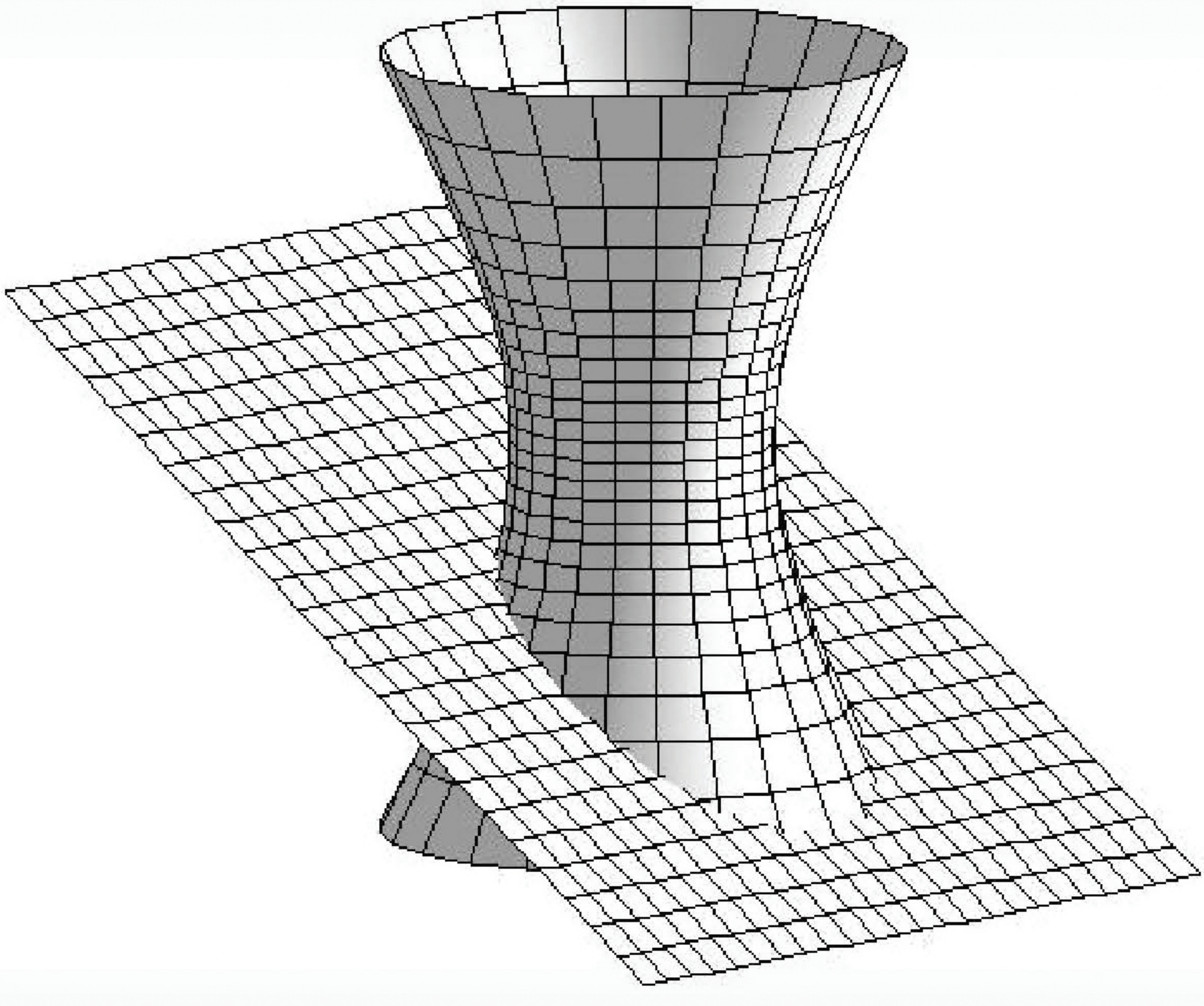
CONSTRUCTIVE GEOMETRY



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STU SLOVENSKÁ TECHNICKÁ UNIVERZITA V BRATISLAVE

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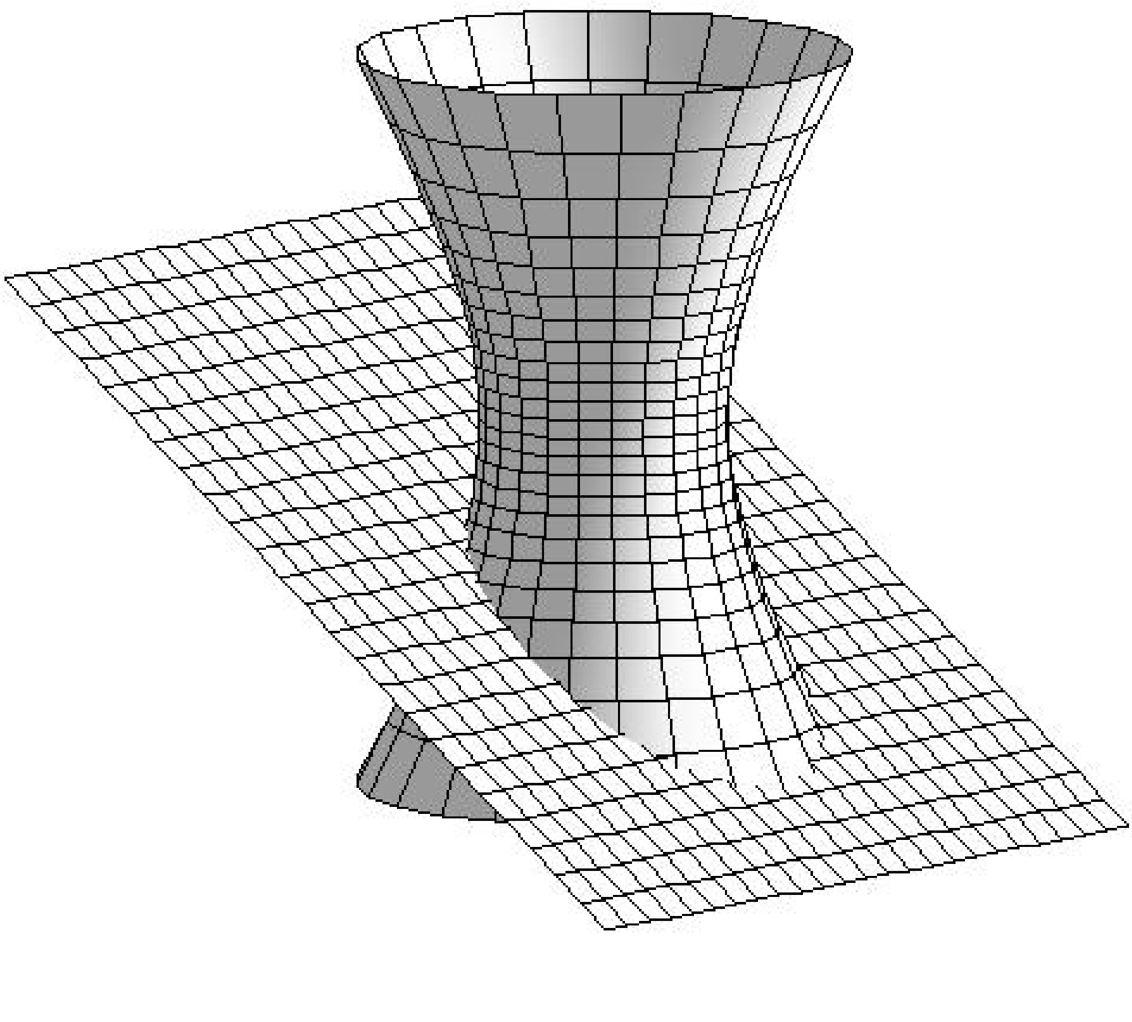

SLOVAK UNIVERSITY OF TECHNOLOGY IN BRATISLAVA, 2011

Učebnica obsahuje výber tém z niekoľkých odvetví konštrukčnej geometrie potrebných na opis a zobrazovanie geometrických objektov a riešenie úloh s geometrickou podstatou, s ktorými sa možno stretnúť v strojárskej konštrukčnej praxi.

Učebnica je určená najmä študentom študujúcim v anglickom jazyku na Strojníckej fakulte STU, môže byť však dobrou pomôckou aj ostatným študentom a všetkým, ktorí sa zaujímajú o danú tému.

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Preface

This book is intended to provide a short insight into the basic methods used in constructive geometry covering geometric topics frequently appearing in various technical engineering applications. The choice and extent of presented concepts were influenced by the original aim of the author to provide information on several possibilities of representation of geometric objects and on various methods used for solving problems of geometric background that engineers may come across, particularly in mechanical engineering.

In technical sciences, constructors and designers cannot do without projections of existing spatial objects, reconstructions of objects mapped in several views, or without sketches and modelling of equipment and mechanisms they design. The first chapter therefore brings basic information on projection methods. Monge method as the multi-view orthographic mapping to several image planes and axonometric mappings are discussed as the most useful and utilised. The latest CAGD systems for geometric modelling and design are in most cases based on these two projection methods.

Conic sections are planar quadratic curves of a tremendous engineering application. Chapter 2 brings basic definitions, equations and properties of these curves in the Euclidean plane, while chapter 3 presents how these curves can be constructed as planar intersections on cylindrical or conical surfaces.

Chapter 4 deals with quadratic surfaces of revolution, a family of rotational surfaces with many interesting properties useful namely in mechanical engineering. Creative laws and equations of quadratic surfaces of revolution are presented, and planar intersections of these surfaces are constructed using properties of conic sections from Chapter 2.

Problems to be solved by readers are included in Chapter 5, and comprise all constructions and solutions studied in individual chapters.

It is my pleasure to express my gratitude and deep thanks to reviewers doc. RNDr. Dagmar Szarková, PhD., RNDr. Margita Vajsáblová, PhD. and Dr. Margareta V. Rebelos, PhD. for their valuable suggestions, inspiring comments and critical remarks, which contributed at a large greatly to the quality improvement of the presented material.

Creative approach to problem solving is the background of success in any human activity. Geometry is one of mathematical disciplines that is a unique composition of exact logical reasoning, creative imagination and pure abstraction. In addition to a good mathematical-physical-technical education constructor’s work requires also a good 3D imagination, logical thinking and creative ideas. Constructive geometry can contribute to the acquisition of these basic attributes of a successful engineer in a large scale.

I wish you a lot of success and enjoyment in “discovering the geometry" of the world around us, in revealing and cultivating your own creative abilities. Readers are invited to address me concerning any questions that might arise. All comments and suggestions are welcome.

Daniela Velichová

Symbols

Points:	$A, B, ..., X, Y$	
Lines:	$a, b, ..., k, l, ...$	
Planes:	$\alpha, \beta, ..., \gamma, \delta, \pi, ...$	
Coincidence:	$A = B$	- points A, B coincide
	$a = b$	- lines a, b coincide
	$\alpha = \beta$	- planes α, β coincide
Difference:	$A \neq B$	- different points A, B
	$a \neq b$	- different lines a, b
	$\alpha \neq \beta$	- different planes α, β
Incidence:	$A \in a$	- A is point on line a
	$A \in \alpha$	- A is point in plane α
	$a \subset \alpha$	- a is line in plane α
Non-incidence:	$A \notin a$	- point A is not on line a
	$A \notin \alpha$	- point A is not in plane α
	$a \not\subset \alpha$	- line a is not in plane α
Parallel figures:	$a \parallel b$	- lines, $a \parallel \alpha$ - line and plane, $\alpha \parallel \beta$ - planes
Intersecting figures:	$a \times b$	- lines, $a \times \alpha$ - line and plane, $\alpha \times \beta$ - planes
Skew lines:	$a \setminus b$	
Perpendicularity:	$a \perp b$	- line a is perpendicular to line b
	$a \perp \alpha$	- line a is perpendicular to plane α
	$\alpha \perp \beta$	- plane α is perpendicular to plane β
Equality:	$a = AB$	- line a is determined by points A, B
	AB	- line segment with endpoins A, B
	$\alpha = Ba$	- plane α determined by line a, point $B \notin a$
	$\alpha = (a \parallel b)$	- plane α determined by paralel lines a, b
	$\alpha = (a \times b)$	- plane α determined by pierce lines a, b
	$\alpha = ABC$	- plane α determined by 3 different points A, B, C that are not on one line
	ΔABC	- triangle with vertices A, B, C
	$k(S, r)$	- circle with centre S and radius r
	$G(S, r)$	- sphere with centre S and radius r

Intersections:	$a \cap b = P$	- point P is the pierce point of lines a, b
	$a \cap \alpha = P$	- point P is the pierce point of line a and plane α
	$\alpha \cap \beta = p$	- line p is the intersection line of planes α, β
Distances:	$\lvert AB \rvert$	- distance of points A and B, line segment AB length
	$\lvert Aa \rvert$	- distance of point A from line a
	$\lvert A\alpha \rvert$	- distance of point A from plane α
Angles:	$\alpha, \beta, \dots, \gamma, \delta, \varphi, \dots$	
	$\angle AVB$	- angle with vertex V and arms VA, VB
	$\angle pq$	- angle of lines p, q
	$\lvert \angle AVB \rvert$	- size of angle AVB
Other symbols:	A_1 , a_1	- orthographic view of point A, line a in ground plane π, top view
	A_2 , a_2	- orthographic view of point A, line a in frontal plane ν, front view
	A_3 , a_3	- orthographic view of point A, line a in side plane μ, side view
	A_0 , a_0	- revolved point A, line a
	p^{α}	- trace of plane α in the ground image plane π
	n^{α}	- trace of plane α in the frontal image plane ν
	m^{α}	- trace of the plane α in the side image plane μ
	${}^1s$, ${}^2s$, ${}^3s$	- slope lines in plane to the image planes π, ν, μ
	$\mathbf{E}^2$, $\mathbf{E}^3$	- Euclidean plane, Euclidean space
	$\{O, x, y\}$	- Cartesian coordinate system of the plane $\mathbf{E}^2$ with centre O and perpendicular coordinate axes x, y
	$\{O, x, y, z\}$	- Cartesian coordinate system of the space $\mathbf{E}^3$ with centre O and perpendicular coordinate axes x, y, z
	e, $\mathbf{E}$	- ellipse, ellipsoid
	p, $\mathbf{P}$	- parabola, paraboloid
	h, $\mathbf{H}$	- hyperbola, hyperboloid

1 PROJECTION METHODS

Projection of space to the plane is a special mapping of space points onto a chosen plane of projection called image plane or projection plane. In general, we can speak about two different types of projections, central projection and parallel projection.

Central projection

Let there be given an arbitrary plane π in the Euclidean space $\mathbf{E^3}$ called **projection plane** and a fixed point S called the **centre of projection**, while S is not a point on the plane π. Image of an arbitrary point $A \neq S$ in space $\mathbf{E^3}$ under the central projection given by S and π is the point $A_S \in \pi$, which is the intersection point of the line $s^A = AS$ passing through points S and A and the projection plane π, $A_S = s^A \cap \pi$. Line s^A is the **projecting line** of the point A, A_S is the view of the point A under the central projection, Fig. 1.1.

The distance d of the centre of projection S and the projection plane π is called the **distance of the central projection**, $d = | S\pi |$.

Centre of projection is the point in space $\mathbf{E^3}$ that has no image in π under the central projection. All points in the plane passing through the centre S and parallel to the image plane π have no views either, so this plane has no view under the central projection in space $\mathbf{E^3}$. Image plane π is invariant under the central projection, all points in π are mapped to the same points in π.

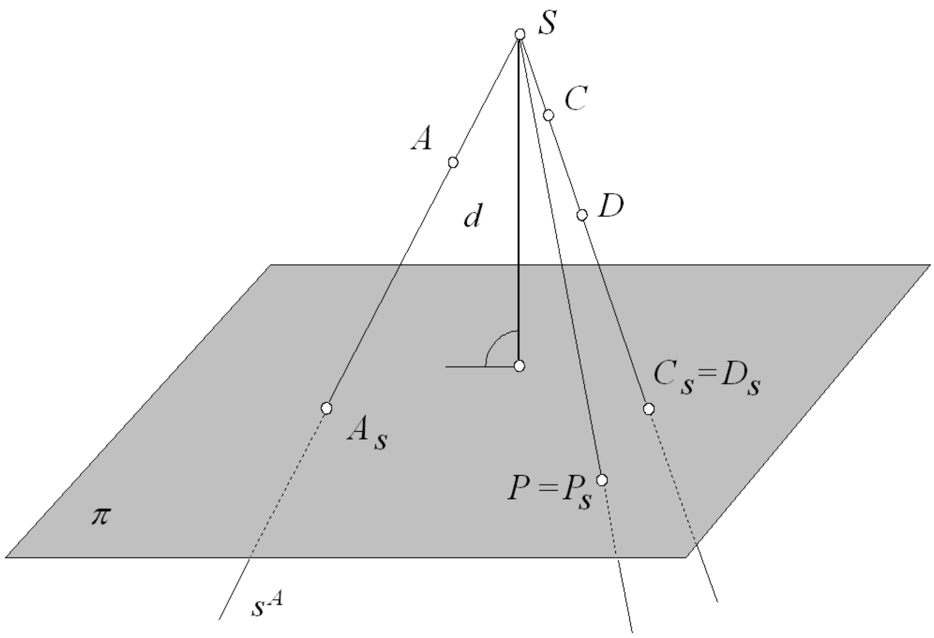


Figure 1.1 Central projection.

Parallel projection

Let there be given an arbitrary plane π in the space called **projection plane** and a fixed direction s called the **direction of projection**, while s is not parallel to the plane π. Image of an arbitrary point A in the space under the parallel projection given by s and π is the point $A_s \in \pi$, which is the intersection point of the line s^A passing through the point A and parallel to the line s and the projection plane π, $A_s = s^A \cap \pi$. Line s^A is the **projecting line** of the point A, A_s is the view of the point A under the parallel projection, Fig. 1.2.
Image plane π is set of all invariant points under the parallel projection of the space $\mathbf{E^3}$ on the plane π in direction s.

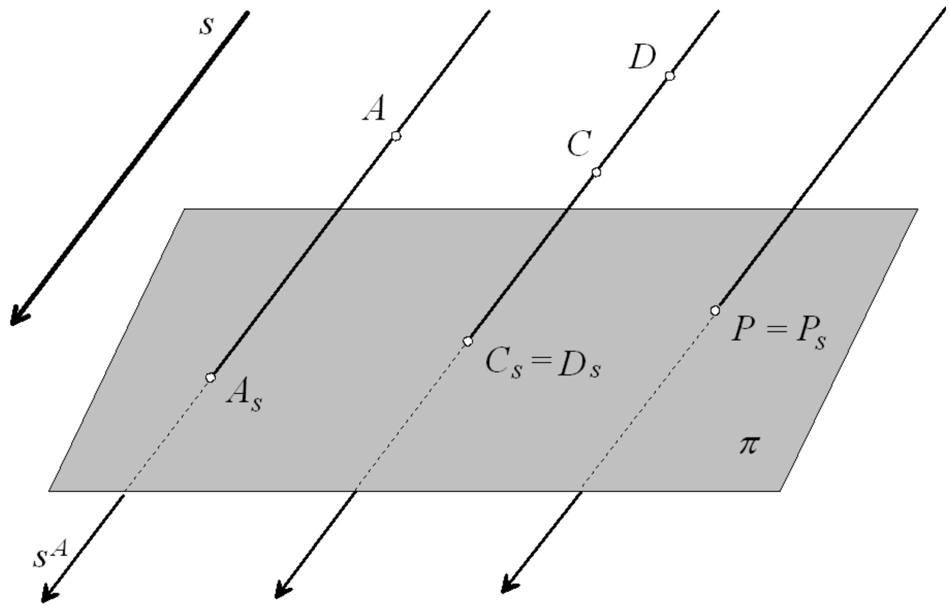


Figure 1. 2 Parallel projection.

1.1 BASICS OF PROJECTIONS

Central and parallel projections are mappings of the Euclidean space $\mathbf{E^3}$ on the image plane π. All points in the image plane are invariant in both projections. All points on one projecting line are mapped to the same point in the image plane π, therefore simple central and parallel projections are not one-to-one (injective) mappings of the Euclidean space $\mathbf{E^3}$ on the image plane π.
View U_S of a geometric figure U is a figure in the image plane, which can be constructed from the views of all points of the figure U, Fig. 1.3.

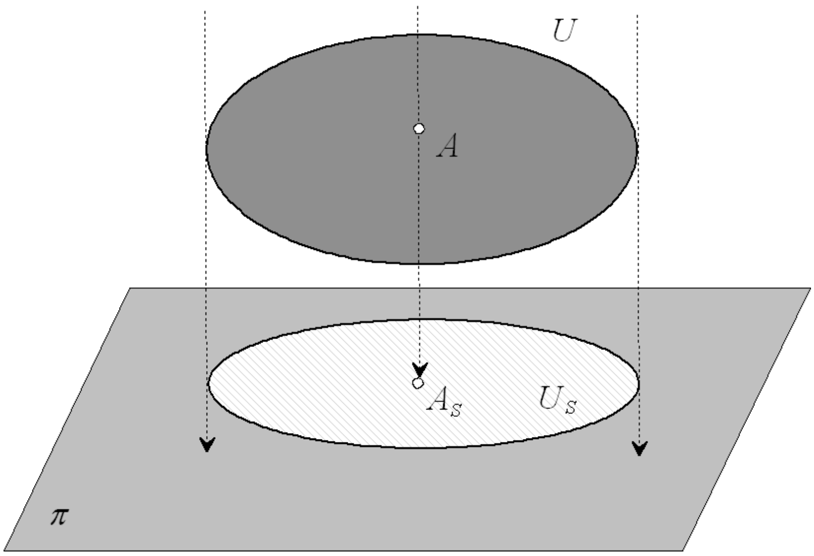


Figure 1.3 View of a geometric figure *U*.

View of a line, which is not a projecting line, is a line (Fig. 1. 4). View of a projecting line is a point. Invariant intersection point of a line and the image plane is the **trace of the line,** point *P*.

All projecting lines of points on one line form the line projecting plane χ. View of a line is the intersection line of the line projecting plane χ and the image plane π.

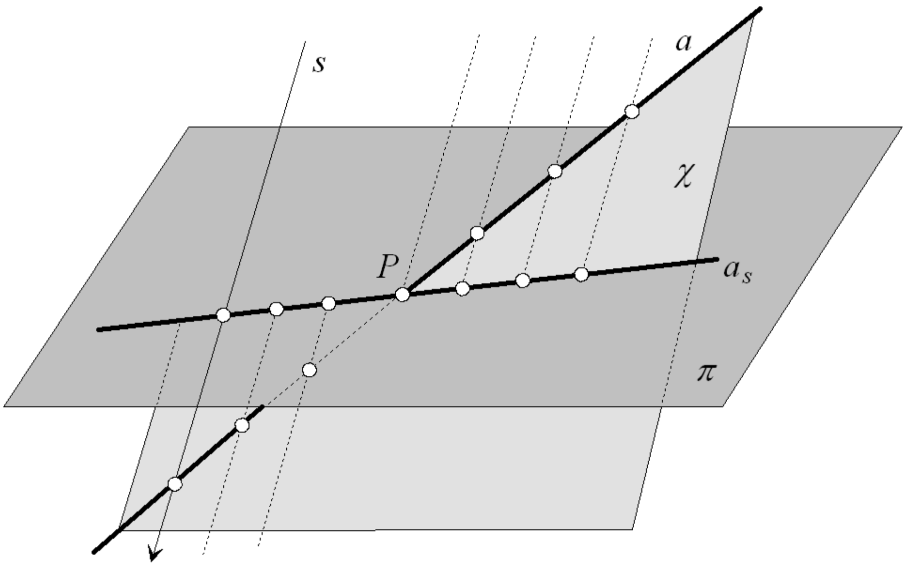


Figure 1. 4 Parallel view of a line.

View of a plane, which is not a projecting plane, is the entire image plane. View of a projecting plane is a line. Intersection line of a plane and the image plane is the **trace of the plane**, line p^{α}. The trace of a plane is the set of traces of all lines in the respective plane.

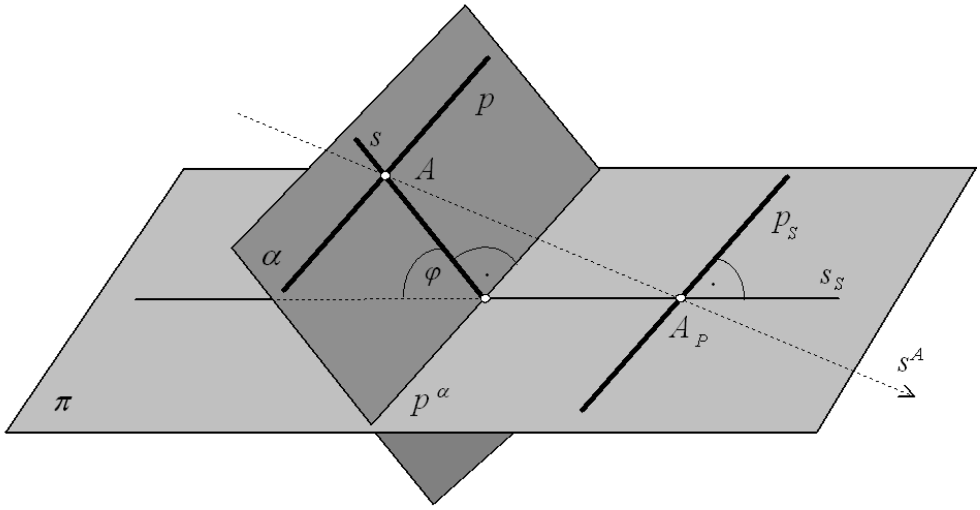


Figure 1. 5 Parallel view of a plane, principle and slope line.

