equal to the negative of the derivative of the potential with respect to some coordinate. These scalar expressions can be rewritten into vector form as

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\boldsymbol{i} + \frac{\partial V}{\partial y}\boldsymbol{j} + \frac{\partial V}{\partial y}\boldsymbol{k} \right). \tag{5.25}$$

This operation is called gradient and thus this equation is usually written as:

$$\boldsymbol{E} = -\operatorname{grad} V . \tag{5.26}$$

5.6 Homogeneous gravitational field of the Earth

Using the second Newton's law of dynamic we can see that if only the gravitational force exerts on a particle, the gravitational field equals to the acceleration of this particle

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{m} = \frac{m\boldsymbol{a}}{m} = \boldsymbol{a} . \tag{5.27}$$

The acceleration of free falling body in gravitational field is called the acceleration of gravity and near the Earth's surface it can be calculated as:

$$g = G\frac{M_E}{R_E^2} = 6.67\times10^{-11}\mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2}\,\frac{5.98\times10^{24}\mathrm{kg}}{\left(6.37\times10^{6}\mathrm{m}\right)^2} = 9.83\mathrm{m}\cdot\mathrm{s}^{-2} . \tag{5.28}$$

The M_E is the mass of the Earth and the parameter R_E represents the mean radius of the Earth. In height h above the Earth's surface the acceleration of gravity varies as:

$$g(h) = G\frac{M_{\mathrm{E}}}{\left(R_{\mathrm{E}} + h\right)^2} . \tag{5.29}$$

For most practical applications the gravitational field of the Earth can be considered as homogeneous field and all quantities are usually calculated using the acceleration of gravity. Then, the gravitational force is

$$\boldsymbol{F} = m\boldsymbol{g} . \tag{5.30}$$

The gravitational potential energy is

$$U = mgh . \tag{5.31}$$

The gravitational potential is

$$V = mg . \tag{5.32}$$

6 Literature

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Appendix A - International system of units

Table A-1: Base units

Quantity	Name	Symbol
Length	meter	M
Mass	kilogram	kg
Time	second	S
electric current	ampere	A
thermodynamics temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Table A-2: Supplementary units

Quantity	Name	Symbol
plane angle	radian	rad
solid angle	steradian	sr

Table A-3: Some derived units

Quantity	Name	Symbol	Equivalent
frequency	hertz	Hz	s^{-1}
force	newton	N	$kg.m.s^{-2}$
pressure	pascal	Pa	$kg.m^{-1}.s^{-2}$
energy	joule	J	$kg.m^{2}.s^{-2}$
power	watt	W	$kg.m^{2}.s^{-1}$
electric charge	coulomb	C	A.s
electric potential	volt	V	$N.m.A^{-1}.s^{-1}$
electric resistance	ohm	Ω	$N.m.A^{-2}.s^{-1}$
capacitance	farad	F	$A^{2}.s^{2}.N^{-1}.m^{-1}$
magnetic flux	weber	Wb	$N.m.A^{-1}$
Inductance	henry	H	$N.m.A^{-2}$
magnetic field	tesla	T	$N.m^{-1}.A^{-1}$

Appendix B – Fundamental physical constants

Table B-1: Values of some fundamental physical constants

Constant	Symbol	Value*	Unit
Speed of light in vacuum	c	2.99792458×10^{8}	$m.s^{-1}$
Gravitational constant	G	$6.67428(67)\times10^{-11}$	$m^{3}.\ kg^{-1}.s^{-2}$
Elementary charge	e	$1.602176487(40)\times10^{-19}$	C
Electric constant	ε_0	$8.854187817...\times10^{-12}$	$F.m^{-1}$
Magnetic constant	μ_0	$1.2566370614...\times10^{-6}$	$N.A^{-2}$
Planck constant	h	$6.62606896(33)\times10^{-34}$	J.s
Avogadro constant	N_A	$6.02214179(30)\times10^{23}$	mol^{-1}
Universal gas constant	R	8.314472(15)	$J.mol^{-1}.K^{-1}$
Boltzmann constant	k	$1.3806504(24)\times10^{-23}$	$J.K^{-1}$
Faraday constant	F	$9.64853399(24)\times10^{4}$	$C\ mol^{-1}$
Stefan-Boltzmann constant	σ	$5.670400(40)\times10^{-8}$	$W.m^{-2}.K^{-4}$
Rydberg constant	R_∞	$1.0973731568527(73)\times10^{7}$	m^{-1}
Fine structure constant	α	$7.2973525376(50)\ )\times10^{-3}$	
Electron rest mass	m_e	$9.109389215(45)\times10^{-31}$	kg
Proton rest mass	m_p	$1.672621637(83)\times10^{-27}$	kg
Neutron rest mass	m_n	$1.674927211(84)\times10^{-27}$	kg
Electron magnetic moment	μ_e	$-9.28476377(23)\times10^{-24}$	$J.T^{-1}$
Proton magnetic moment	μ_p	$1.410606662(37)\times10^{-26}$	$J.T^{-1}$

* According to the National Institute of Standards and Technology.

Appendix C – Basic differentiation rules

Constant multiplication rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[cf(x)\right]=c\frac{\mathrm{d}f(x)}{\mathrm{d}x} \tag{C-1}$$

Addition rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)+g(x)\right]=\frac{\mathrm{d}f(x)}{\mathrm{d}x}+\frac{\mathrm{d}g(x)}{\mathrm{d}x} \tag{C-2}$$

Multiplication rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)g(x)\right]=g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}+f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x} \tag{C-3}$$

Division rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}-f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}}{f(x)g(x)} \tag{C-4}$$

Table C-1: Derivatives of fundamental functions

$f(x)$	$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$
x^n	nx^{n-1}
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$

Appendix D – Basic integration rules

Constant multiplication rule:

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x \tag{D-1}$$

Addition rule:

$$\int\left[f(x)+g(x)\right]\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x \tag{D-2}$$

Per-partes method:

$$\int\left[f(x)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right]\mathrm{d}x = f(x)g(x) - \int\left[g(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right]\mathrm{d}x \tag{D-3}$$

Substitution method:

$$\int f\left(g(x)\right)\frac{\mathrm{d}g(x)}{\mathrm{d}x}\left[\begin{array}{c} u = g(x) \\ \mathrm{d}u = \dfrac{\mathrm{d}g(x)}{\mathrm{d}x}\mathrm{d}x \end{array}\right] = \int f(u)\,\mathrm{d}u = F(u) = F\left(g(x)\right) \tag{D-4}$$

Table D-1: Integrals of fundamental functions

$f(x)$	$\int f(x)\,\mathrm{d}x$
x^n	$\dfrac{x^{n+1}}{n+1}+c;\ n\neq -1$
$\dfrac{1}{x}$	$\ln x$
a^x	$\dfrac{a^x}{\ln a}+c$
$\ln x$	$x\ln x - x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\tan x$	$-\ln\lvert\cos x\rvert + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$

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