Any line located in the plane and parallel to the image plane is called **principle line** in the plane (it has no trace in the image plane). View of a principle line *p* in the plane is a line parallel to the plane trace (Fig. 1. 5).

Any line in the plane perpendicular to the plane trace (and to all principle lines in the same direction) is called **slope line** in the plane. Acute angle φ, which the slope line *s* in the plane forms to the image plane π, is the angle of the plane to the image plane. We refer to this angle when we speak about the **slope** of the plane to the image plane.

Properties of the parallel projection

Theorem 1. View of the figure *U*, which is located in the plane parallel to the image plane, is the figure congruent to the figure *U*.

Theorem 2. Parallelism is an invariant property of the parallel projection. Parallel lines that are not in the direction *s* of the projection are mapped to the parallel lines in views. Parallel planes have parallel plane traces and views of principle and slope lines.

Theorem 3. Ratio of three points on one line is invariant, $\lambda(ABC) = \lambda(A_1\ B_1\ C_1)$. Centre of a figure is mapped to the centre of the respective figure view (Fig. 1.6).

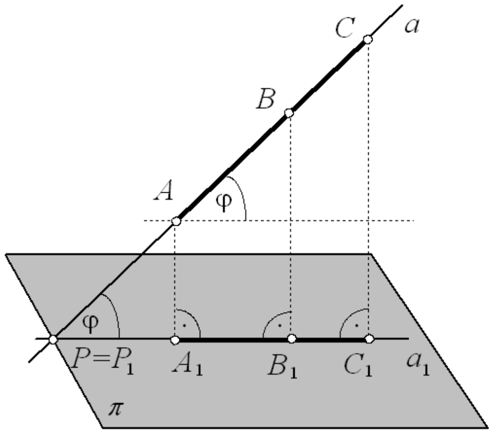


Figure 1. 6 View of a line segment.

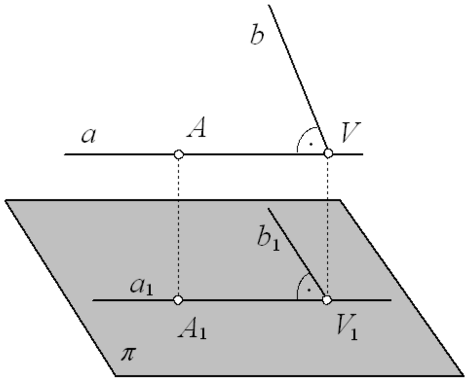


Figure 1. 7 View of a right angle.

Orthogonal projection is a parallel projection, while the projecting lines in the direction *s* are perpendicular to the image plane.
In addition to properties 1 - 3 of the general parallel projection, special properties are valid for the orthogonal projection and orthographic views of geometric figures under this projection.

Theorem 4. Let AB be a line segment on the line a, forming the angle φ to the image plane π. Length of the orthographic view A_1B_1 satisfies the following:

$$|A_1 B_1| = |AB| \cos \varphi$$

The length of the orthographic view of a line segment is therefore lesser then the length of the original line segment for values of angle $\varphi \in (0°, 90°)$ (Fig. 1. 6, $|AB| > |A_1 B_1|$), for $\varphi = 90°$ it is zero ($A_1 = B_1$), and it is equal to the length of the original line segment for $\varphi = 0°$ (Fig. 1. 7, $|A_1 V_1| = |AV|$).

Theorem 5. The right angle is mapped as the right angle, if at least one arm of the angle is parallel to the image plane and none of the arms is perpendicular to the image plane (Fig. 1. 7).

Slope line s in the plane α, which is not perpendicular to image plane, is mapped to the line s_1 perpendicular to the plane trace p^α (and views of all principle lines in the plane). Orthographic view k_1 of the line k perpendicular to the plane α is perpendicular to the plane trace p^α (and views of all principle lines in the plane) (Fig. 1.8). Therefore orthographic views of both, the slope line s in the plane α and the line k perpendicular to the plane α and intersecting the slope line s in α (their common point in Fig. 1. 8 is point K), coincide, and appear in one line $s_1 = k_1$.

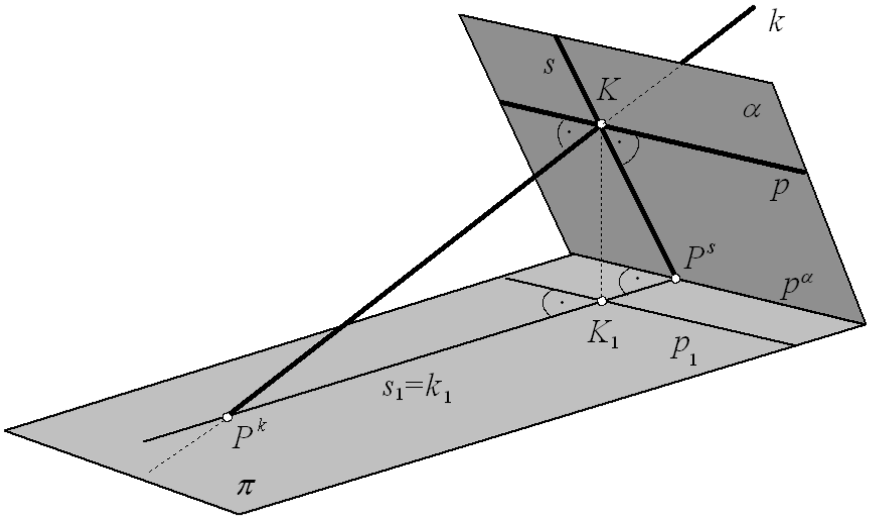


Figure 1. 8 View of a slope line in a plane and line perpendicular to the plane.

The idea to map three-dimensional objects to the plane originated in practical needs of human activities and is very important in many fields. For the technician, it is not enough just to map the object, but he also needs to find out some features of the mapped object right from the views, or to reconstruct it entirely and unambiguously as a three dimensional object. This problem can be solved using different types of projection methods suitable for expected tasks.
The most frequently used types of projection methods are:

altitudinal projection

orthogonal projection to two orthogonal image planes - Monge method

orthogonal and oblique axonometric method

central projection and linear perspective

stereoscopic projection (with two centres).

The choice of the proper projection method depends on the utilisation of the created views. In art, architecture and civil engineering, the commonly used methods are linear perspective, Monge method, axonometry, or altitudinal projection. In machine engineering, Monge and axonometric methods are used mostly. Creative work of engineers as constructors of new machines, equipments and machine parts, but also the work of designers and technicians who design and construct these new objects on the basis of technical documentation, is unthinkable without a good knowledge of projection methods. New methods in constructions and computer aided design even stressed the importance and accuracy of spatial abilities and reconstruction capabilities of 3D objects designed and visualized by projecting in some of the projection methods, and plot on the computer screen, or using any other computer graphics device (plotter, drawing machine).

1.2 MONGE METHOD

Let us consider three dimensional Euclidean space $\mathbf{E}^3$ with Cartesian coordinate system $\{O, x, y, z\}$. Monge method is a composition of two orthographic mappings of the space $\mathbf{E}^3$ on two perpendicular image planes, **ground plane** $\pi = xy$, and **frontal plane** $\nu = xz$, see in (Fig. 1. 9).

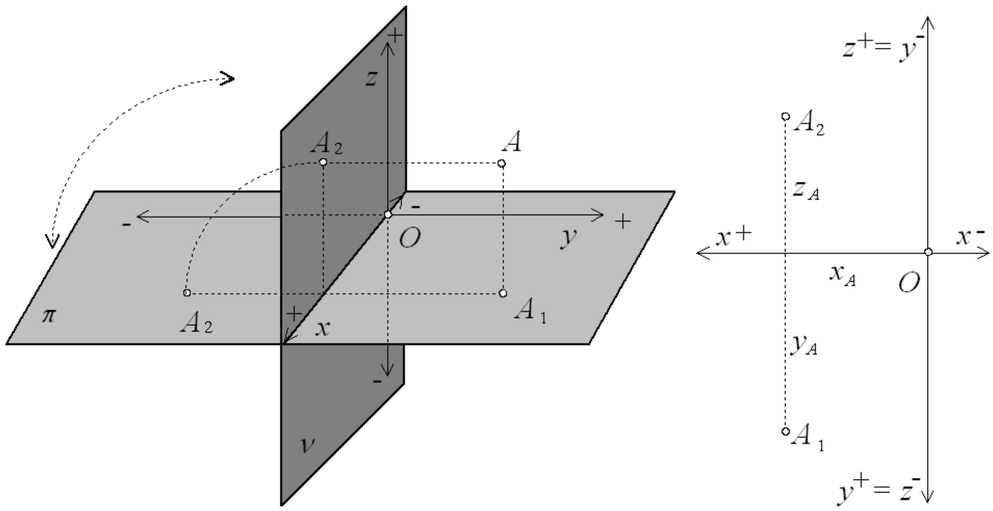


Figure 1. 9 Monge method.

Any point A in the space $\mathbf{E}^3$ can have attached a pair of related views (A_1, A_2), where A_1 is the orthographic view of point A in the ground plane π called the **top (plan, ground) view** of the point A, whereas A_2 is the orthographic view of point A in the frontal plane ν called the **front view** of the point A.
Additional orthographic views can be attached onto the additional image planes, if necessary, for views of more complex and difficult objects.
Choosing new image planes perpendicular to the ground plane and frontal plane, the third and fourth views can be determined, in the right or the left **side (profile) plane**. The fifth image plane is parallel to the ground plane and provides the view from the bottom (Fig. 1. 10).
In the multi-view orthographic method, views of the point $A = [x_A, y_A, z_A]$ are determined by the following Cartesian coordinates in the respective image planes:

$A_1 = [x_A, y_A]$ - groung view, $A_2 = [x_A, z_A]$ - front view

$A_3 = [y_A, z_A]$ - left side view, $A_4 = [y_A + d, z_A]$ - right side view

$A_5 = [x_A, y_A + d]$ - bottom view

for the distance $d \neq 0$ of the two side image planes, or the ground image plane and also the fifth image plane.

Distribution of separate views on a technical drawing according to the European norms is given in Fig. 1. 10 on the right, whereas mapped objects are usually located in the coordinate trihedron $Ox^{+}y^{+}z^{+}$.

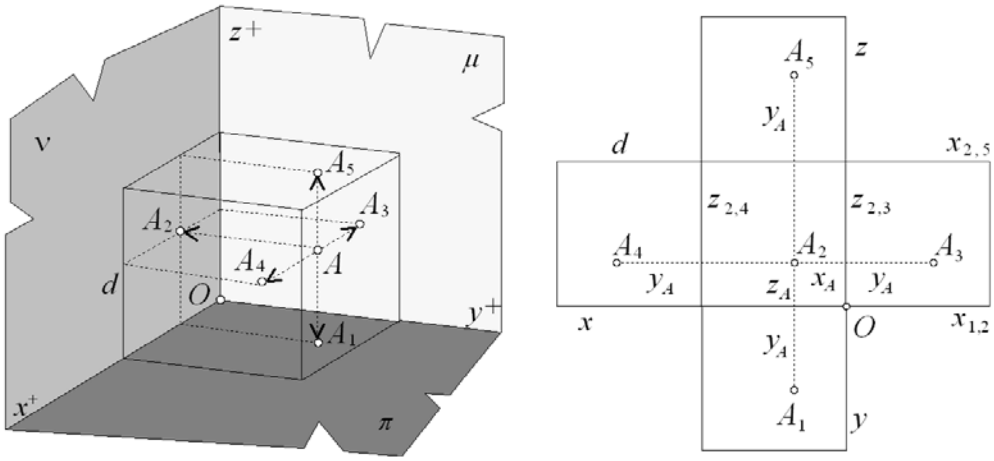


Figure 1. 10 European norm for multi-view projections.

In the American normalisation, mapped figures are located into the coordinate trihedron $Ox^{-}y^{+}z^{-}$ and distribution of the views on a technical drawing is in illustrated the Fig. 1. 11 on the right.

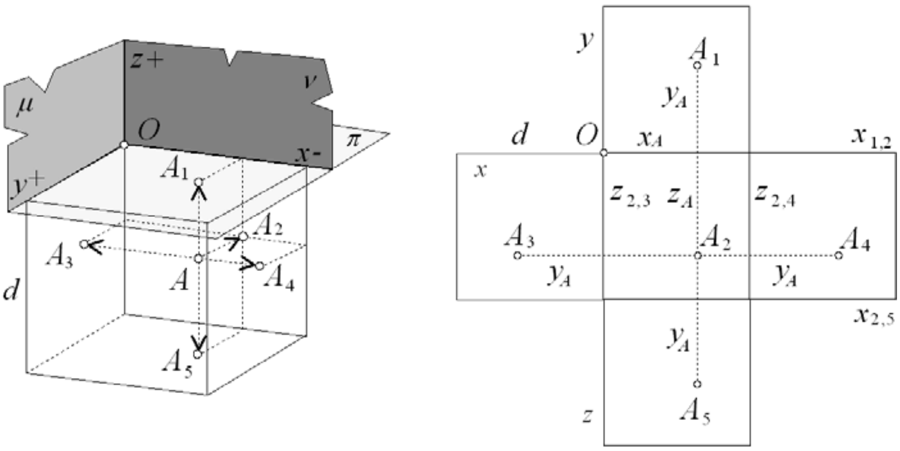


Figure 1. 11 American norm for multi-view projections.

Basic image planes, ground plane π and frontal plane ν, can be placed into one plane by revolving one of them to the other one about the common line in the coordinate axis x. In this way, we get the views‘ relative positions in a one-plane sheet of drawing paper, pairs of point A related views (A_1, A_2), while $A_1A_2 \perp x_1$, $x_1 = x_2$ (line x is located in both image planes, ground plane π and frontal plane ν, therefore its orthographic views to these planes coincide). View of the coordinate axis x is denoted as the reference line $x_{1,\ 2}$. Top view and front view of a point are related in the direction

perpendicular to the reference line (Fig. 1. 9). Top view A_1 of the point A is determined by point Cartesian coordinates x_A and y_A, while front view A_2 is determined by coordinates x_A and z_A.
Perpendicular image planes π and ν divide the space into four quadrants. Coordinates of points located in the separate quadrants satisfy the following inequalities:

$$\text{I. } y_A > 0,\ z_A > 0,\ \text{II. } y_B < 0,\ z_B > 0,\ \text{III. } y_C < 0,\ z_C < 0,\ \text{IV. } y_D > 0,\ z_D < 0.$$

Related views of the points A, B, C, D are presented in Fig. 1. 13.

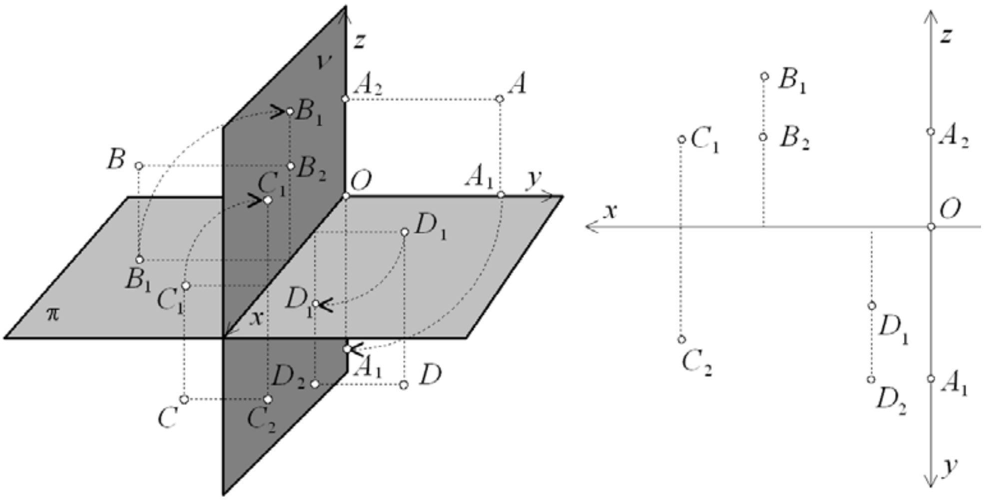


Figure 1. 13 Related views of points A, B, C, D in different quadrants.

Front view of the ground plane and top view of the frontal plane are in the reference line $x_{1,2}$. Any point in the ground plane, $P \in \pi$, has got the front view on the reference line $x_{1,2}$ $(|P\ \pi| = 0)$, and any point in the frontal plane, $N \in \nu$, has also got the top view on the reference line $x_{1,2}$ $(|N\ \nu| = 0)$ (Fig. 1.14).

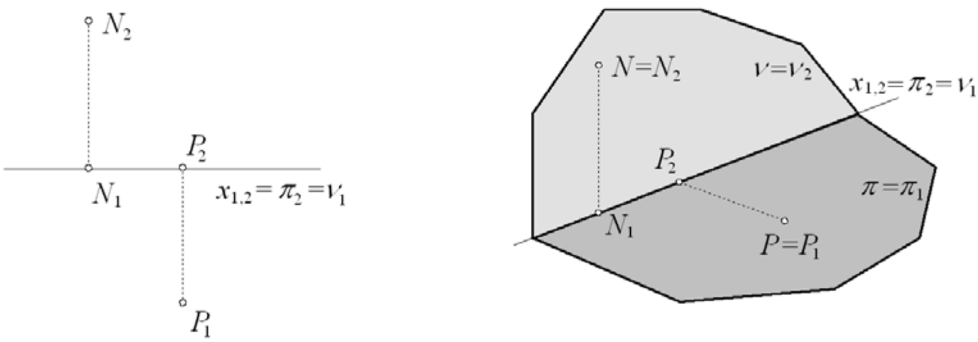


Figure 1. 14 Related views of points P and N from image planes.

1.3 AXONOMETRIC METHOD

Axonometry is a parallel projection onto one image plane - **axonometric image plane** ρ, which does not coincide with any of the coordinate planes π, ν, μ. Space figures can be mapped onto the axonometric image plane together with their orthographic views to the coordinate planes, and the entire coordinate trihedron $Ox^+y^+z^+$. The direction of projection s can be perpendicular to the axonometric image plane ρ, and then we refer to **orthogonal axonometry**.
Axonometric image plane ρ intersects all coordinate planes and axes, Fig. 1. 15.

$$\rho \cap \pi = XY,\ \rho \cap \nu = XZ,\ \rho \cap \mu = YZ$$

Triangle XYZ, with vertices in the intersection points of the coordinate axis x, y, z and axonometric image plane ρ, is the **axonometric triangle**, known also as **Pelz triangle**, as it was discovered by Czech geometer Karel Pelz.

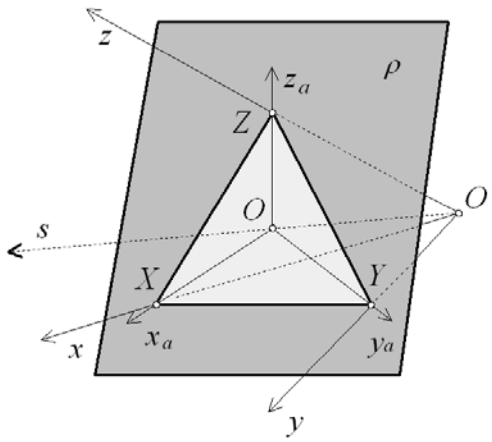


Figure 1. 15 Axonometric projection method.

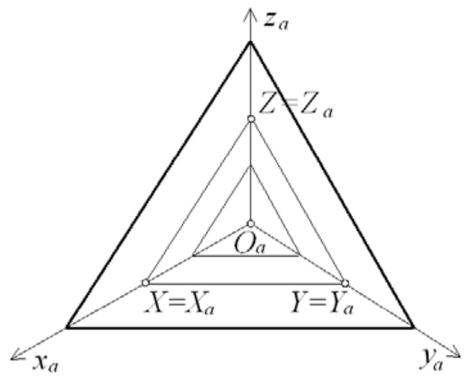


Figure 1. 16 Axonometric triangles.

Orthographic views of the coordinate axis x, y, z in the axonometric image plane ρ are lines x_A, y_A, z_A in the altitudes of the axonometric triangle XYZ. Orthographic view of the origin O in ρ is the orthocentre of the triangle XYZ.
Figure $O_ax_ay_az_a$ is denoted the **axonometric axial cross**. Axonometric image plane ρ can be translated in the direction s, while the size of the axonometric triangle XYZ will change. This occurs with respect to the different distance of the plane ρ and the origin O, nevertheless, the view of the axonometric axial cross will remain unchanged, as illustrated in the Fig. 1. 16. All Pelz triangles are similar to each other, they are scaled by a ratio in the homothety transformation with the centre in the origin O, and they determine the same axonometric projection. The axonometric triangle XYZ is a triangle with all acute angles, in the case of the orthogonal axonometry.
Point A in the space can be mapped orthogonally to the top view A_1 in the ground plane. Orthographic view A_a - **axonometric view** of the point A in the axonometric

image plane ρ, and the orthographic view A_{1a} - **axonometric top view** of the top view A_1 in ρ, form a pair of axonometric views of the point A related in direction of the axis z view, which is perpendicular to the side XY of the axonometric triangle.
If the drawing paper coincides with the axonometric image plane, the indication of the axonometric views can be omitted, and therefore the index "*a*" will not be used in all of the following figures (Fig. 1. 17, on the left).

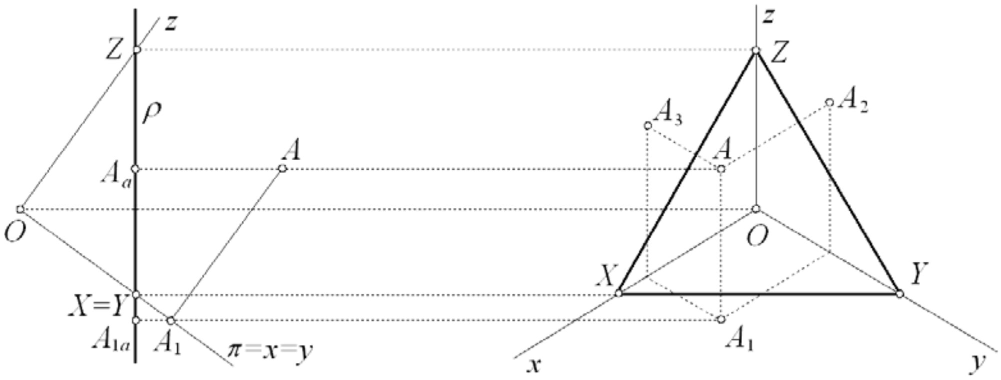


Figure 1. 17 Axonometric projection method.

Point A is unambiguously determined by a pair of its axonometric views (A, A_1). Similarly, any point A in the space can be determined by an ordered pair of axonometric views (A, A_2), which are axonometric view and **axonometric front view**, or alternatively by (A, A_3), axonometric view and **axonometric side view**.
Axonometric top view and axonometric view of the ground plane coincide (z = 0 for all points), axonometric front view of the ground plane is in the view of the axis x, and axonometric side view in the view of the axis y (Fig. 1. 18). All points in the ground plane have coincidental axonometric views and axonometric top views, as $P = P_1$. Similarly, for all points in the frontal plane $N = N_2$, axonometric ground view of the frontal plane is in the view of the axis x, and axonometric side view in the view of the axis z. Axonometric top view of the side plane is in the view of the axis y, axonometric front view in the view of the axis z, and $M = M_3$ for all points in the side image plane.
Visibility in axonometry is applied with respect to the trihedron $Ox^+y^+z^+$, and is related to the axonometric view of the mapped figure. Naturally, those parts of projected objects are visible, which are located in the larger distances from the origin O of the coordinate system. Axonometric view is the view to the part of the space $\mathbf{E}^3$ included in the orthogonal coordinate trihedron $Ox^+y^+z^+$, as mapped objects are usually located in this position with respect to the European norms for technical drawings. Views from other directions can also be obtained, and we speak about special views from the bottom, or from the top, while the coordinate axes are no more mapped to form the axonometric axial cross.

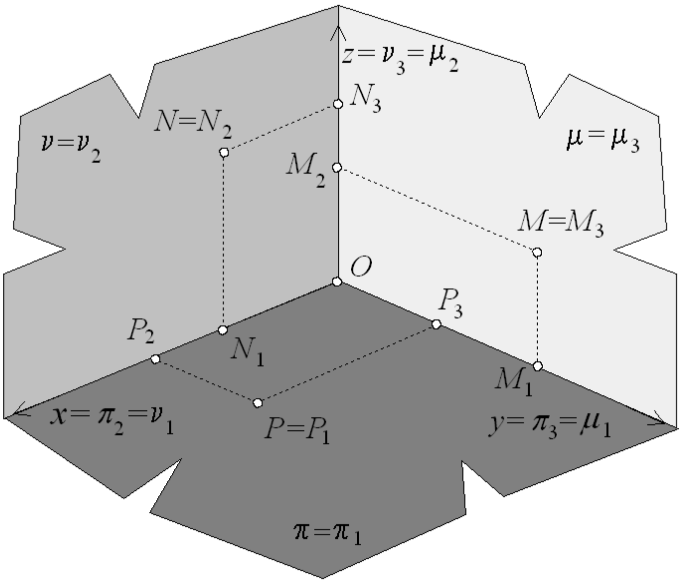


Figure 1. 18 Axonometric views of points in coordinate planes.

Coordinate axes x, y, z form specific angles to the axonometric image plane ρ .
If all these 3 angles are different, then the length of the views of unit line segments j_x, j_y and j_z on coordinate axes are different, axonometric triangle XYZ is a scalene triangle and axonometric projection is a general axonometry - trimetry (Fig. 1. 19a).
If two of the coordinate axes are at the same angles to the axonometric image plane ρ, the unit line segments on these axes are projected equally, the axonometric triangle is an isosceles triangle with two equal sides and the axonometry is called a dimetry (Fig. 1. 19b).
If all coordinate axes x, y, z are at the same angles to the axonometric image plane ρ, views of the unit line segments are equal on all axes, the axonometric triangle is an equilateral triangle and the axonometry is called an isometry (Fig. 1.19c).

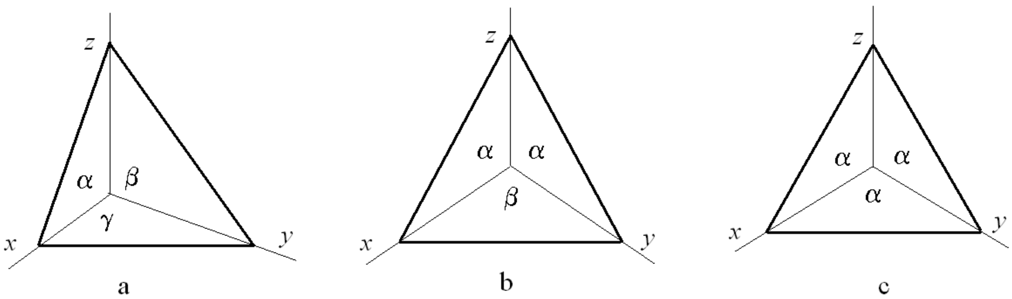


Figure 1. 19 Axonometric triangles for different angles of coordinate axes.

Let the general axonometric projection be given by the image plane ρ and direction of projection s, $s \not\subset \rho$. Numbers p, q, r determining the ratios of the length of views j_x, j_y, j_z of unit line segments j on the coordinate axes

$$p = \frac{j_x}{j}, \quad q = \frac{j_y}{j}, \quad r = \frac{j_z}{j}$$

are **scaling coefficients** (showing the ratio of shortening or lengthening on axonometric views of separate coordinate axis).
The scaling coefficients satisfy the following equation:

$$p^2 + q^2 + r^2 = 2 + \mathrm{cotg}^2\varphi, \ \varphi = \angle\, s\rho$$

where φ is the angle that direction of projection s forms to the image plane ρ .
In the orthogonal axonometry ($s \not\subset \rho$, $\varphi = 90°$), the equation is in the form

$$p^2 + q^2 + r^2 = 2.$$

On technical drawings, where ground views, or side views and profiles of the mapped figures are the most important ones, different oblique axonometric projections are widely used.

Military perspective is a projection method suitable for technical drawings in urban architecture, in design of dwelling s and suburbs. Construction of views of complex objects with complicated ground views and difficult to survey structures is relatively easy. Views of coordinate axes x and y are perpendicular, and the scaling coefficients satisfy the ratio

$p : q : r = 1 : 1 : 1$, it means $j_x = j_y = j_z = k.j$, $k > 0$ (Fig. 1. 20a).

Cavalieri perspective is an oblique projection with the image plane parallel to the frontal plane or the side plane. It was widely used in the 16th and 17th century, for constructions of maps (called *vedutas*) of important towns and settlements. Views of coordinate axes x and z, or y and z are perpendicular, and

$p : q : r = 1 : 1 : 1$, $j_x = j_y = j_z = k.j$, $k > 0$ (Fig. 1. 20b).

Both above projections were used for military purposes, the practical advantage of easy constructions overruled the readability of the pictures and realism of mappings, in both methods angle $\varphi = 45°$.

Oblique projection is a slightly more realistic mapping used mostly on technical drawings in mechanical engineering. Views of coordinate axes x and z, or y and z are perpendicular, and ratio of scale coefficients is

$1 : q : 1$, or $p : 1 : 1$ (while q, or p is from the interval $(0, 1)$).

In so called technical projection, the oriented angle of the coordinate axis y to the view of the coordinate axis x equals to 135°, and $j_y = 1 : 2$, (Fig. 1. 20c).

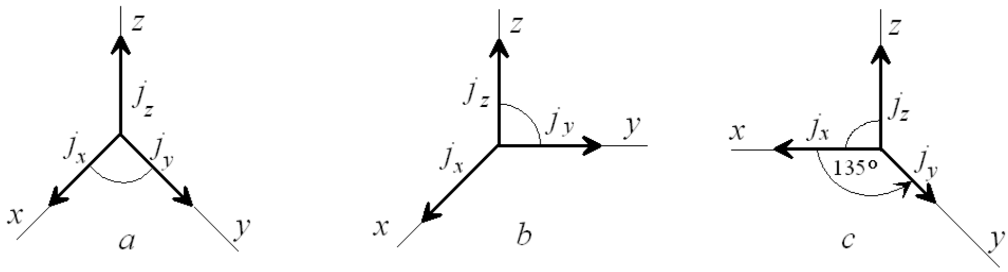


Figure 1. 20 Views of unit line segments on axonometric axes.

Eckhardt intersection method

Axonometric view of the figure can be constructed easily from two given orthographic views, ground view and front view (Fig 1. 21).

By revolving ground plane π about the line XY to the axonometric image plane, the revolved coordinate system $\{{}^1O_0, {}^1x_0, {}^1y_0\}$ can be determined appearing in the true view. Revolved ground plane can be translated in the drawing sheet in the direction of the coordinate axis z, out of the axonometric triangle XYZ, to make the drawing readable, to $\{{}^1O, {}^1x, {}^1y\}$, and the position of the ground view of the point A can be determined by the coordinates x_A , y_A.

Similarly, the frontal plane ν can be revolved about the line XZ to the axonometric image plane, and the revolved coordinate system $\{{}^2O_0, {}^2x_0, {}^2z_0\}$ appearing in the true view can be determined. Revolved frontal plane can be translated in the direction of the coordinate axis y out of the triangle XYZ to the new position $\{{}^2O, {}^2x, {}^2z\}$. Using true values of coordinates x_A and z_A the front view of the point A can be determined.

Axonometric view of the point A can be constructed from the positioned true ground view and front view respectively, using lines in direction of the two used translations. Therefore, axonometric view of the point A is the intersection point of the line passing through the point ${}^1A$ and parallel to the coordinate axis z, and the line passing through the point ${}^2A$ and parallel to the coordinate axis y.

The presented Eckhardt intersection method is widely used for constructions of realistic views of different objects determined by the two orthographic views. In the skew axonometric projection, the position of orthographic views of figures and the direction of translations O^1O, O^2O can be determined arbitrarily, with respect to the realism of the achieved skew axonometric view.

In Fig. 1. 22, the skew axonometric view of a machine part determined by the object front view and its side view is presented. Position of the views and the directions of translations are chosen with respect to the realistic view of a cube.

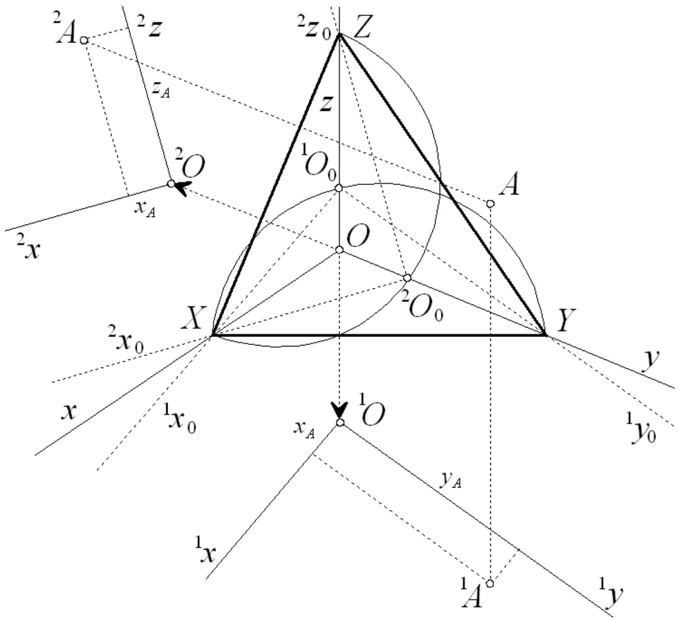


Figure 1. 21 Eckhardt intersection method for orthogonal axonometry.

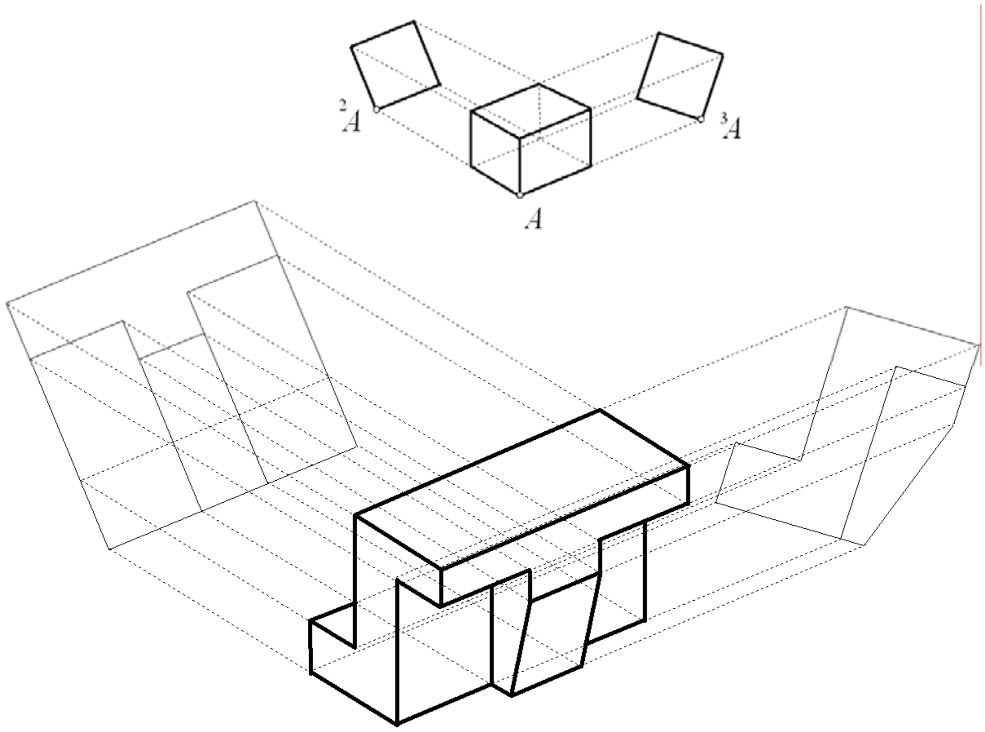


Figure 1. 22 Eckhardt intersection method for skew axonometry constructed from front view and side view of the object.

2 CONIC SECTIONS

Conic sections (or conics, for short) are planar curves with special properties.
In the 3-dimensional geometric space conic sections form a group of curves that might be determined as different forms of conical surface planar intersections, i.e. intersections of conical surface by planes in special superpositions to the surface and its generating lines.
These graceful curves were well-known to ancient Greek geometers, who recognized their great practical importance and discovered many of their interesting geometric properties.
Analytic geometry and calculus enhanced further study of conic sections and introduced many relations between their synthetic geometric representations and analytic representations in the form of equations determining Cartesian coordinates of points on conic sections and their relations.

2.1 ELLIPSE

Ellipses are curves that are of a great practical importance in many fields ranging from art through physical and various technical applications to astronomy.
Any circular object viewed in general angle forms an ellipse, all orbiting satellites, natural or artificial, move in on elliptic paths. Oval elliptic forms are the most commonly used decorative patterns in architecture, and flashlight illuminates areas with mostly elliptic boundaries and shadows.

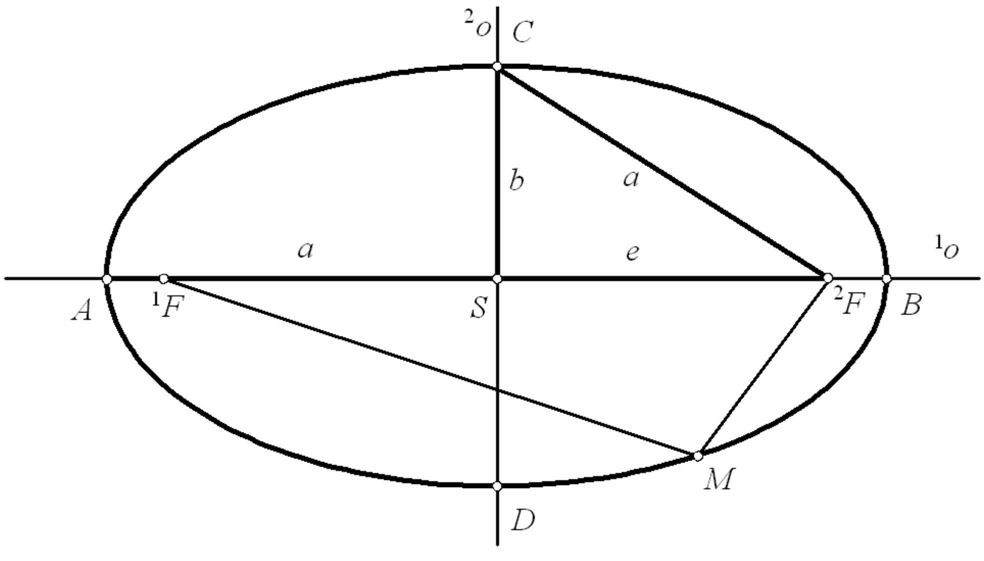


Figure 2. 1 Definition of ellipse in Euclidean plane.

Definition 1. An ellipse is a set of all points in the Euclidean plane such that the sum of the distances from any point M to two fixed points ${}^1F$ and ${}^2F$ is constant. Here ${}^1F$ and ${}^2F$ are called the focal points, or the foci of the ellipse.

Midpoint S of the line segment ${}^1F\,{}^2F$ is called the **centre** of the ellipse, Fig. 2. 1.
The distance between the centre S and either **focus** ${}^1F$ or focus ${}^2F$ is called the **linear eccentricity** of the ellipse and it is denoted by e. Notice that the ellipse is symmetric about the line through the foci ${}^1F$ and ${}^2F$.
Let A and B be the points where the line through points ${}^1F$ and ${}^2F$ intersects the ellipse. Centre S bisects the line segment AB. The ellipse is symmetric also with respect to the line through S and perpendicular to the line AB. Let C and D be the points where this perpendicular intersects the ellipse. The four points A, B, C, D are called the **vertices** of the ellipse, here A, B are **major vertices** and C, D are **minor vertices**. The line $AB = {}^1o$ is called the **major axis**, and the line $CD = {}^2o$ the **minor axis** of the ellipse.
Let a denote the length of the line segment $SA = SB$, and b the length of the line segment $SC = SD$. The number a is called the **semi-major axis** and the number b the **semi-minor axis** of the ellipse. Hence

$$|{}^1F\,C| + |{}^2F\,C| = 2a \text{ and } |{}^1F\,D| + |{}^2F\,D| = 2a,$$

the distance between minor vertex C or D and either focus ${}^1F$ or ${}^2F$ equals to a.
Applying the Pythagorean Theorem to the right angled triangle $CS{}^2F$ we find that

$$a^2 = b^2 + e^2,$$

which is the geometric equation of the ellipse.

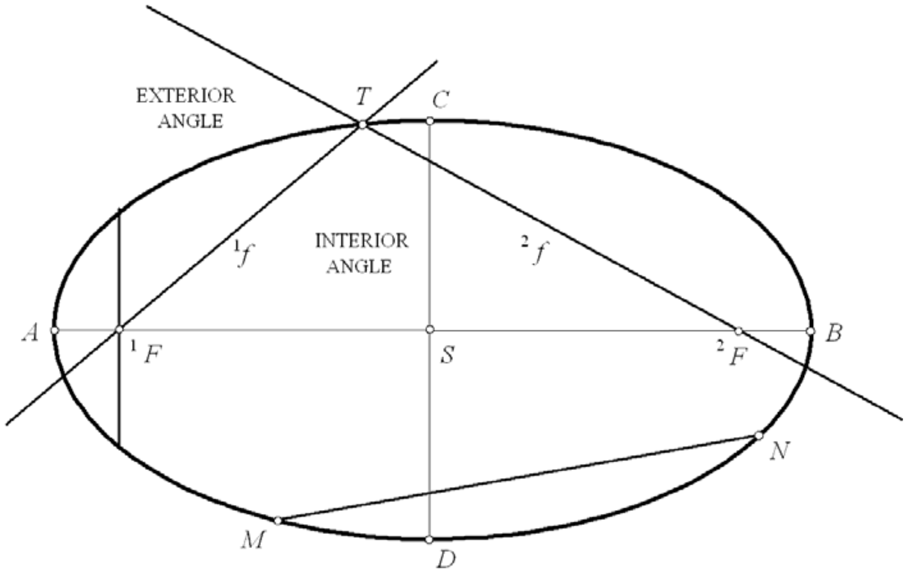


Figure 2. 2 Focal lines and chord of ellipse.

Lines passing through an arbitrary point T of the ellipse and either focus ${}^1F$ or ${}^2F$, ${}^1f = {}^1FT$, ${}^2f = {}^2FT$, are called focal lines. Line segments ${}^1FT$ and ${}^2FT$ located on the focal lines in the point T indicate the distances of the ellipse point T to the foci, the sum of which is constant and according to the ellipse definition equals $2a$.

$$|{}^1F\,T| + |{}^2F\,T| = 2a$$

Focal lines in the point T form two angles with the common vertex in their common point T on the ellipse. These are:

1. interior angle in which the centre of the ellipse is located,
2. exterior angle in which the major vertices A and B are located,

as illustrated in Fig. 2. 2.
Any line segment MN determined by endpoints M, N on the ellipse is called the **chord** of the ellipse. A focal chord is passing through a focus perpendicularly to the major axis of the ellipse.
If the ellipse is placed in Euclidean plane $\mathbf{E_2}$ with defined Cartesian coordinate system $\{O, x, y\}$ so that the centre S is located in the origin O, and the two fixed points, foci ${}^1F = [-e, 0]$ and ${}^2F = [e, 0]$, both lie on negative and positive portions of the coordinate axis x respectively, then the analytic equation of the ellipse can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a is the semi-major axis, b is the semi-minor axis, and $a^2 = b^2 + e^2$.

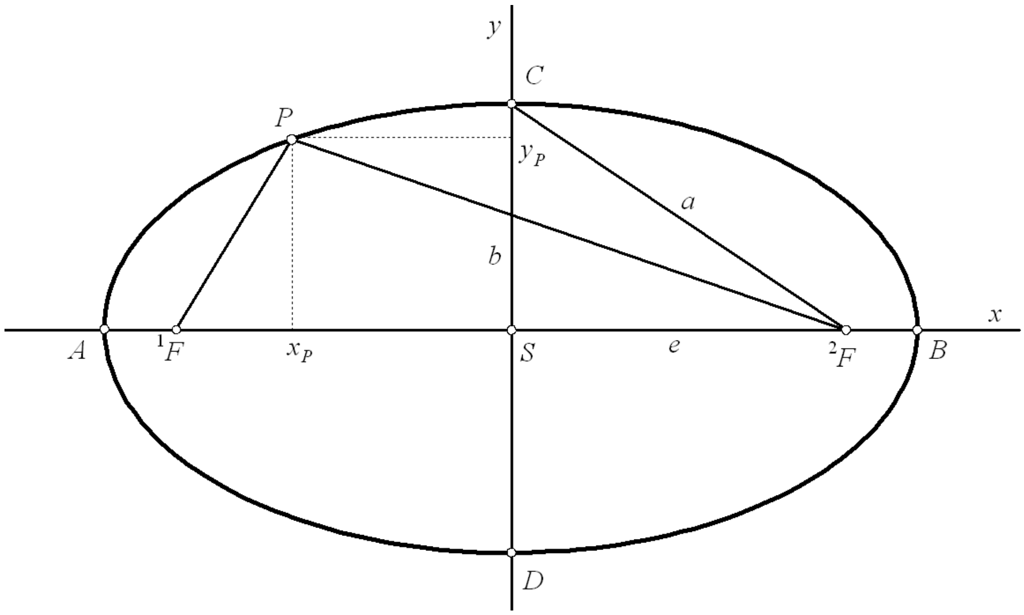


Figure 2. 3 Ellipse in Cartesian plane.

Let $P = [x_P, y_P]$ be a point on the ellipse, see Fig. 2. 3. Then the equation holds

$$|{}^1F\,P| + |{}^2F\,P| = 2a,$$

from which follows

$$\sqrt{(x_P + e)^2 + y_P^2} + \sqrt{(x_P - e)^2 + y_P^2} = 2a\,.$$

Squaring the above equation and after some manipulations we have

$$ex_P - a^2 = -a\sqrt{(x_P - e)^2 + y_P^2}\,.$$

Squaring the last equation we obtain

$$a^2\left(a^2 - e^2\right) = \left(a^2 - e^2\right)x_P^2 + a^2 y_P^2\,.$$

Since $a^2 = b^2 + e^2$, then $a^2 - e^2 = b^2$, and the equation above can be rewritten as

$$a^2 b^2 = b^2 x_P^2 + a^2 y_P^2\,.$$

Dividing both sides of the above equation by a^2b^2, we receive

$$1 = \frac{x_P^2}{a^2} + \frac{y_P^2}{b^2}\,,$$

which is the presented equation of the ellipse.

Conversely, it can be shown that if the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ holds, then the point $P = [x, y]$ is located on the ellipse with the foci at ${}^1F = [-e, 0]$, ${}^2F = [e, 0]$ and

$$|{}^1F\,P| + |{}^2F\,P| = 2a.$$

Note: If a and b are positive constants, $a > b$, the above Cartesian equation is called the standard form for the equation of an ellipse with centre at the origin O and with horizontal major axis ${}^1o$ in coordinate axis x. The standard equation of an ellipse with the same centre at the origin O but with vertical major axis in coordinate axis y as in Fig. 2. 4, is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\,,\; a > b\,.$$

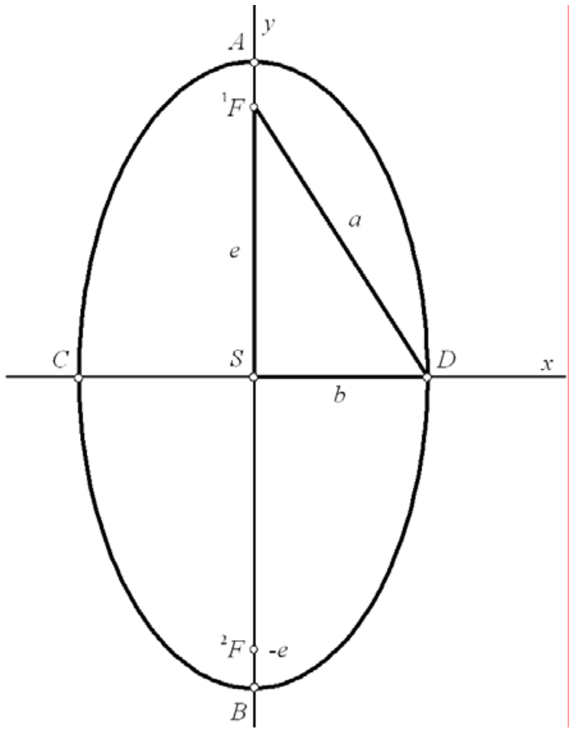


Figure 2. 4 Ellipse with vertical major axis.

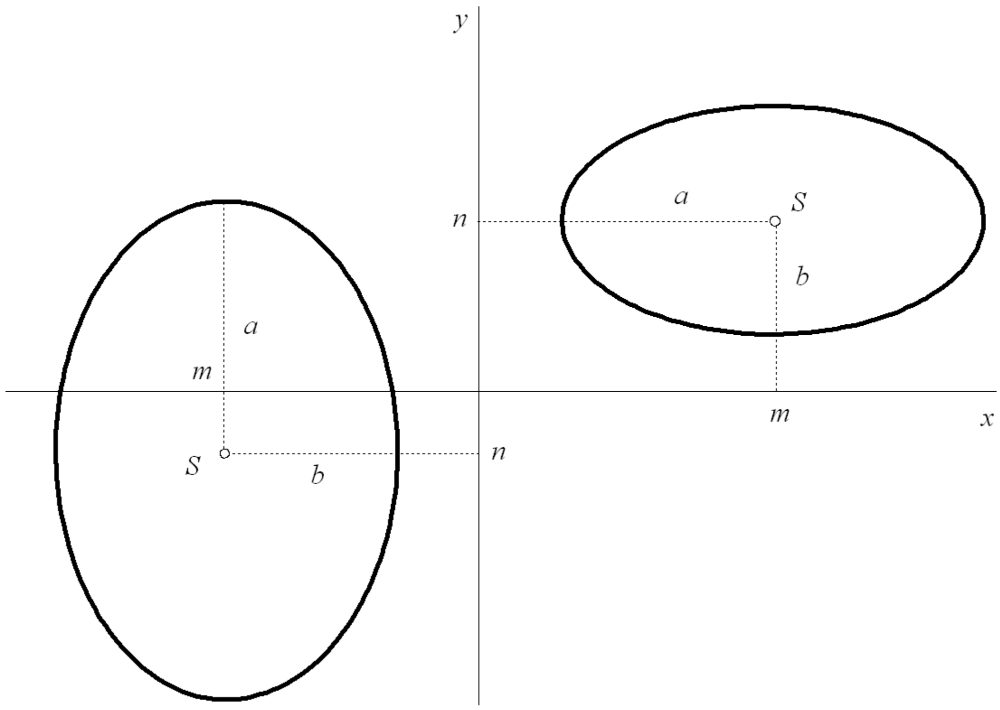


Figure 2. 5 Ellipses with shifted centres.

Let the ellipse with centre S at the origin, semi-major axis a, semi-minor axis b, $a > b$, and axes in the coordinate axes x and y be shifted so that the centre is at the point $S = [m, n]$. The equation of the ellipse shifted into a new position will have one of the following standard forms

$$\frac{(x-m)^2}{a^2}+\frac{(y-n)^2}{b^2}=1, \quad \frac{(x-m)^2}{b^2}+\frac{(y-n)^2}{a^2}=1$$

for the horizontal major axis, and vertical major axis respectively, Figure 2. 5.

Parametric equations of an elliptic arc with centre in the origin of the coordinate system O corresponding to the central angle $\varphi \in \langle 0, 2\pi \rangle$, with axes in the coordinate axis x and y and semi-axes a and b (Fig. 2. 6) are in the following vector form

$$\mathbf{r}(u) = (x(u), y(u)) = (a\cos\varphi u, b\sin\varphi u), \quad u \in \langle 0,1 \rangle,$$

while vector equation of the ellipse with the shifted centre in the point $S = [m, n]$ is

$$\mathbf{r}(u) = (x(u), y(u)) = (m + a\cos\varphi u, n + b\sin\varphi u), \quad u \in \langle 0,1 \rangle.$$

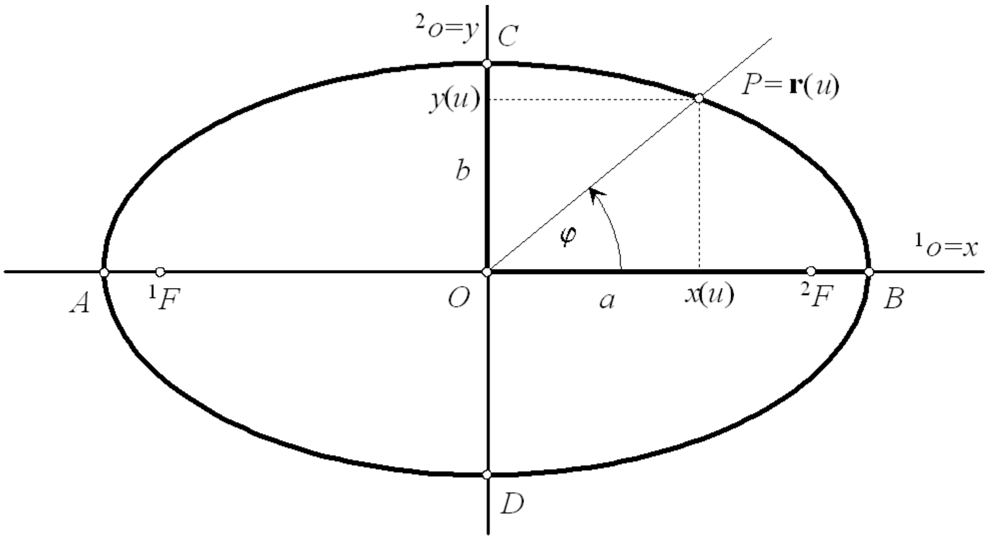


Fig. 2. 6 Vector form of the parametric equations of an ellipse.

To draw an ellipse, we can find 4 circles that approximate ellipse in its four vertices, which are called hyper-osculating circles. These circles are symmetric in pairs with respect to the axes of the ellipse, and have equal radii. The 2 circles in the major vertices are symmetric according to the minor axis, and the other 2 circles in the minor vertices are symmetric with respect to the major axis. The following construction of their centres and radii is illustrated in Fig. 2. 7.

Construction 1.

1. Through points C and B construct lines parallel to the axes of ellipse – lines AB and CD.
2. Find intersection point K and construct a perpendicular line k to the line CB through the point K.
3. Find intersection points of line k and axes of ellipse, point S_B on the axis AB and point S_C on the axis CD.
4. Find point S_A symmetric to S_B on axis AB with respect to the centre S, and point S_D symmetric to S_C on axis CD with respect to the centre S.
5. Construct 4 circles with centres S_A, S_B, S_C, S_D and passing through vertices of ellipse A, B, C, D, respectively.

Any line in a general position can:

1. be tangent to the ellipse in one point,
2. intersect the ellipse in two different points,
3. have no common points with the ellipse.

All three possible positions of an ellipse and a line are illustrated in Fig. 2. 8.

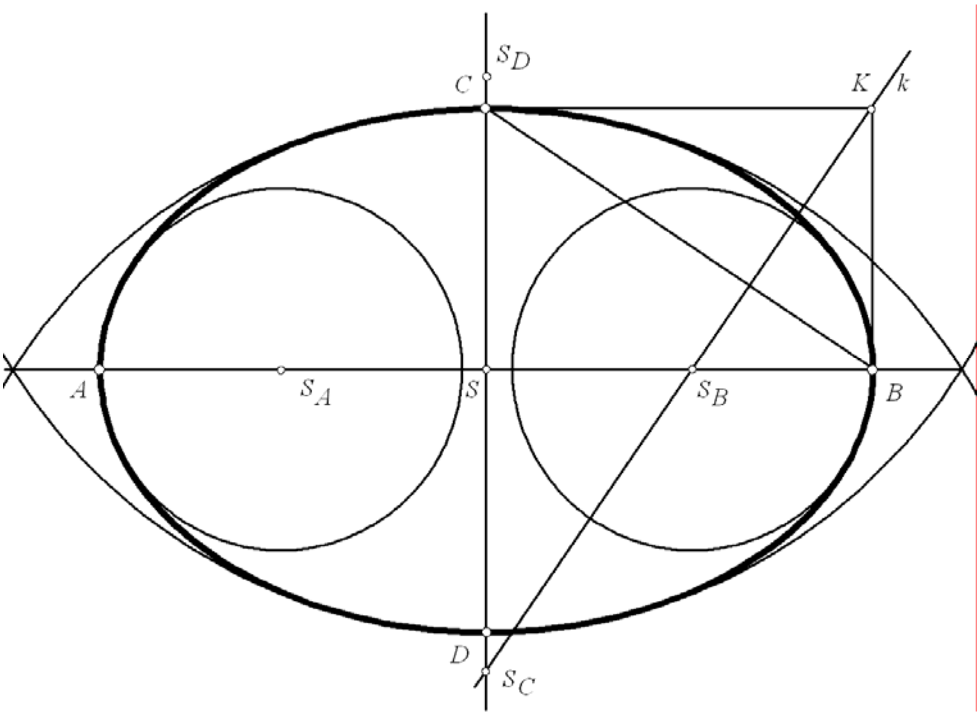


Figure 2. 7 Hyper-osculating circles in vertices of ellipse.

Tangent line (or tangent, for short) to the ellipse contains no interior points of the ellipse and it has one single common point with the ellipse called tangent point.

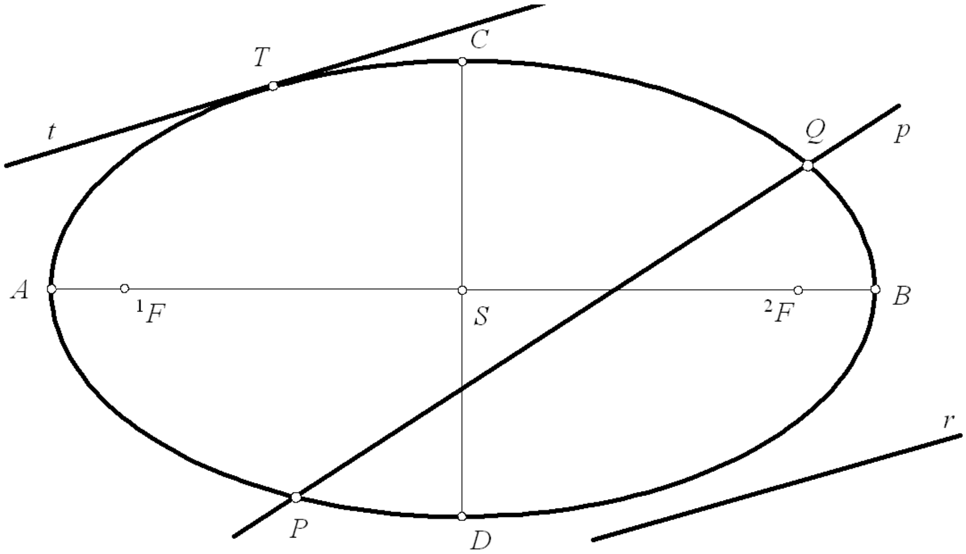


Figure 2. 8 Line and ellipse: *t* is tangent in the point *T*, *p* intersects ellipse in points *P* and *Q*, and *r* does not intersect the ellipse.

Definition 2. Line *t* is tangent to the ellipse in the point *T* iff it is the axis of the exterior angle formed by the focal lines in the tangent point *T*.

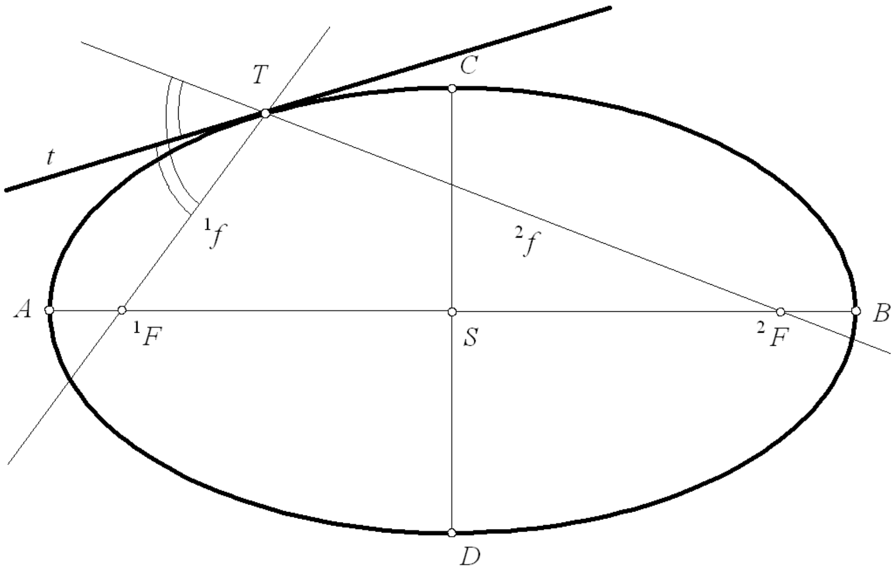


Figure 2. 9 Tangent to the ellipse.

Theorem 1. All intersection points of tangents to the ellipse and their perpendiculars through the ellipse foci ^1F and ^2F are located on one circle with a centre in the centre of the ellipse and radius equal to the ellipse semi-major axis. Circle $v(S, a)$ is denoted the **auxiliary circle**.

Theorem 2. All points symmetric to one focus of the ellipse with respect to all tangents to the ellipse are located on one circle with a centre in the other focus and radius equal to the doubled semi-major axis of the ellipse. Circle $^1g\ (^2F, 2a)$ is denoted the **control circle** and it is related to the focus ^1F, circle $^2g(^1F, 2a)$ is related to the focus ^2F.

Theorem 3. There exist two different tangent lines to the ellipse that are passing through an arbitrary exterior point of the ellipse, or that are parallel to an arbitrary direction.

Corollary 1. Tangent points of two tangent lines to the ellipse that are parallel to each other form a line segment with endpoints on the ellipse and midpoint in the centre of the ellipse.

Illustration of auxiliary circle $v(S, a)$ that is circumscribing the ellipse is presented in Fig. 2. 10, where two points ^1P and ^2P are constructed on perpendiculars passing through both foci ^1F, ^2F to the tangent line t in point T on the ellipse. The control circle $^1g\ (^2F, 2a)$ related to the focus ^1F of the ellipse is illustrated in Fig. 2. 11.

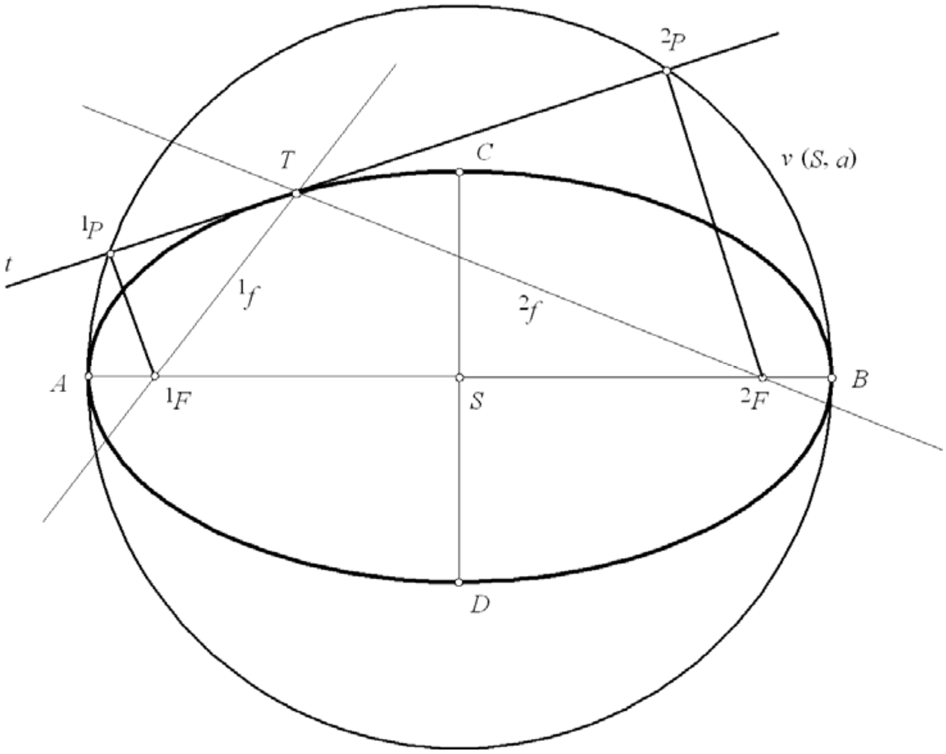


Figure 2. 10 Auxiliary circle.

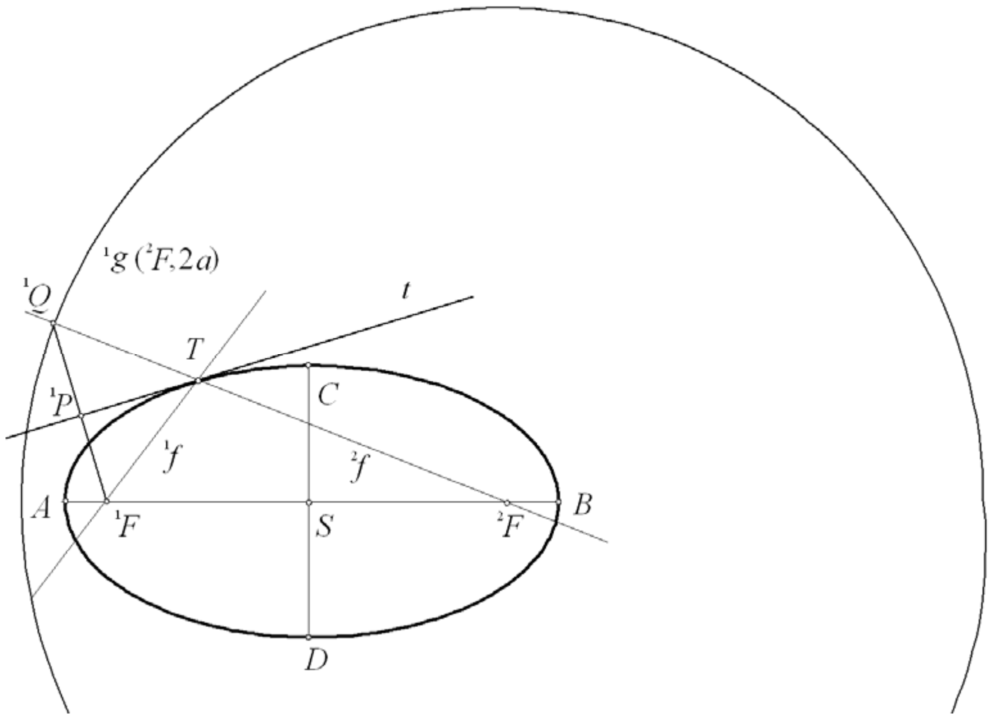


Figure 2. 11 Control circle.

A line that is not tangent to the ellipse and has at least one common point with the ellipse intersects the ellipse in one more point.
Lines passing through the centre of the ellipse, and intersecting ellipse in two points, are called **diameters**.
Distance of the two intersection points of an ellipse and its diameter varies in interval $\langle 2b, 2a \rangle$, where a and b are the semi-axes of the ellipse.

Definition 3. Any two diameters of an ellipse are called **related or conjugate diameters**, if they satisfy the following property: Tangent lines to the ellipse in the endpoints of one of the diameters are parallel to the other diameter.

Corollary 2. Tangent lines at the endpoints of a pair of related diameters of an ellipse form a tangent parallelogram subscribed to the ellipse.

In Fig. 2. 12 tangent lines p and q at ellipse points P and Q are parallel, while tangent points form diameter PQ, which is parallel to tangent lines t and r in the ellipse points T and R. These two tangent points form diameter TR of the ellipse that is parallel to the tangent lines p and q. Therefore diameters PQ and RT are conjugate diameters of the ellipse.

Note: Semi-major and semi-minor axes form the only pair of related diameters of an ellipse such that the two diameters are perpendicular to each other, and consequently the subscribed tangent parallelogram is a rectangular oblong.

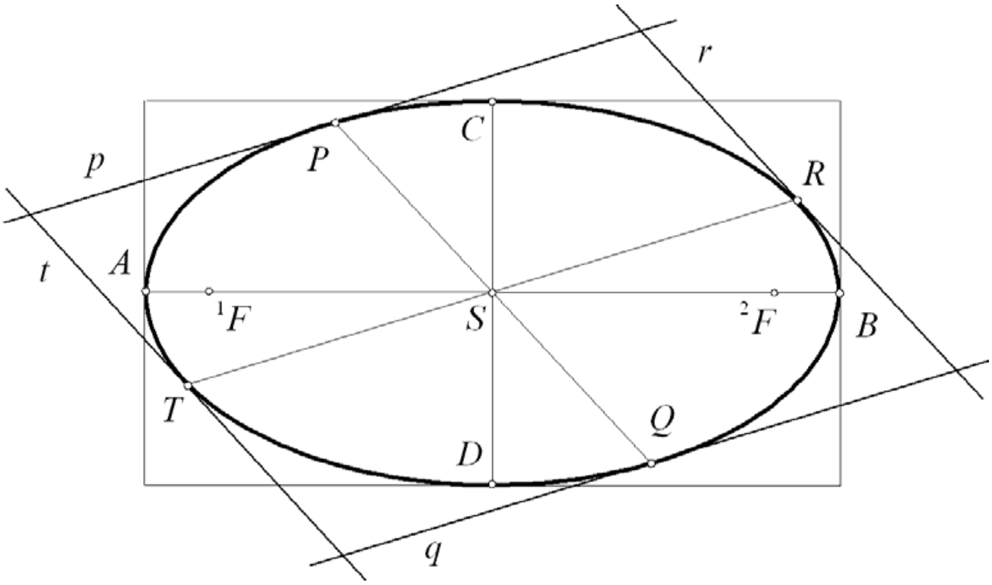


Figure 2. 12 Conjugate (related) diameters of ellipse.

2.2 PARABOLA

Parabolas are planar curves with a specific reflecting property.
The light, sound, or electromagnetic radiation emanating from the focus of a parabolic reflector is always reflected parallel to the axis of the parabola. Thus, if an intense source of light such as a carbon arc or an incandescent filament is placed at the focus of a parabolic mirror, the light is reflected and projected in a parallel beam.
The same principle is used in reverse in a reflecting telescope - parallel rays of light from a distant object are brought together at the focus of a parabolic mirror and in satellites.
A ball thrown up at an angle travels along a parabolic arc, a main cable in a suspension forms an arc of a parabola, and the familiar "dish antennas" have parabolic cross sections.

Definition 4. A parabola is a set of all points in the Euclidean plane such that the distance from *M* to a fixed point *F* is equal to the distance from *M* to a fixed line *d*. Here point *F* is called the focus of the parabola, and line *d* is called the directrix of the parabola.

Fig. 2. 13 shows the fixed point *F* and the fixed line *d* and the point *M* of the parabola in the same distance from them. Here we have

$$|MF| = |M\,d|\,.$$

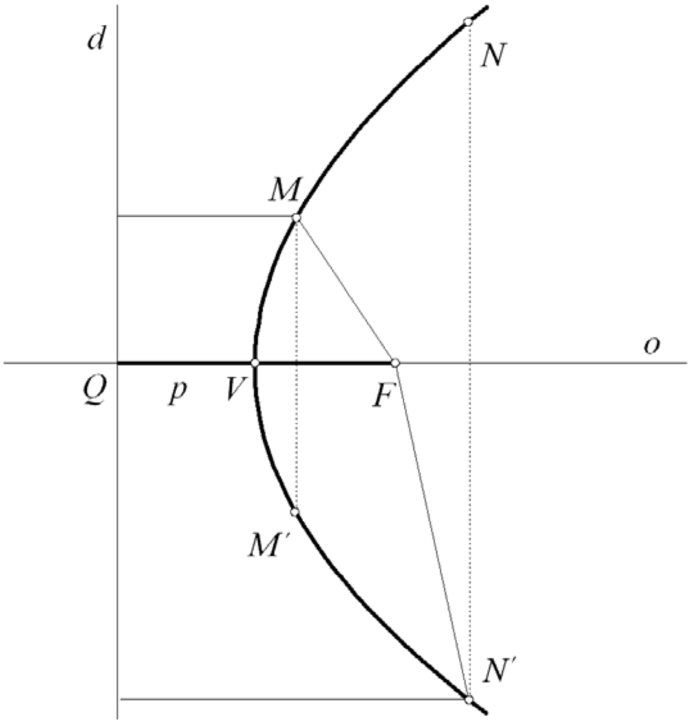


Figure 2. 13 Definition of parabola in plane.

The distance between the **directrix** d and **focus** F is called the **parameter of parabola,** and it is denoted by $p = |FQ|$. Midpoint V of the line segment FQ perpendicular to line d and passing through focus F is called the **vertex** of the parabola, see Fig. 2. 13. The distance of the vertex V to the directrix d is

$$|VF| = |Vd| = |VQ| = \frac{p}{2}.$$

Notice that the parabola is symmetric about the line through focus F and perpendicular to directrix d, symmetric pairs of points M, M' and N, N' are shown in Fig. 2. 13. This axis of symmetry is called **axis of parabola** and denoted by o. Axis o intersects parabola in the vertex V of parabola.

Lines passing through the arbitrary point M of the parabola and through the focus F or that are perpendicular to the directrix d (parallel to the axis o) are called **focal lines** in the point M, ${}^1f = FM$, $M \in {}^2f \parallel o$. Line segments FM and MQ located on the focal lines ${}^1f$, ${}^2f$ in the point M indicate the distances of the parabola point M to the focus and directrix, and they are equal. Focal lines in the point M form two angles with common vertex in their common point M on the parabola, exterior angle is the one in which the vertex V of the parabola is located, as illustrated in Fig. 2. 14.

Any line segment MN determined by endpoints M, N on the parabola is called the **chord** of the parabola. A focal chord is passing through a focus perpendicularly to the axis of the parabola.

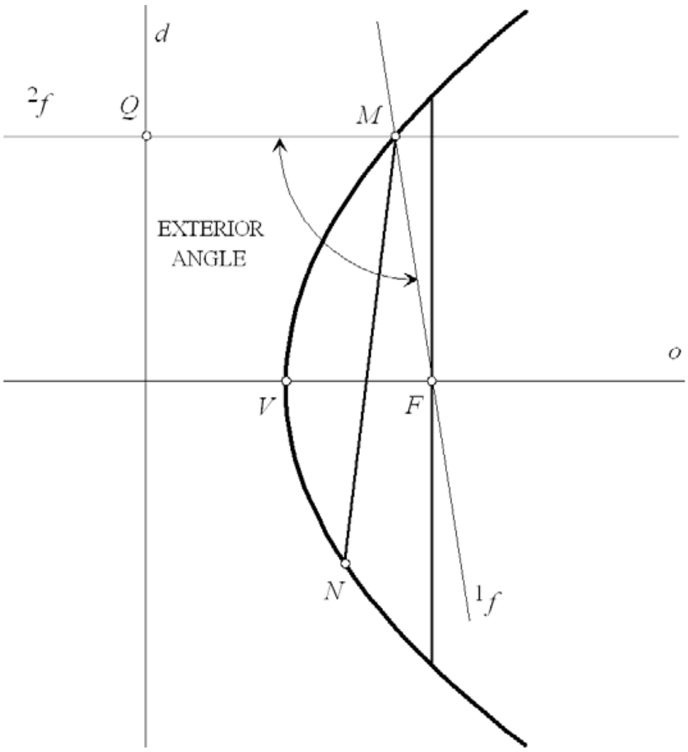


Figure 2. 14 Focal lines and chord of a parabola.

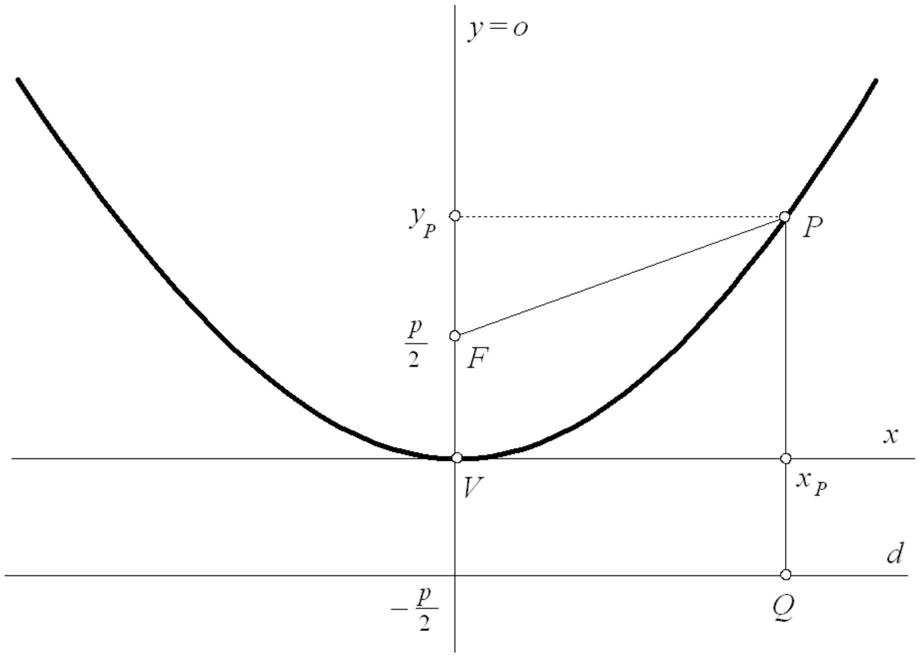


Figure 2. 15 Parabola in Cartesian plane.

If a parabola is placed in the Euclidean plane $\mathbf{E}_2$ with defined Cartesian coordinate system $\{O, x, y\}$ so that the vertex V is located at the origin O, and the focus F lies on the positive portion of the coordinate axis y, in the point $F = [0, \frac{p}{2}]$, and the directrix is the line with equation $y = -\frac{p}{2}$, then the analytic equation of the parabola can be derived in the form

$$x^2 = 2py\text{, or } y = \frac{x^2}{2p}$$

where $p > 0$ is the parameter of parabola.

Let $P = [x_P, y_P]$ be any point of the parabola, and $Q = [x_P, -\frac{p}{2}]$ be a point at the foot of the perpendicular from P to the directrix d. Then the equation from the parabola definition holds, $|PF| = |PQ|$, from which follows

$$\sqrt{x_P^2 + \left(y_P - \frac{p}{2}\right)^2} = \sqrt{\left(y_P + \frac{p}{2}\right)^2}$$

Squaring the above equation we receive

$$x_P^2 + y_P^2 - py_P + \frac{p^2}{4} = y_P^2 + py_P + \frac{p^2}{4}$$

and finally the presented equation of the parabola can be derived

$$x_P^2 = 2py_P\text{, or } y_P = \frac{x_P^2}{2p}.$$

Conversely, it can be shown that if the equation $y = \frac{x^2}{2p}$ holds for the coordinates of the point $P = [x_P, y_P]$, then point P is located on the parabola with the focus at $F = [0, \frac{p}{2}]$ and directrix with equation $y = -\frac{p}{2}$, and the equality $|PF| = |Pd|$ holds.

Note: Obviously, modifications of the position of fixed elements provide Cartesian equations for the parabolas downward, modifications of argument provide parabolas opened to the right or to the left. We assume that vertex V of the parabola is at the origin and $p > 0$ for each situation.

The standard equation of the parabola with axis in the coordinate axis y, focus in the point $F = [0, -\frac{p}{2}]$ and directrix in the line $y = \frac{p}{2}$ in Fig 2.16 is

$$x^2 = -2py \text{, or } y = -\frac{x^2}{2p}$$

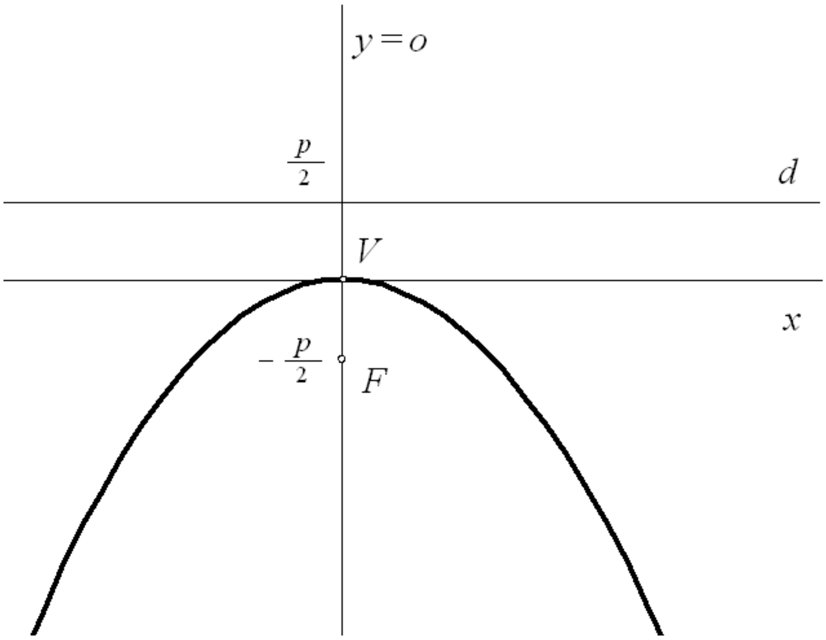


Figure 2. 16 Parabola opened downwards.

The standard equation of the parabola with the horizontal axis in the coordinate axis x, focus in the point $F = [-\frac{p}{2}, 0]$ and directrix in the line x = $\frac{p}{2}$ in Fig 2.16 on the left is

$$y^2 = -2px.$$

If the parabola is placed in the Cartesian plane so that the vertex V is located at the origin O, and the focus F lies on the positive portion of the coordinate x in the point $F = [\frac{p}{2}, 0]$ in Fig 2.16 on the right, while directrix is the line with equation $x = -\frac{p}{2}$, where p is the parameter of parabola, then the analytic equation of the parabola can be written as

$$y^2 = 2px.$$

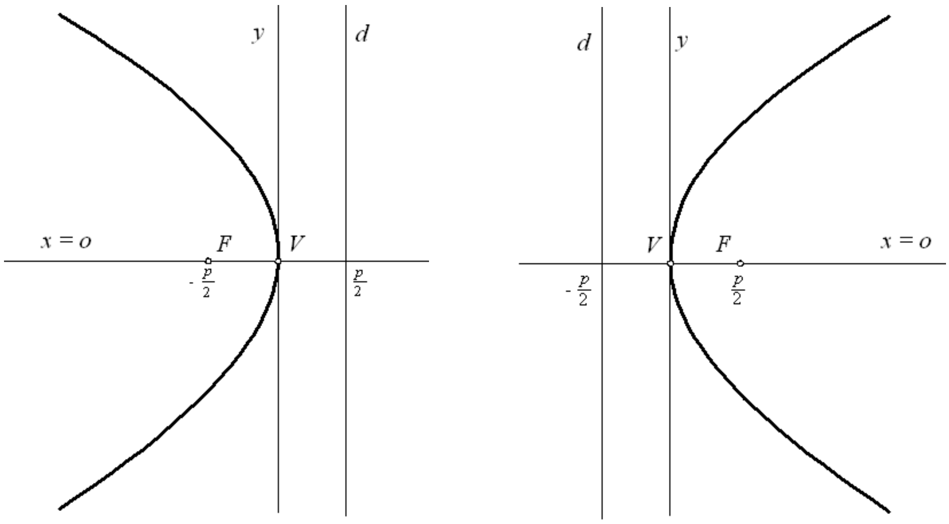


Figure 2. 17 Parabolas in Cartesian plane with horizontal axes.

Let the parabola with the vertex V at the origin, parameter p, and axis in the coordinate axes x or y be shifted so that the vertex is at the point $V = [m, n]$. The equation of a parabola shifted into a new position will have one of the following standard forms

$$(y-n)^2 = \pm 2p(x-m), \quad (x-m)^2 = \pm 2p(y-n)$$

for the horizontal axis, and vertical axis respectively, Fig. 2. 18.

Parametric equations of a parabolic arc with vertex in the origin of the coordinate system O, axis in the coordinate axis y and parameter p corresponding to the interval $\langle -a, a\rangle$ on the coordinate axis x (Fig. 2. 19) can be written in the vector form as

$$\mathbf{r}(u) = (x(u), y(u)) = \left(a(2u-1), \pm\frac{u^2}{2p}\right), \quad u \in \langle 0,1\rangle .$$

Vector equation of a parabolic arc with vertex in the point $V = [m, n]$, axes in the line parallel to the coordinate axis y and passing through the points $A = [a, 0]$ and $B = [2m - a, 0]$ on coordinate axis x (Fig. 2. 20) is in the following vector form

$$\mathbf{r}(u) = (x(u), y(u)) = \left(2(m-a)u + a, 4n\left(-u^2 + u\right)\right), \quad u \in \langle 0,1\rangle .$$

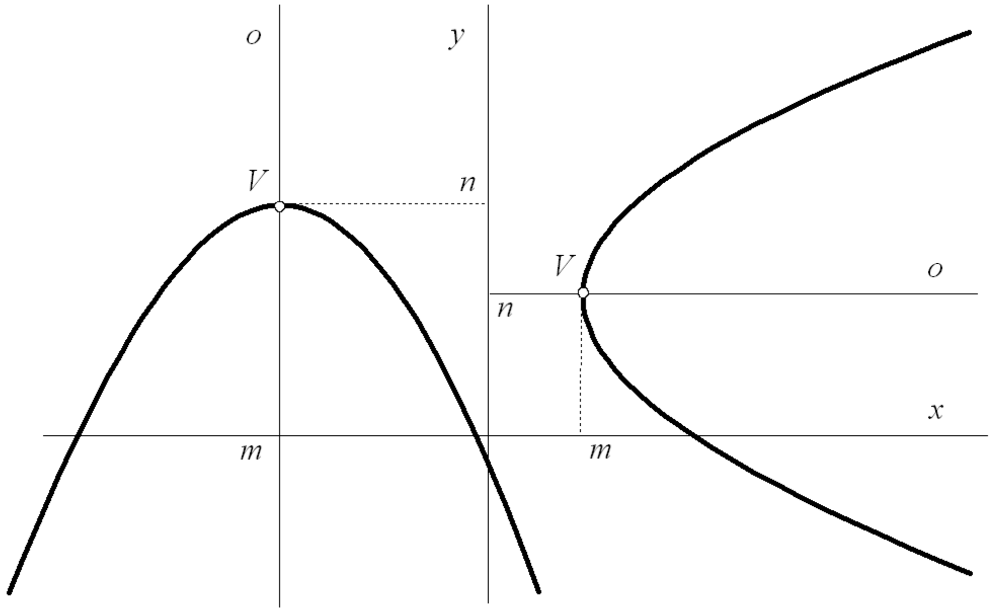


Figure 2. 18 Parabolas in Cartesian plane with shifted vertices.

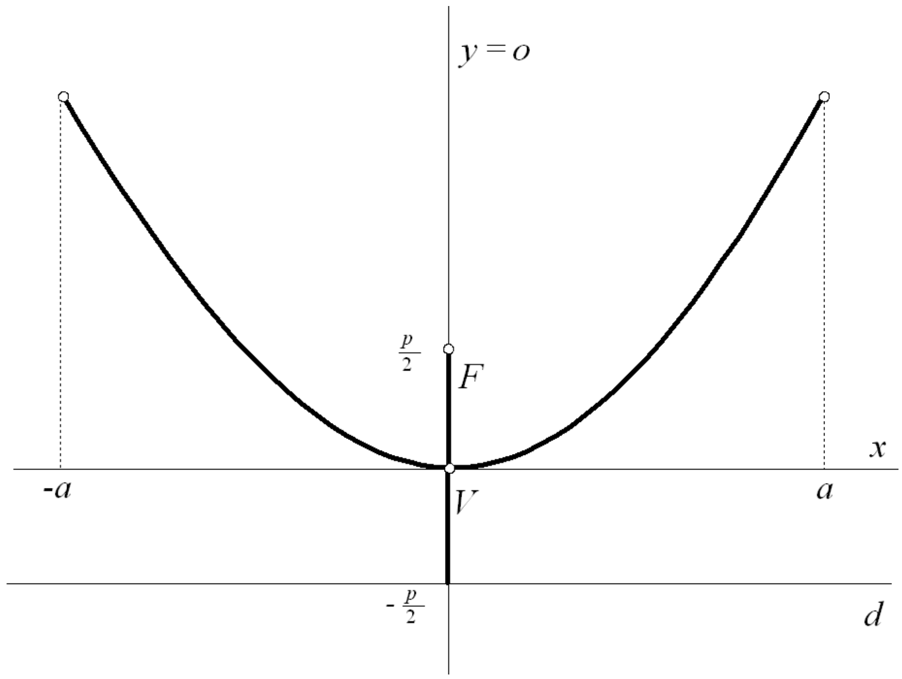


Figure 2. 19 Vector equation of parabola in Cartesian plane.

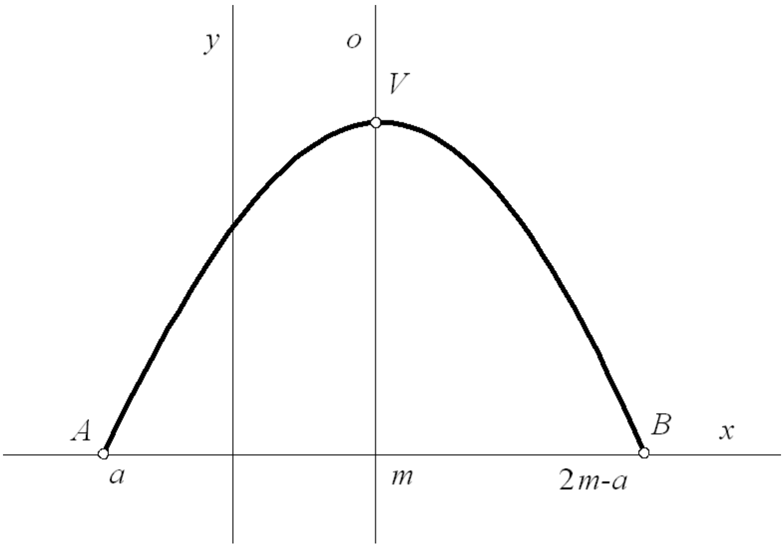


Figure 2. 20 Vector equation of parabola with shifted vertex.

To draw a parabola, we can find a circle approximating it in the vertex that is called hyper-osculating circle, and some points on the parabola can also be constructed, Fig. 2. 21.

Construction 2.

1. Centre *S* of the osculating circle is on the axis *o* of a parabola, and it is in a distance *p* from the vertex of parabola *V*. Parameter *p* is the radius of the hyper-osculating circle.
2. Find points *P* and *P´* on the parabola, which are in the same given distance *a* from the focus *F* and directrix *d*.
3. Construct a line parallel to directrix *d* in a distance *a* from it.
4. Construct a circle with centre in the focus *F* and with radius equal to the distance *a*.
5. Find two intersection points of the parallel line and the circle – these are points *P* and *P´* on the parabola.
6. To find other two symmetric points *Q* and *Q´* on the parabola, which are in a given distance *b* from the focus *F* and directrix *d*, follow the steps 3 – 5 using the distance *b*.

Any line in a general position can:

1. be tangent to the parabola in one point,
2. intersect the parabola in two different points,
3. have no common points with the parabola.

All three possible positions of a parabola and a line are illustrated in Fig. 2. 22.

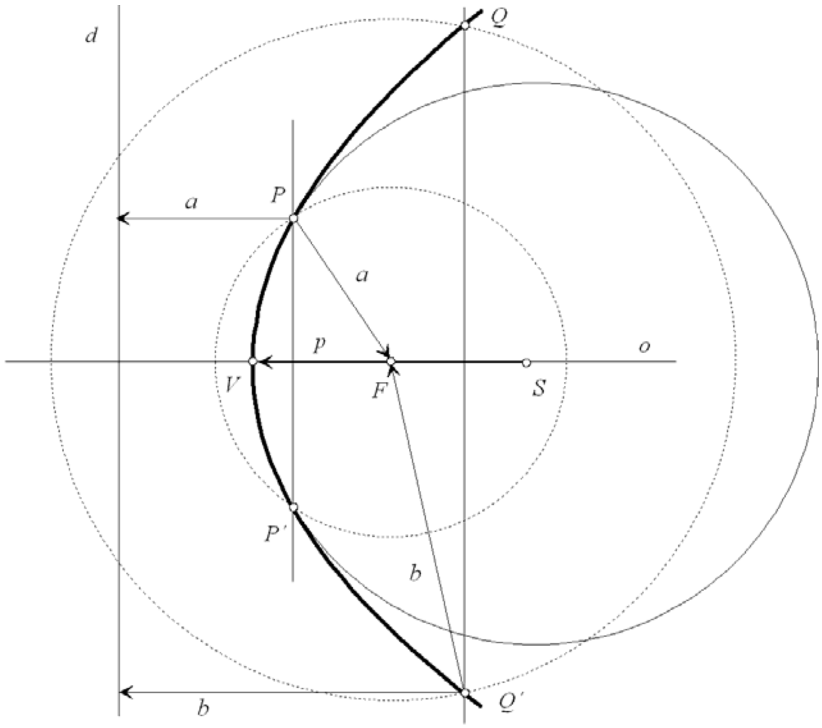


Figure 2. 21 Hyper-osculating circle and points on parabola.

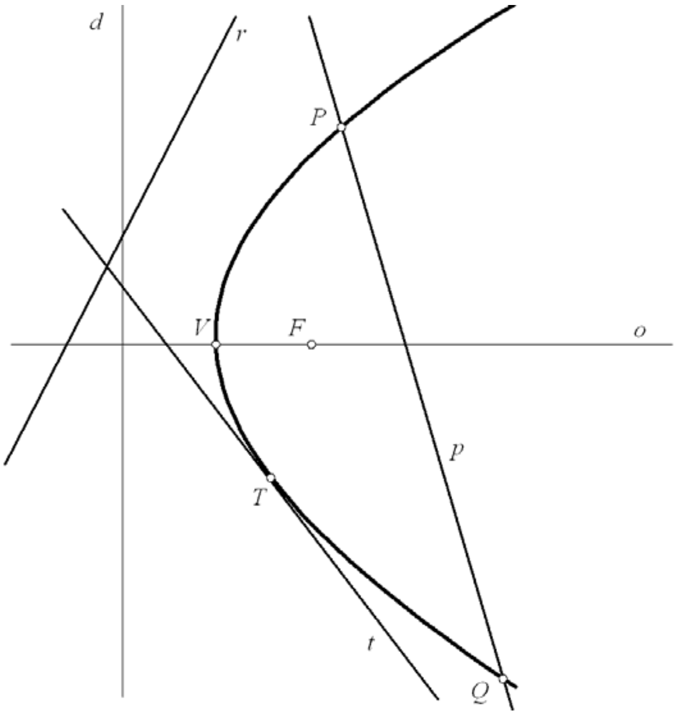


Figure 2. 22 Line *t* is tangent to the parabola in point *T*. Line *p* intersects the parabola in points *P* and *Q*. Line *r* does not intersect the parabola.

Tangent line (or tangent, for short) to the parabola contains no interior points of the parabola and it has a single common point with the parabola called the tangent point, see Figure 2. 23. Any ray passing through the parabola focus *F* and intersecting the parabola in the point *T* (focal line ^1f in the point *T*) is reflected to its mirror image with respect to the parabola tangent into the line passing in direction of the parabola axis *o* (focal line ^2f in the point *T*). Reflexion property of the parabola is a consequence of the definition of tangent line to parabola in its arbitrary point *T*.

Definition 5. Line *t* is tangent to the parabola in point *T* iff it is the axis of the exterior angle formed by the focal lines in the tangent point *T*.

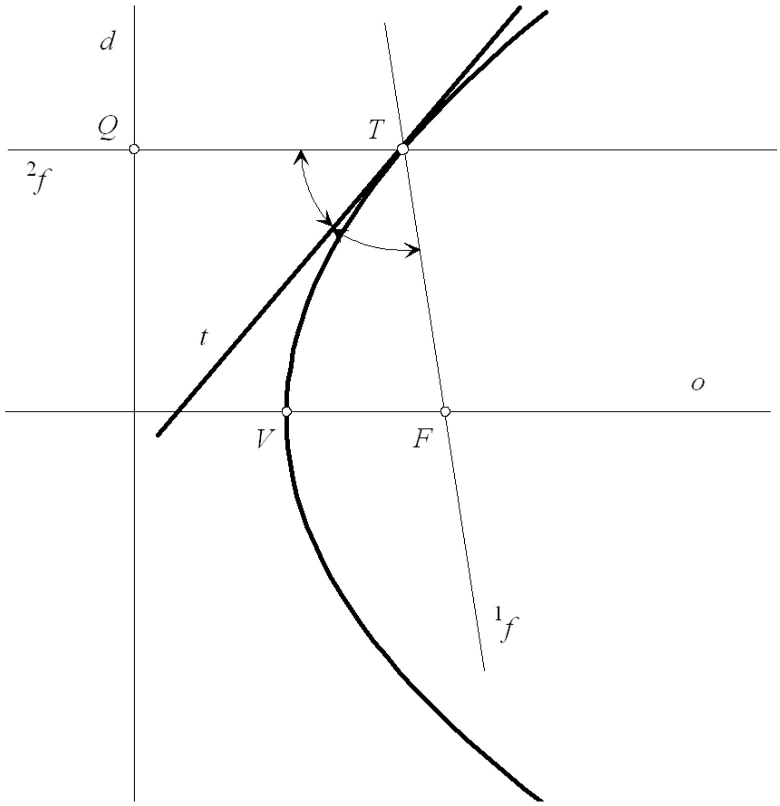


Figure 2. 23 Tangent to parabola.

Theorem 4. All intersection points of tangents to the parabola and their perpendiculars through the parabola focus *F* are located on the tangent to the parabola in its vertex called **vertex tangent** *v*.

Theorem 5. All points symmetric to the focus *F* of the parabola with respect to all tangents to the parabola are located on the directrix of the parabola *d*.

Theorem 6. There exist two different tangent lines to the parabola that are passing through an arbitrary exterior point, but only one tangent line that is parallel to an arbitrary direction.

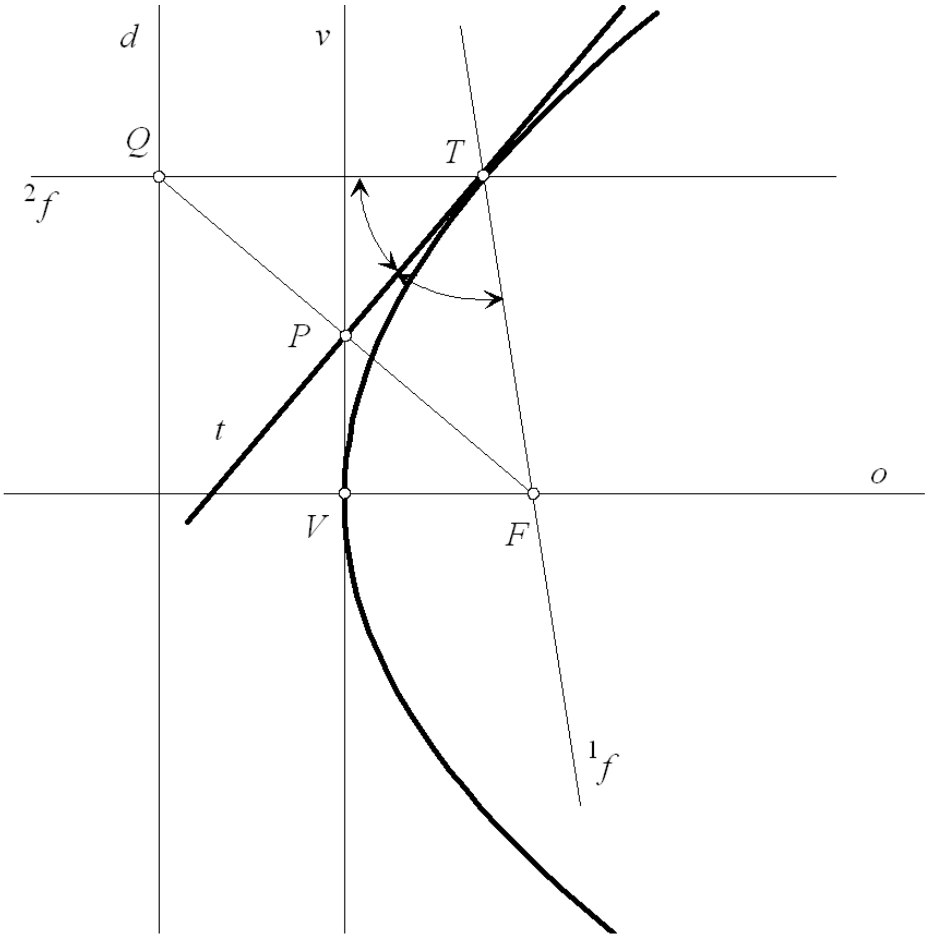


Figure 2. 24 Tangent to parabola at vertex and directrix as set of points symmetric to parabola focus.

Let the tangent line t in an arbitrary point T on parabola intersect axis o of parabola in point R, and the chord of parabola passing perpendicularly to the parabola axis through tangent point T intersect the axis o in the pierce point X. Let the normal n, which is the line perpendicular to the tangent line t in the tangent point T, intersect axis o of the parabola in point Y, as illustrated in Fig. 2. 25.

Definition 6. Line segment RX on the axis of parabola, which is the orthogonal image of the line segment RT on the tangent line in the point T to the parabola axis, is called **subtangent**.

Definition 7. Line segment XY on the axis of parabola, which is the orthogonal image of the line segment TY on the normal line in the point T to the parabola axis, is called **subnormal.**

Theorem 7. Vertex V of the parabola is the centre of subtangent, line segment RX, and

$$|RV| = |VX|.$$

Theorem 8. Focus of the parabola is the centre of the line segment, which is the union of subtangent and subnormal, and

$$|RF| = |FY|.$$

Corollary 3. Tangent lines to the parabola in symmetric points *T* and *T'* that are forming the chord of the parabola perpendicular to the parabola axis *o* meet in a common point *R* on the axis.

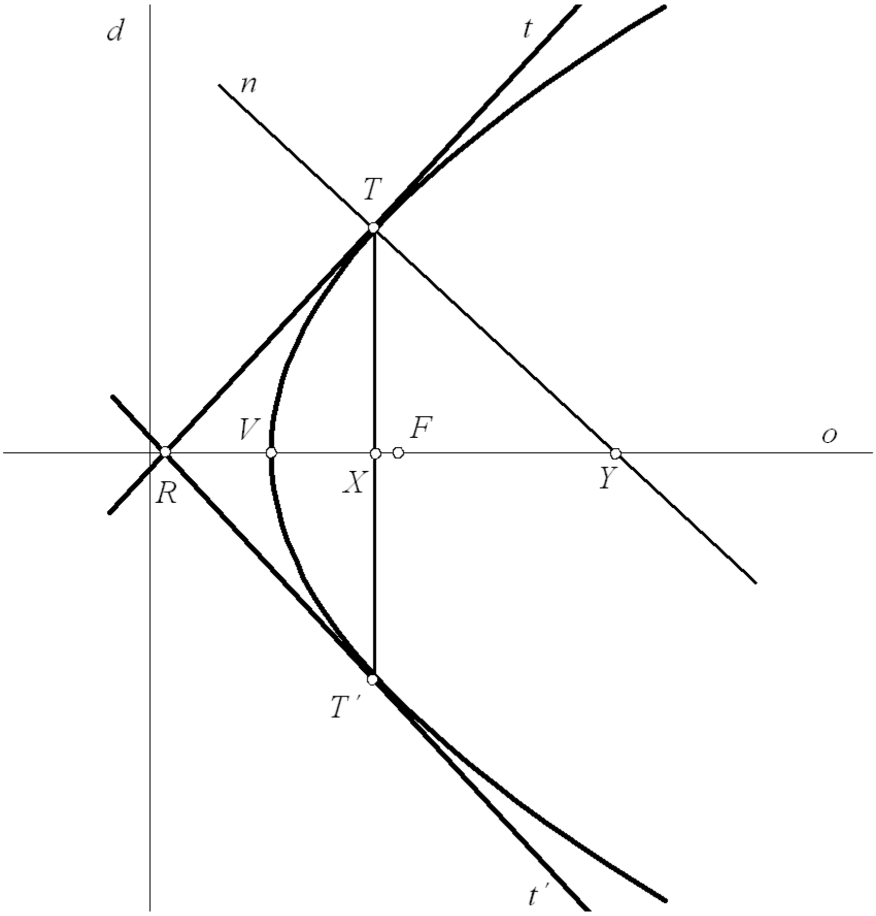


Figure 2. 25 Subtangent and subnormal, and tangents in symmetric points on parabola.

Theorem 9. Tangent lines to the parabola in the end points *T* and *T'* of an arbitrary chord meet in a point *R*. Centre *O* of the chord *TT'* and point *R* determine the line *s* in the direction of the parabola axis *o*.

This property illustrated in Fig. 2. 26 can be used for construction of a parabola determined by two arbitrary tangent lines with attached tangent points on them. The construction is shown in details in Fig. 2. 27.

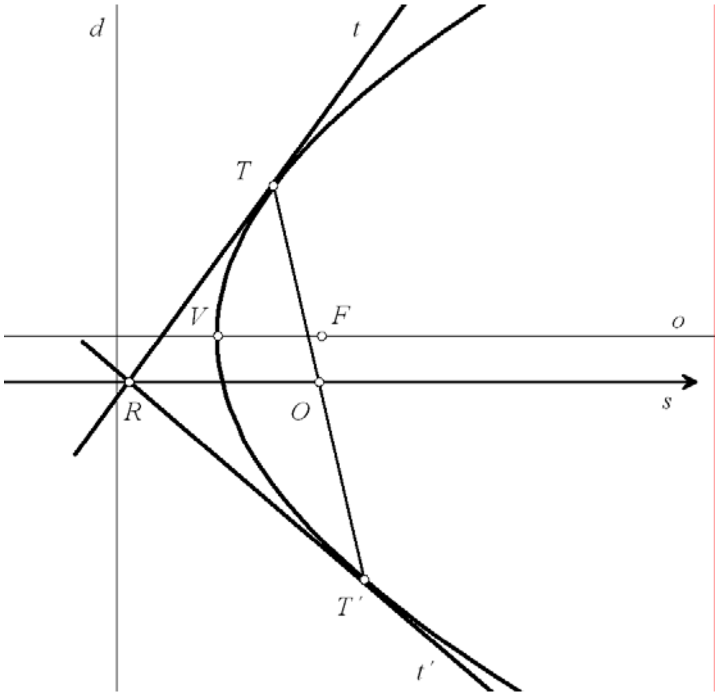


Figure 2. 26 Tangents in endpoints of a parabola chord.

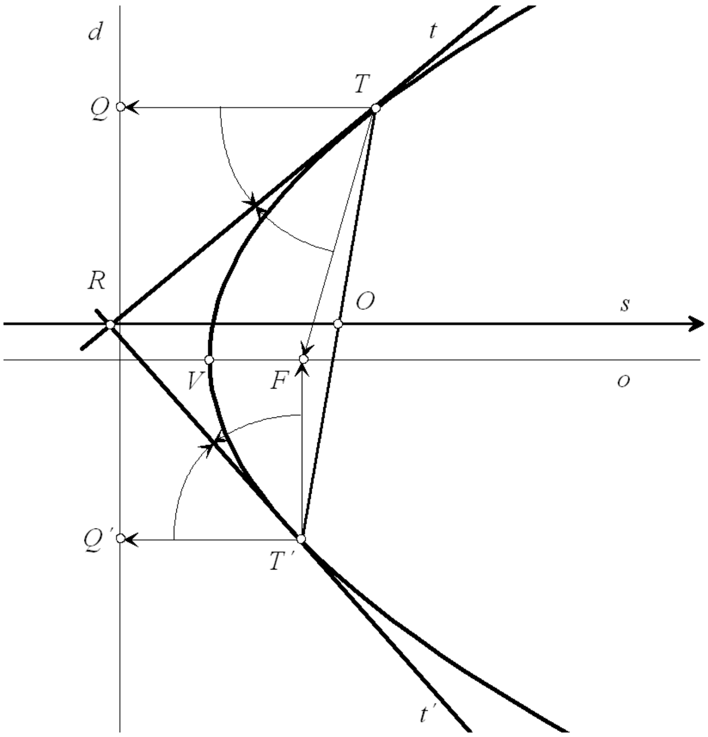


Figure 2. 27 Construction of parabola determined by 2 tangents with tangent points.

2. 3 HYPERBOLA

Hyperbolas are of practical importance in fields ranging from engineering to navigation. The natural-draft evaporative cooling towers used at large electric power stations are in the shape of 1-sheet hyperboloids with hyperbolic cross sections. A comet or other object moving with sufficient kinetic energy to escape the solar gravitational pull traces out one branch of a hyperbola.

Definition 8. A hyperbola is a set of all points P in Euclidean plane such that the absolute value of the difference of the distances from P to two fixed points ${}^1F$ and ${}^2F$ is a constant positive number. Here ${}^1F$ and ${}^2F$ are called the focal points, or the foci, of the hyperbola.

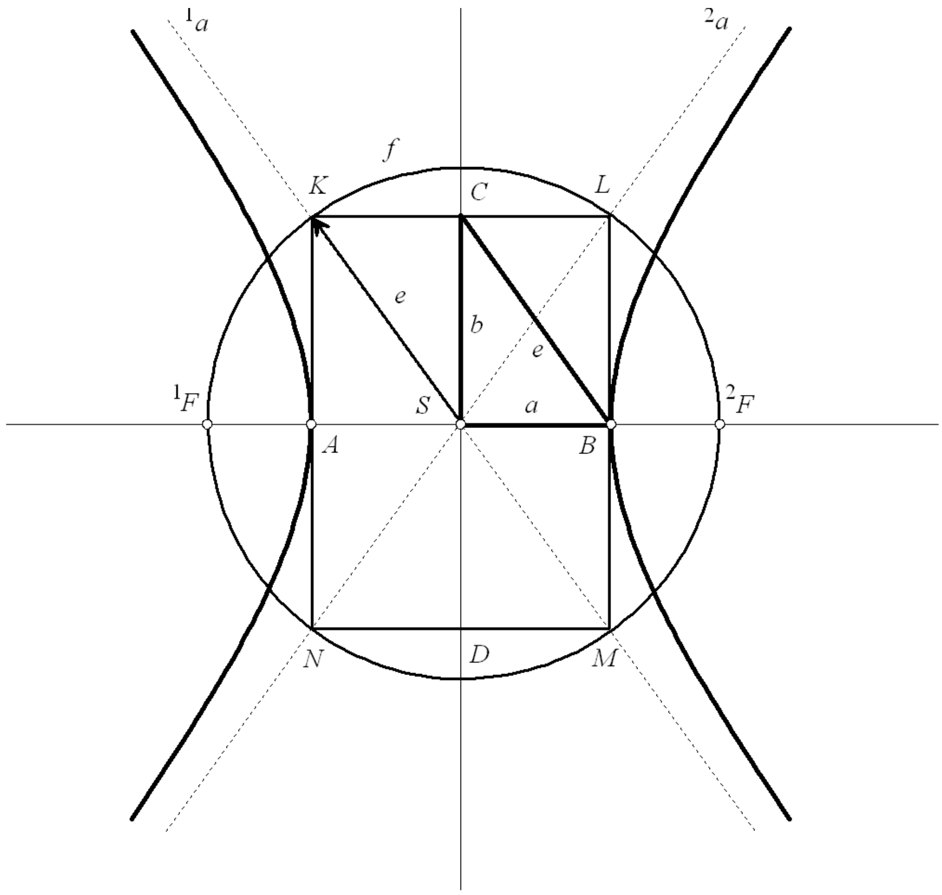


Figure 2. 28. Definition of hyperbola in plane.

The midpoint S of the line segment ${}^1F\,{}^2F$ is called the **centre** of the hyperbola, Fig. 2.28. The distance between the centre S and either **focus** ${}^1F$ and ${}^2F$ is called the **linear eccentricity** of the hyperbola and it is denoted by e. Notice that the hyperbola is symmetric about a line through the foci ${}^1F$ and ${}^2F$. Let A and B be the points where the

line through points ${}^1F$ and ${}^2F$ intersects the hyperbola. Centre S bisects the line segment AB and hyperbola is symmetric also with respect to the line through S and perpendicular to the line AB. Let C and D be points on sides of rectangular $KLMN$, with sides passing in direction of the line through points ${}^1F$ and ${}^2F$. The four points A, B, C, D are called the **vertices of the hyperbola**. The line AB is called the **major axis**, and the line CD the **minor - transverse axis** of the hyperbola. Let a denote the length of the line segment $SA = SB$, and b the length of the line segment $SC = SD$. The numbers a and b are called the **semi-major (semi-transverse) axis** and the **semi-minor (semi-conjugate) axis** of the hyperbola.
Applying the Pythagorean theorem to the right triangle CSB (Fig. 2. 28) we find that

$$e^2 = a^2 + b^2,$$

which is called the geometric equation of the hyperbola.

Lines passing through an arbitrary point T of the hyperbola and either focus ${}^1F$ or ${}^2F$, ${}^1f = {}^1FT$, ${}^2f = {}^2FT$, are called **focal lines**. Line segments ${}^1FT$ and ${}^2FT$ located on the focal lines in the point T indicate the distances of the hyperbola point T to the foci, the difference of which is constant and according to the definition of hyperbola this is equal to $2a$.

$$\left|\,|{}^1F\,T| - |{}^2F\,T|\,\right| = 2a$$

The focal lines in the point T form two angles with the common vertex in their common point T on the hyperbola. These are:

3. interior angle in which the centre of the hyperbola is located,
4. exterior angle in which the major vertices A and B are located,

as illustrated in Fig. 2. 29.

Any line segment MN determined by endpoints M, N on the hyperbola is called the **chord of the hyperbola**. A focal chord is passing through a focus perpendicularly to the major axis of the hyperbola.

If the hyperbola is placed in the Euclidean plane $\mathbf{E}_2$ with defined Cartesian coordinate system $\{O, x, y\}$ so that centre S is located at the origin O, and the two foci ${}^1F = [-e, 0]$ and ${}^2F = [e, 0]$ both lie on the negative and positive portions of the coordinate axis x respectively, then the analytic equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a is the semi-major axis, b is the semi-minor axis.

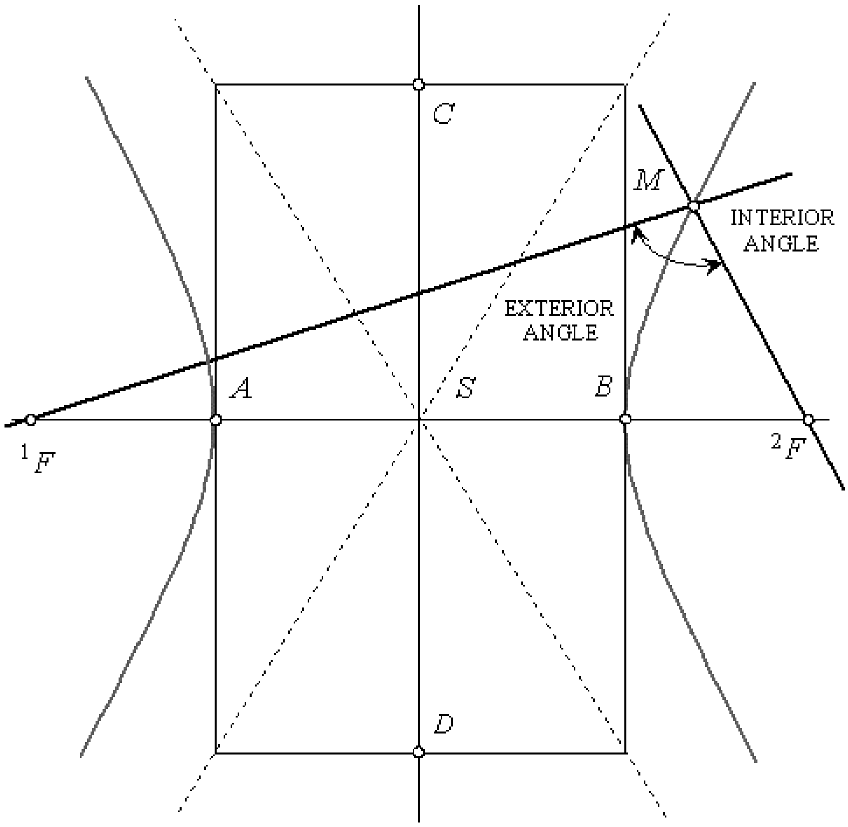


Figure 2. 29. Angles of hyperbola focal lines.

Let $P = [x_P, y_P]$ be a point on the hyperbola, see Fig. 2. 30. Then the equation holds

$$|{}^1F\,P| - |{}^2F\,P| = 2a,$$

from which follows

$$\sqrt{(x_P + e)^2 + y_P^2} - \sqrt{(x_P - e)^2 + y_P^2} = 2a\,.$$

Squaring the above equation, and after some manipulations we have

$$ex_P - a^2 = a\sqrt{(x_P - e)^2 + y_P^2}\,.$$

Squaring the last equation we obtain

$$a^2\left(a^2 - e^2\right) = \left(a^2 - e^2\right)x_P^2 + a^2 y_P^2.$$

Since $e^2 = a^2 + b^2$, then $e^2 - a^2 = b^2$, and the equation above can be rewritten as

$$-a^2b^2 = -b^2x_P^2 + a^2y_P^2.$$

Dividing both sides of the above equation by a^2b^2, we receive

$$1 = \frac{x_P^2}{a^2} - \frac{y_P^2}{b^2},$$

which is the presented equation of the hyperbola.

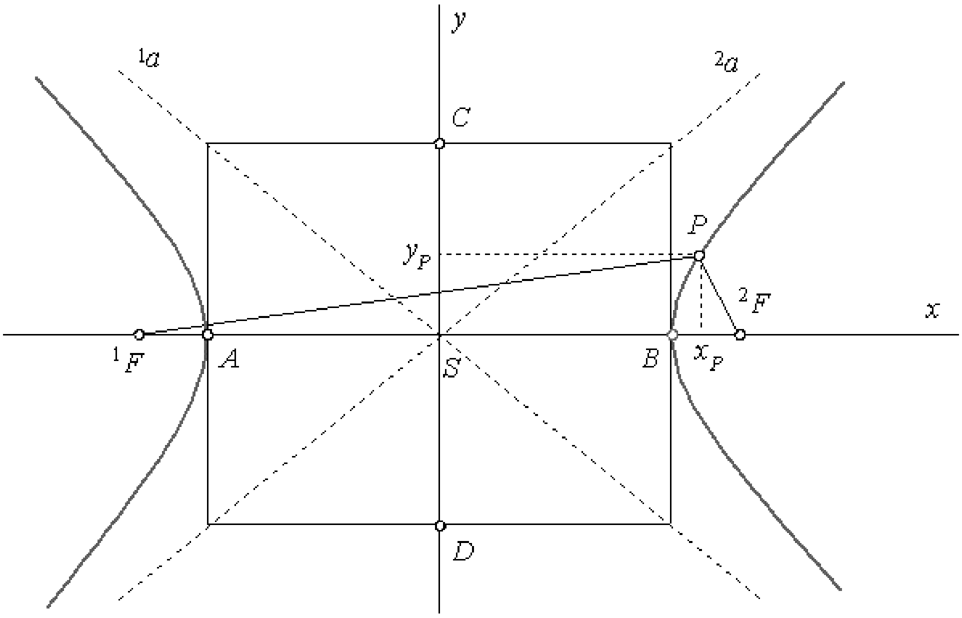


Figure 2. 30. Hyperbola in Cartesian plane, with horizontal major axis.

Conversely, it can be shown that if the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ holds, then the point $P = [x, y]$ is located on the hyperbola with the foci at ${}^1F = [-e, 0]$, ${}^2F = [e, 0]$ and

$$|\,|{}^1F\,P| - |{}^2F\,P|\,| = 2a.$$

Note: If a and b are positive constants, $a > b$, the above Cartesian equation is called the standard form for the equation of a hyperbola with the centre at the origin O and with the horizontal major axis ${}^1o$ in the coordinate axis x. The standard equation of the hyperbola with the same centre but with the vertical major axis in coordinate axis y is

$$-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Let the hyperbola with the centre S at the origin, semi-major (semi-transverse) axis a, semi-minor (semi-conjugate) axis b, and axes in the coordinate axes x and y be shifted so that the centre is at the point $S = [m, n]$. The equation of the hyperbola shifted into a new position, see Fig. 2. 31 for example, will have one of the following standard forms

$$\frac{(x-m)^2}{b^2} - \frac{(y-n)^2}{a^2} = 1, \quad \text{if the major axis is horizontal}$$

$$-\frac{(x-m)^2}{b^2} + \frac{(y-n)^2}{a^2} = 1, \quad \text{if the major axis is vertical.}$$

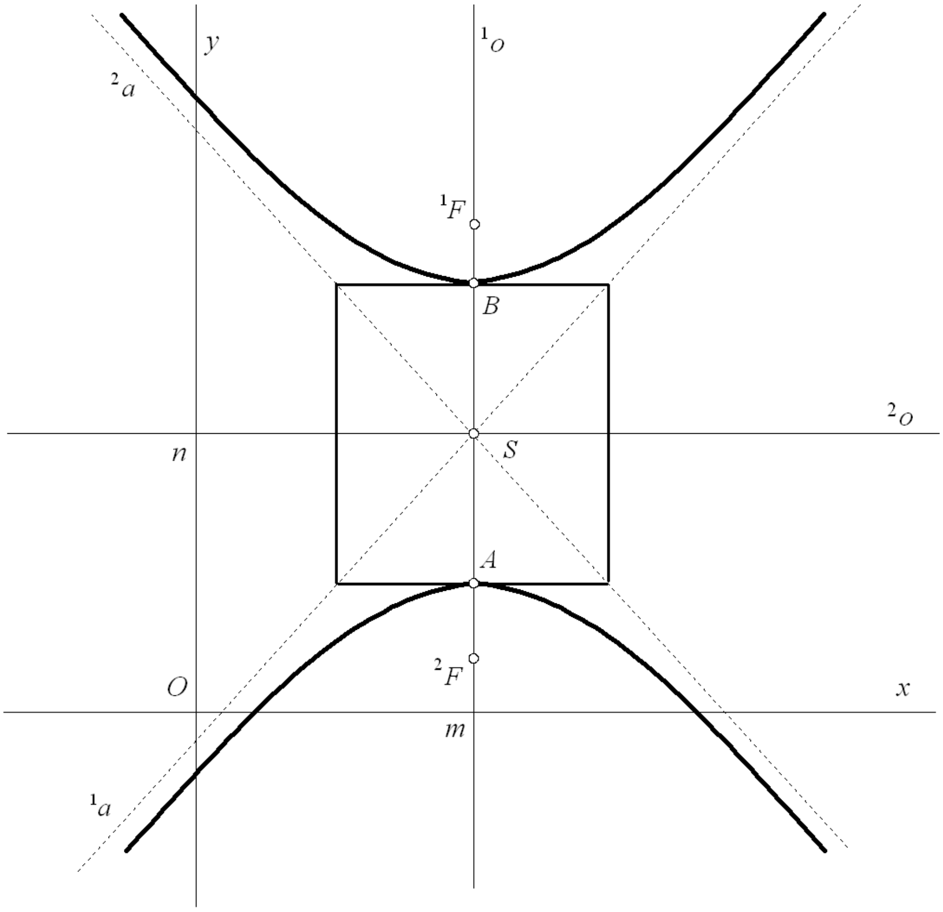


Figure 2. 31 Hyperbola with shifted centre and vertical major axis.

Parametric equations of a hyperbolic arc with centre in the origin of the coordinate system O corresponding to the central angle $\varphi \in \langle 0,2\pi \rangle$, with axes in the coordinate axis x and y and semi-axes a and b (Fig. 2. 32) are in the following vector form

$$\mathbf{r}(u) = (x(u), y(u)) = \left(\frac{a}{\cos \varphi(1-2u)}, b \tan \varphi(1-2u) \right), \quad u \in \langle 0,1 \rangle,$$

and vector equation of the hyperbola with the shifted centre in the point $S = [m, n]$ is

$$\mathbf{r}(u) = (x(u), y(u)) = \left(m + \frac{a}{\cos \varphi(1-2u)}, n + b \tan \varphi(1-2u) \right), \quad u \in \langle 0,1 \rangle.$$

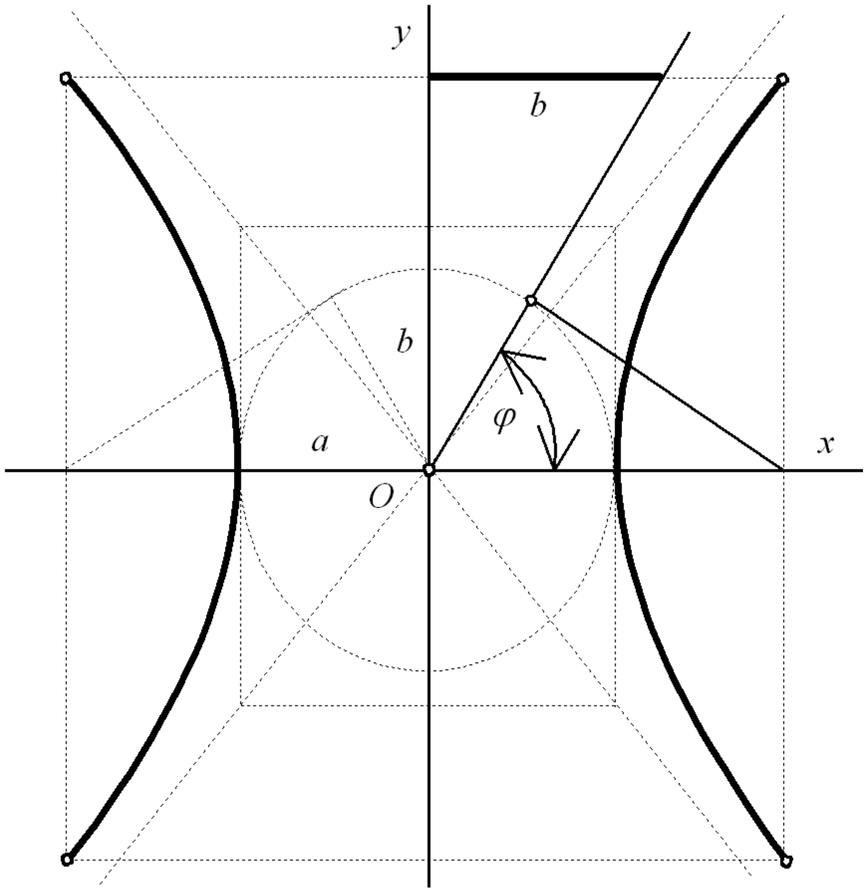


Figure 2. 32 Vector form of parametric equations of a hyperbola.

To draw a hyperbola, we can find two circles that approximate it in vertices and we call them hyper-osculating circles. Diagonals of the rectangular *KLMN* are called asymptotes of hyperbola, and they show the hyperbola direction to infinity.

Construction 3.

1. Draw rectangular *KLMN* with sides parallel to hyperbola axes.
2. Through point *K* and *L* construct lines perpendicular to asymptotes of the hyperbola.
3. Find intersection points of these lines and the major axis of the hyperbola *AB*.
4. These are points S_A and S_B on axis *AB*.
5. Construct 2 circles with centres S_A , S_B passing through the hyperbola vertices *A*, *B*.

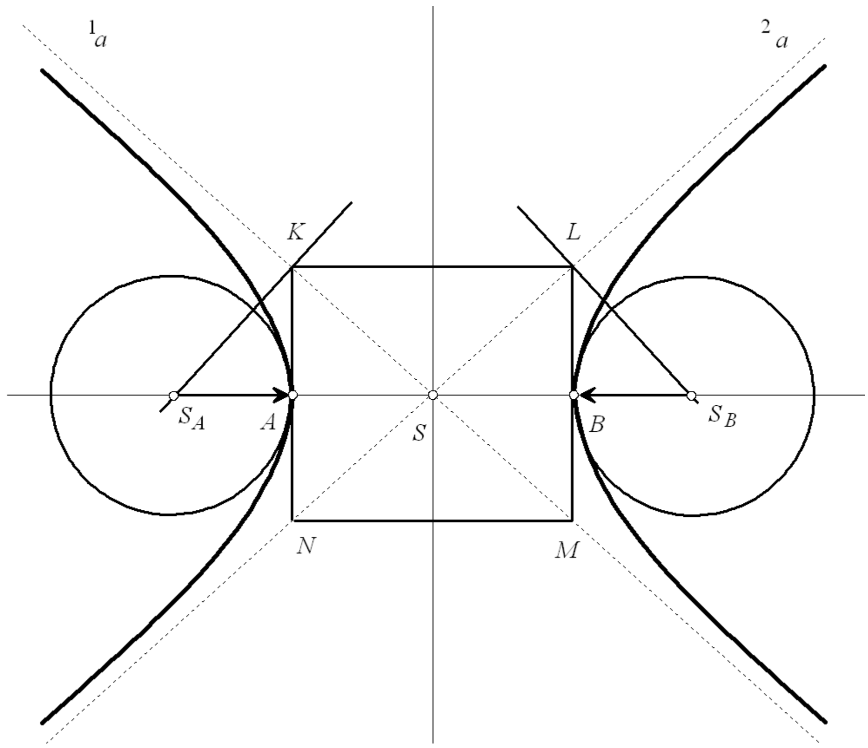


Figure 2. 33. Osculating circles of hyperbola.

Any line in a general position can:

1. be tangent to the hyperbola in one point,
2. intersect the hyperbola in two different points,
3. have no common points with the hyperbola.

All three possible positions of a hyperbola and a line are illustrated in Fig. 2. 34. Line *t* is tangent to the hyperbola in the tangent point *T*, line *p* intersects hyperbola in two different points *P* and *Q*, while line *r* does not intersect the hyperbola. Lines passing through the centre of the hyperbola and located in the same angle of the asymptotes as the conjugate (imaginary) axis, do not intersect the hyperbola.

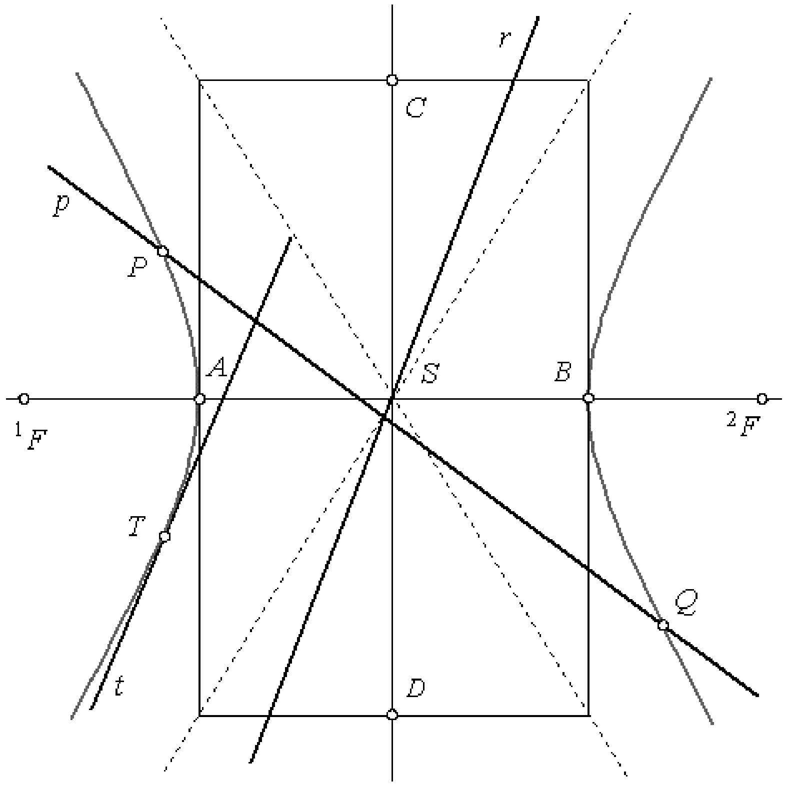


Figure 2. 34. Superposition of line and hyperbola.

Definition 9. Line *t* is **tangent to the hyperbola** in point *T* if it is the axis of the exterior angle formed by the focal lines in the tangent point *T*.

Note: Tangent line to the hyperbola contains no hyperbola interior points.

Theorem 10. All intersection points of tangents to the hyperbola and their perpendiculars through the hyperbola foci ^1F and ^2F are located on a circle with centre in the centre of the hyperbola and radius equal to the hyperbola semi-transverse (semi-major) axis. Circle $v(S, a)$ is called the **auxiliary circle**.

Theorem 11. All points symmetric to one focus of the hyperbola with respect to all tangents to the hyperbola are located on a circle with centre in the other focus and radius equal to the doubled semi-transverse (semi-major) axis of the hyperbola. Circle $^1g = (^2F, 2a)$ is called the **control circle** related to the focus ^1F.

Note: The control circle related to the focus ^2F has a centre in the focus ^1F and the same radius equal to the double semi-major axis of the hyperbola, $^2g = (^1F, 2a)$.

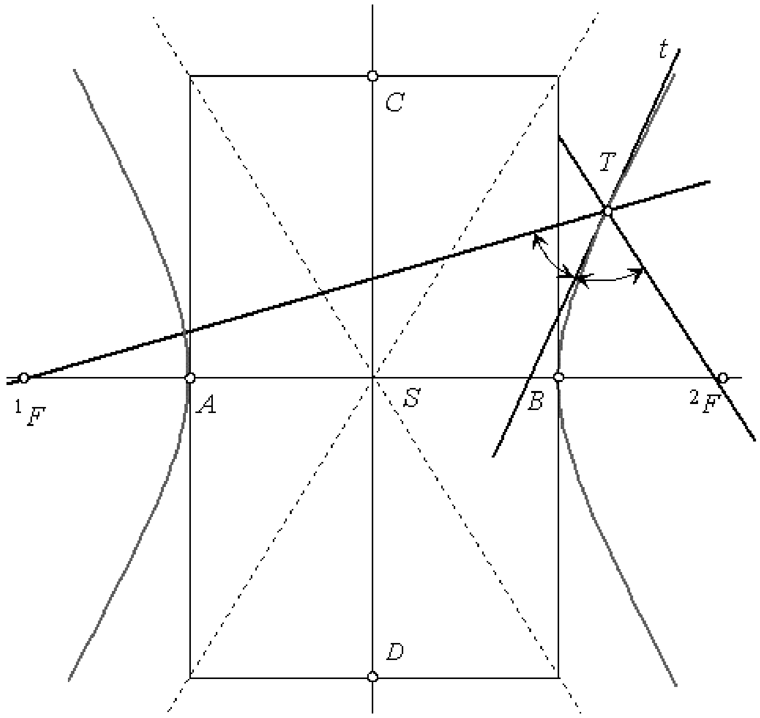


Figure 2. 35 Tangent to the hyperbola.

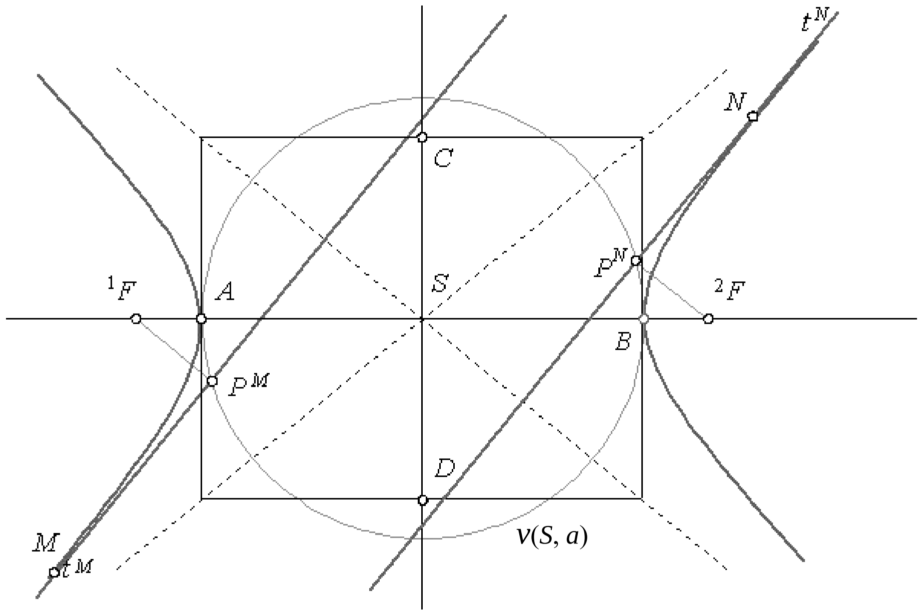


Figure 2. 36 Auxiliary circle

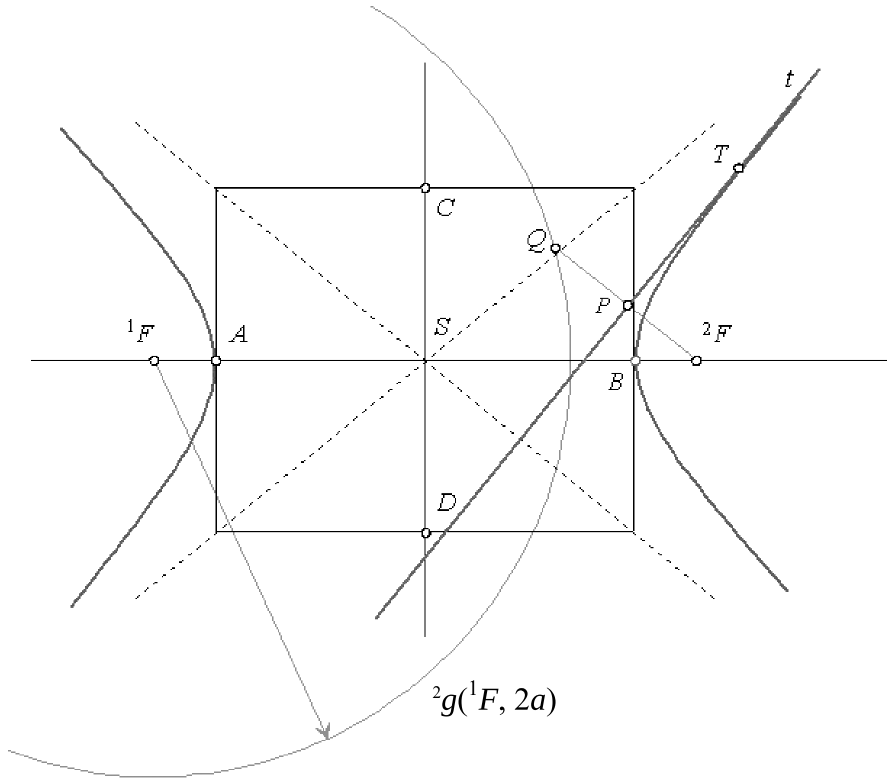


Figure 2. 37 Control circle.

Theorem 12. There exist two different tangent lines to the hyperbola that are passing through an arbitrary exterior point of the hyperbola, or that are parallel to an arbitrary direction, but asymptotes of the hyperbola.

Corollary 4. Tangent points of two tangent lines to the hyperbola that are parallel to each other form a line segment with endpoints on the hyperbola and midpoint in the centre of the hyperbola.

A line that is not tangent to the hyperbola, is not parallel to any of the two asymptotes and is not passing through the hyperbola centre and has at least one common point with the hyperbola, intersects hyperbola in one more point. Lines passing parallel to any of the asymptotes intersect the hyperbola in one real point.

Any line passing through the hyperbola centre is called the **diameter of the hyperbola**. Diameters of the hyperbola passing in that angle formed by asymptotes, in which major vertices and foci are located, intersect hyperbola in two different real points. Line segment determined by these two intersection points is called the **chord of the hyperbola**.

Distance of the two intersection points of the hyperbola and its diameter varies in the interval $\langle 2a, \infty)$, where a is the semi-major axes of the hyperbola.

Definition 10. Any two diameters of the hyperbola are called **related or conjugate diameters**, if they satisfy the following property: Tangent lines to the hyperbola in the endpoints of one of the diameters are parallel to the other diameter, which is not intersecting the hyperbola.

Note: Semi-major and semi-minor axes form the only pair of related diameters of the hyperbola such that the two diameters are perpendicular lines.

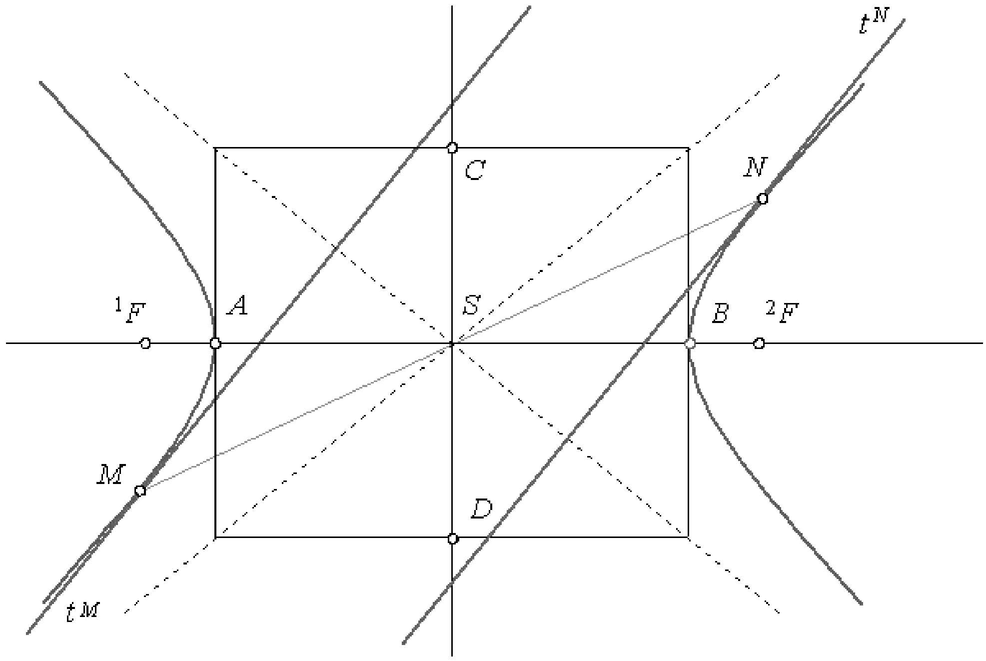


Figure 2. 38 Hyperbola chord formed by tangent points of two parallel tangents.

3 CONIC SECTIONS ON CYLINDRICAL AND CONICAL SURFACES

Planar intersections of circular cylindrical surfaces or circular conical surfaces are conic sections of different types, depending on the position of the intersection plane to the surface generating lines. Any of these conic sections can be determined by basic elements, and its views can be drawn using supporting hyper-osculating circles. Single points of the intersection conics can be constructed as pierce points of the intersection plane and specific generating lines of concerning surfaces. Other determining elements of the intersection conics might be axes or conjugate diameters for an ellipse, whereas for a parabola it is possible to determine its axis, vertex and two tangent lines with attached tangent points on them, and for a hyperbola its vertices, axes, and asymptotes can be found. Visibility of the resulting conic section images must be determined in all views.

3.1 ELLIPSE ON CYLINDRICAL SURFACES

Circular cylindrical surface can be determined as a set of all lines that are passing in a given direction *s*, while each of them is passing through one point on a given **basic circle** $k(S, r)$ with centre *S* and radius *r*, which is located in plane α not parallel to the direction *s*. Any line *h* forming the surface is called the **generating line**. Line *o* passing through the centre *S* of the basic circle *k* in the given direction *s* is called the **axis of the circular cylindrical surface**. If the given direction *s* is perpendicular to the plane α of the basic circle *k*, we refer to the **cylindrical surface of revolution**. The name is derived from the fact that this rotational surface can also be determined by revolving an arbitrary line *h* about the chosen parallel line *o*, which stands for the axis of revolution, and consequently, the axis of the cylindrical surface of revolution.
If the circular cylindrical surface is placed in the Euclidean space $\mathbf{E^3}$ with the Cartesian coordinate system $\{O, x, y, z\}$ so that its axis *o* is parallel to the coordinate axis *z* and the basic circle $k(S, r)$ is in the coordinate plane $\pi = xy$, while its centre is the point $S = [m, n, 0]$, then the analytic equation of the surface is

$$(x-m)^2 + (y-n)^2 = r^2, z \in R$$

Axonometric view of the circular cylindrical surface is presented in Fig. 3. 1.
Axonometric view and axonometric top view of the basic circle $k(S, r)$ in the coordinate plane $\pi = xy$ coincide in an ellipse $k = k_1$, whose centre is in the incident views of point *S* in π. Major axis *AB* is perpendicular to the view of the coordinate axis *z*, semi-major axis equals to the radius *r*. Minor axis *CD* is in the direction of the coordinate axis *z*, and semi-minor axis *b* can be expressed by means of the slope φ of the axonometric image plane to the coordinate plane π, $b = r \cos \varphi$.

Axonometric view of the axis o determines the direction of the views of cylindrical surface generating lines. Outlines f and g are tangent lines to the ellipse $k = k_1$ in the given direction. Tangent points F and G form one diameter of the ellipse, and they determine change of visibility of the basic circle k in the axonometric view. Axonometric front view of the surface can be outlined by front views of generating lines h and l that are parallel to the axonometric front view o_2 of the cylindrical surface axis. Their axonometric views are passing through such points on the basic circle k, which form its diameter parallel to the coordinate axis x. Axonometric front view of the basic circle k is a line segment k_2 on the coordinate axis x with centre S_2 as the axonometric front view of the centre S.

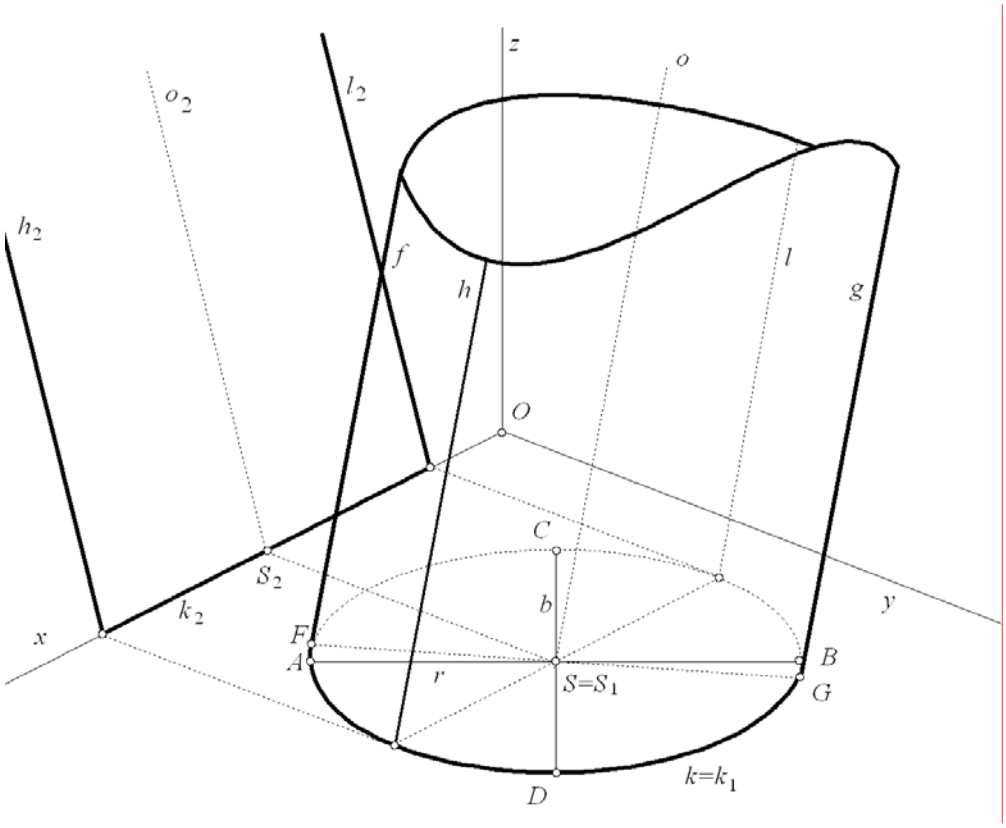


Figure 3. 1 Circular cylindrical surface in axonometric projection.

Planar intersections on the circular cylindrical surface can be of the following types:

1. circle - intersection plane ρ is parallel to the plane α of the basic circle k
2. 1 or 2 generating lines - intersection plane ρ is parallel to the surface axis o, while in case of just 1 common line plane ρ is called the tangent plane
3. ellipse - intersection plane ρ is neither parallel to the surface axis o, nor parallel to the plane α of its basic circle.

Illustration of an elliptic intersection on circular cylindrical surface by arbitrary plane generated in mathematical software Maple is presented in Fig 3.2.

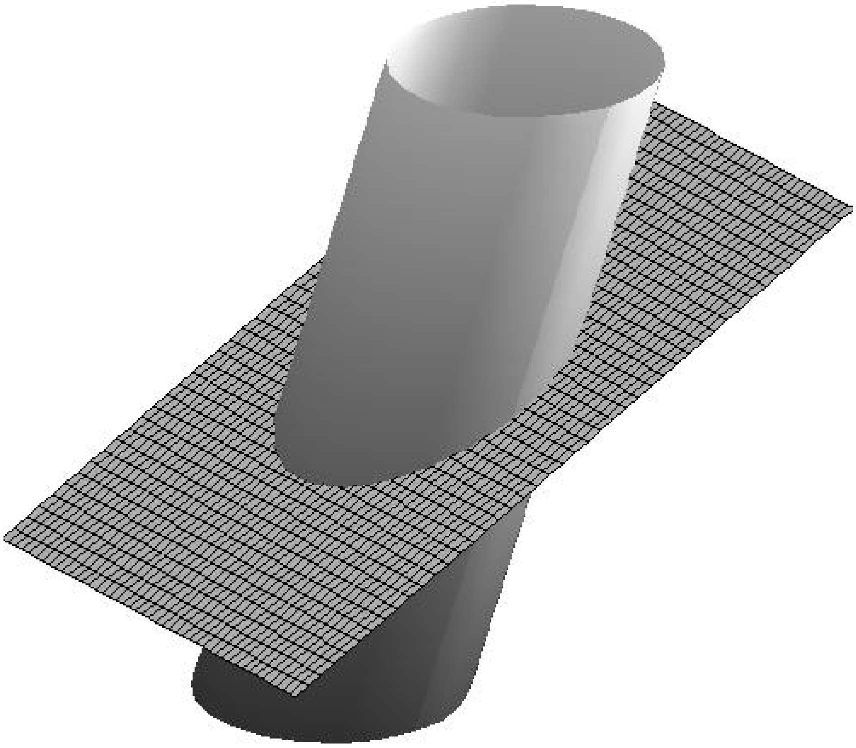


Figure 3. 2 View of elliptic intersection on circular cylindrical surface.

Related views of the circular cylindrical surface with basic circle $k(S, r)$ in the coordinate plane $\pi = xy$ and axis o determining the direction of the cylindrical surface generating lines are shown in the Monge method in Fig. 3. 3. Choosing the intersection plane ρ perpendicular to the front image plane, therefore appearing in the edge view ρ_2, the front view of the intersection ellipse e appears as a line segment e_2. Top view e_1 of the intersection can be determined by the related diameters. End points H_1 and L_1 on one diameter are top views of line segment e_2 end points H_2 and L_2 on surface generating lines h and l. These are passing through such points on the basic circle k which form its diameter parallel to the coordinate axis x. Centre O of the line segment HL is the centre of ellipse e, common point of the surface axis o and intersection plane ρ. Its views are centres of intersection ellipse views - line segment e_2 and ellipse e_1. End points of the related diameter MN are on generating lines m and n forming projecting plane perpendicular to the frontal image plane. Their front views coincide with O_2 and top views can be easily found on top views m_1 and n_1 of respective lines m and n. Visibility in the top view is defined by top views of points F and G, which are common points of the outlines f and g and the intersection plane ρ directly accessible in the front view.

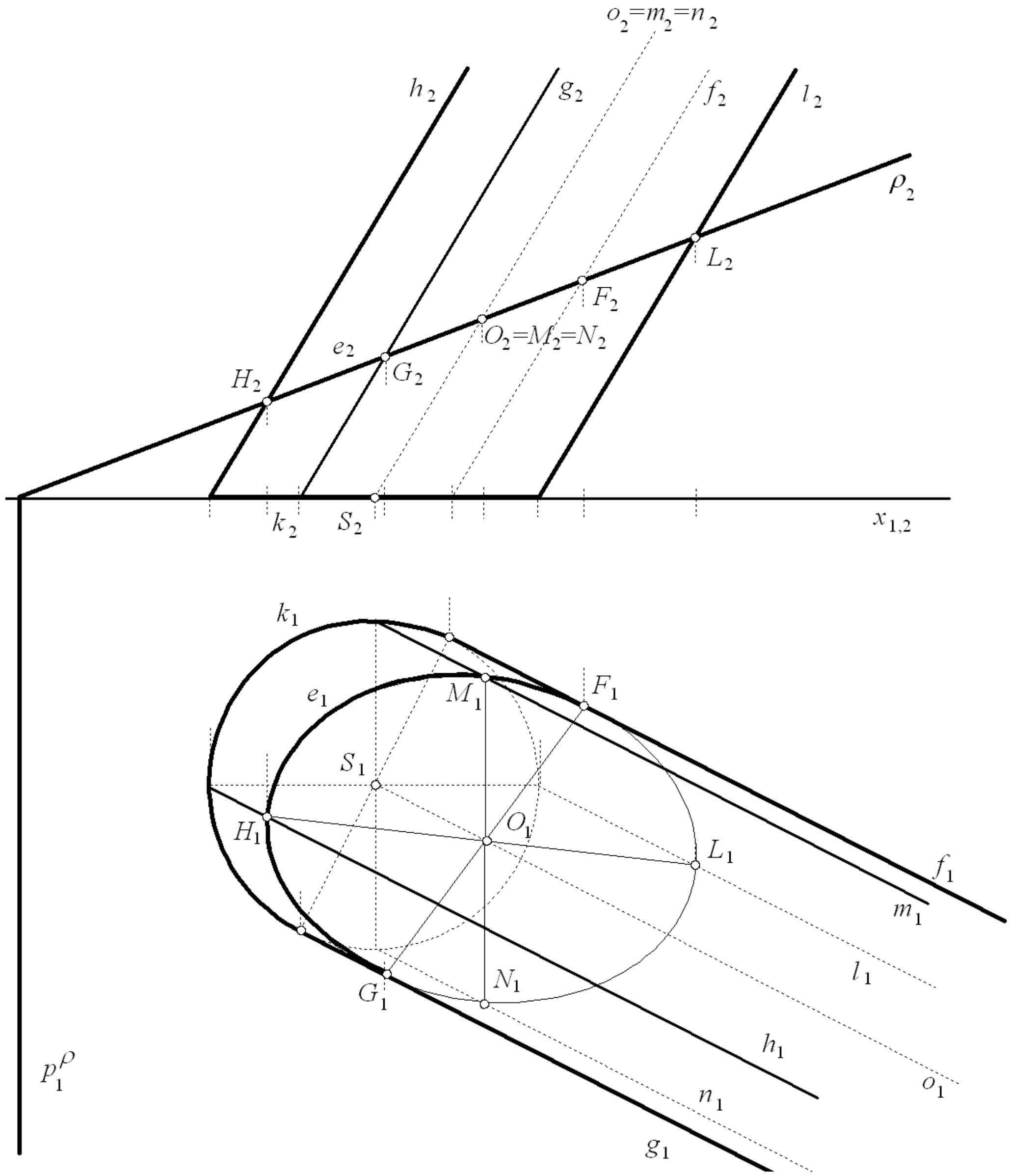


Figure 3. 3 Related views of elliptic intersection on circular cylindrical surface.

Axonometric view of the elliptic intersection on the circular cylindrical surface can be determined from the axonometric front view in Fig. 3. 4. Let the intersection plane ρ be perpendicular to the frontal image plane, therefore its traces, p^ρ in the coordinate plane $\pi = xy$ and m^ρ in the coordinate plane $\mu = yz$, are parallel to the coordinate axis y, and this remains true also for their axonometric views.

Axonometric front view e_2 of the intersection ellipse appears as line segment, whose end points are determining end points H and L on one diameter of the axonometric view of the intersection *e*. These points are located on the surface generating lines *h* and *l*. Centre *O* of the line segment *HL* is the centre of ellipse *e*. Generating lines *m* and *n* forming plane perpendicular to the frontal image plane appear in front views in one line with the axonometric front view of the surface axis, $o_2 = m_2 = n_2$. Their intersection points *M* and *N* in the plane ρ are end points of the related diameter *MN* of the intersection ellipse *e*.

Visibility of the intersection ellipse in the axonometric view can be determined by points *F* and *G* located on outlines *f* and *g* of the surface axonometric view and forming additional diameter of the intersection ellipse view. Lines *f* and *g* are mapped to the ellipse *e* tangent lines in the axonometric view, while axonometric views of tangent points on them can be derived from their clearly visible axonometric front views F_2 and G_2 in the common points of axonometric front views of lines f_2 , g_2 and intersection plane edge view ρ_2 .

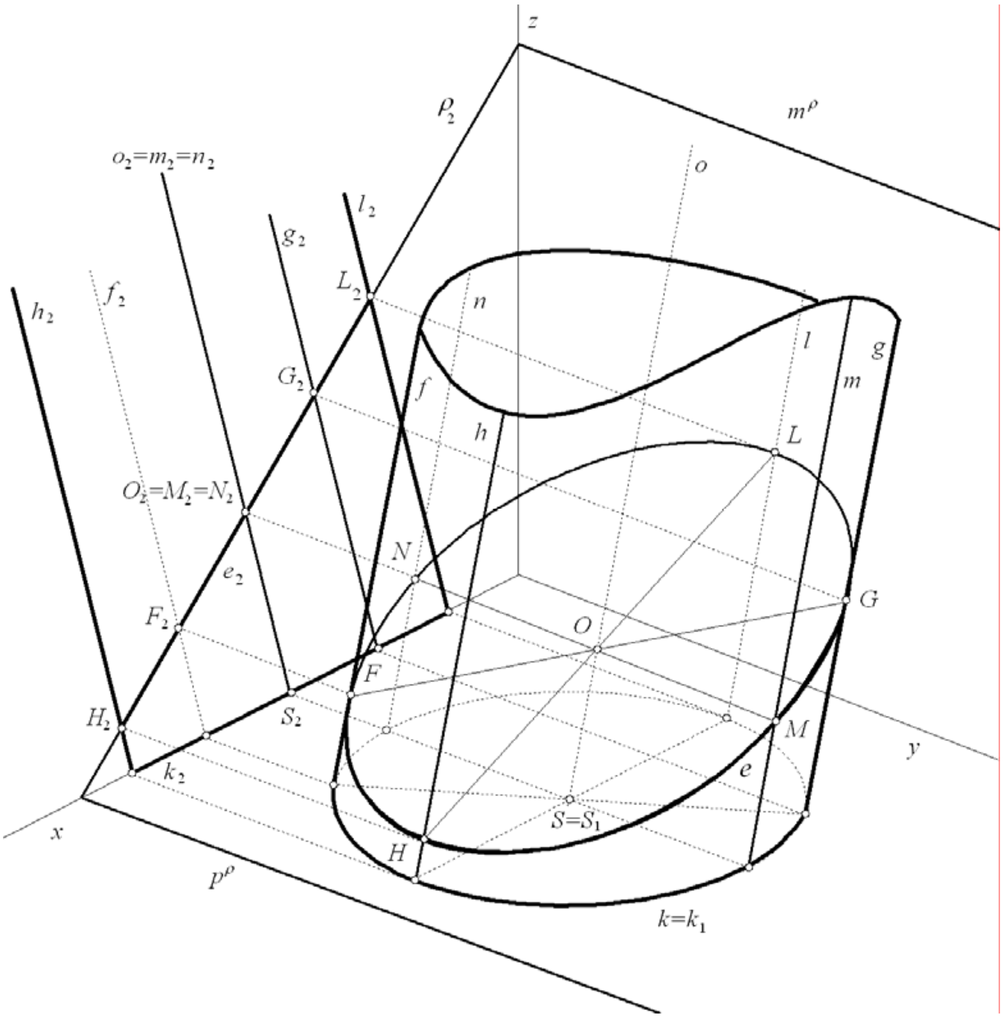


Figure 3. 4 Axonometric view of elliptic intersection on circular cylindrical surface.

3.2 CONIC SECTIONS ON CONICAL SURFACES

Circular conical surface can be determined as a set of all lines that are passing through a given point *V*, each intersecting in one single point a given circle $k(S, r)$ with centre *S* and radius *r* located in plane β not passing through the point *V*. Lines *h* forming the surface are called **generating lines**, given point *V* is the **main vertex** and the circle *k* is the **basic circle**. Line *o* passing through the centre *S* of the basic circle *k* and the main vertex *V* is called the **axis of the circular conical surface**, Fig. 3. 5. If the axis *o* is perpendicular to the plane β of the basic circle *k*, we refer to **conical surface of revolution**. The name stems from the fact that this rotational surface can also be determined by revolving an arbitrary line *h* about the chosen line *o*, the axis of revolution, while lines *h* and *o* are meeting in one common point *V*.
If the circular conical surface is placed in the Euclidean space $\mathbf{E^3}$ with the Cartesian coordinate system $\{O, x, y, z\}$ so that its vertex is the point $V = [m, n, d]$ and the basic circle $k(S, r)$ is in the coordinate plane $\pi = xy$, while its centre is the point $S = [m, n, 0]$, therefore its axis o is parallel to the coordinate axis *z*, then the analytic equation of the surface is

$$(x-m)^2 + (y-n)^2 = k(z-d)^2, k \in R$$

Axonometric view of the circular conical surface is presented in Fig. 3. 5.
Axonometric views of the basic circle $k(S, r)$ in the coordinate plane $\pi = xy$ can be determined in the same way as views of the basic circle of the circular cylindrical surface. Outlines *f* and *g* are tangent lines to the ellipse $k = k_1$ passing through the axonometric view of the main vertex *V*. Tangent points *F* and *G* do not form a diameter of the ellipse, but they determine change of visibility of the basic circle *k* in its axonometric view. Axonometric front view of the surface is outlined by front views of generating lines *h* and *l* that are passing through the axonometric front view V_2 of the conical surface vertex. Their axonometric views are determined by those points on the basic circle *k*, which are forming its diameter parallel to the coordinate axis *x*. Axonometric front view of the basic circle *k* is a line segment k_2 on the coordinate axis *x* with centre S_2 as the axonometric front view of the centre *S*.
Conic sections form a group of curves that might be determined as different forms of circular conical surface planar cuttings, i.e. intersections of conical surface by planes in special superposition to the surface generating lines.
With respect to the position of the intersecting plane ρ to the circular conical surface of revolution, different types of regular conic sections can be distinguished, when plane ρ is not passing through the surface main vertex:

1. circle - plane ρ is perpendicular to the surface axis
2. ellipse - plane ρ intersects all generating lines on the surface
3. parabola - plane ρ intersects all but one generating line on the surface
4. hyperbola - plane ρ is parallel to 2 different generating lines on the surface.

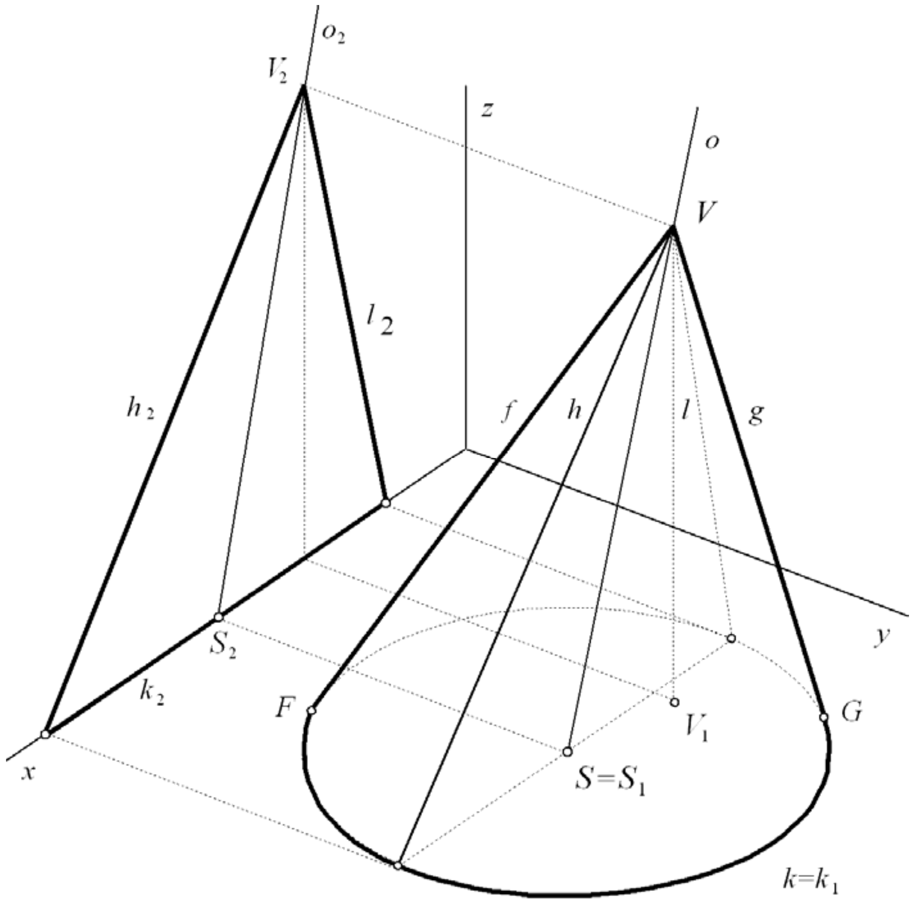


Figure 3. 5 Circular conical surface.

In Fig. 3.6 the illustration on the left is the circular intersection mapped in the Monge method, which is the orthographic view onto 2 different perpendicular image planes (frontal image plane and ground plane). On the right, the same circular intersection of a cone by plane α is presented in axonometric view. Plane α is perpendicular to the axis of the conical surface of revolution, it is parallel to the plane $\pi = xy$ of its basic circle k with centre S, therefore the surface intersection is a circle mapped in the axonometric view as an ellipse. This ellipse can be determined by related diameters AB, CD intersecting in the centre O of the ellipse, while the end points A, B, C, D can be constructed as intersections of surface generating lines mapped in the axonometric views and in the axonometric front views. Point O, centre of the intersection circle, is the intersection point of the conical surfaces axis o and the intersection plane α. Points E and F are points of the change of visibility in the axonometric view, which can be determined as intersections of the plane α and surface lines forming the outlines of the conical surface. These can be found in the axonometric front view, where the plane α perpendicular to the front image plane appears in the edge view.

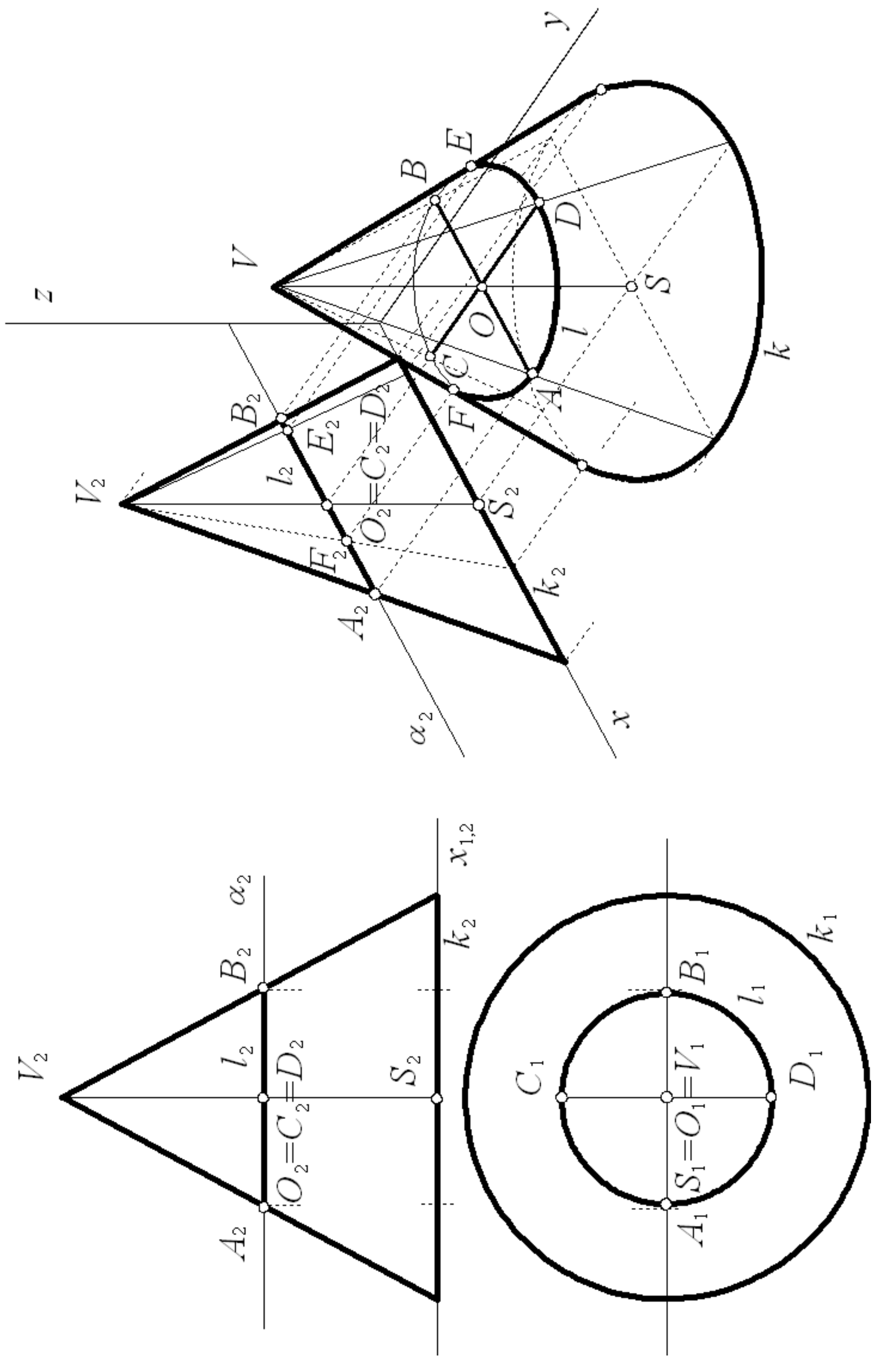


Figure 3. 6 Views of circular intersection on circular conical surface

Ellipse can be one of the possible planar intersections of the circular conical surface, provided the intersection plane α is intersecting all generating lines of the surface. View of the elliptic intersection on the circular conical surface by arbitrary plane not passing through the surface vertex generated in software Maple is in Fig 3. 7.

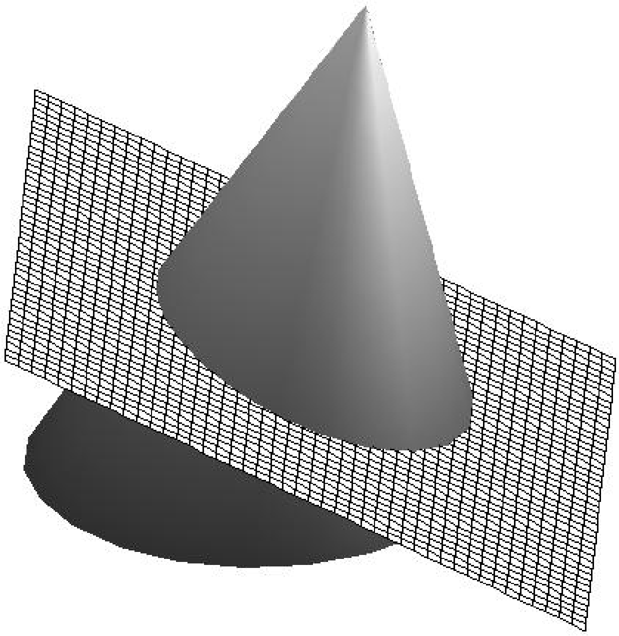


Figure 3. 7 Elliptic intersection on circular conical surface.

Construction of the elliptic intersection on the conical surface of revolution is presented in Fig. 3. 8. The illustration on the left is the elliptical intersection mapped in the Monge method, presented are orthographic views in the frontal image plane and in the ground plane. Intersection plane α perpendicular to the frontal image plane appears in the front view as a line α_2. Intersection ellipse can be determined directly by four vertices *A*, *B*, *C*, *D* and axes, while their common point is its centre *O* , which is not the point on the axis *VS* of the conical surface.

On the right the same elliptic intersection on conical surface of revolution by plane α is presented in the axonometric view. Here the intersection ellipse is determined by related diameters *AB*, *CD* intersecting in the centre *O* of the intersection ellipse. The end points *A*, *B*, *C*, *D* can be constructed as intersections of the respective surface generating lines mapped in the axonometric views and in the axonometric front views. Diameter *AB* is parallel to the frontal image plane, so $AB \,//\, A_2B_2$ .Points *E* and *F* of the change of visibility in the axonometric view of the intersection ellipse are received from the axonometric front view, where the ellipse is mapped to a line segment A_2B_2. These are tangent points on conical surface outlines, which are the views of generating lines mapped to the tangents of the basic circle axonometric view.

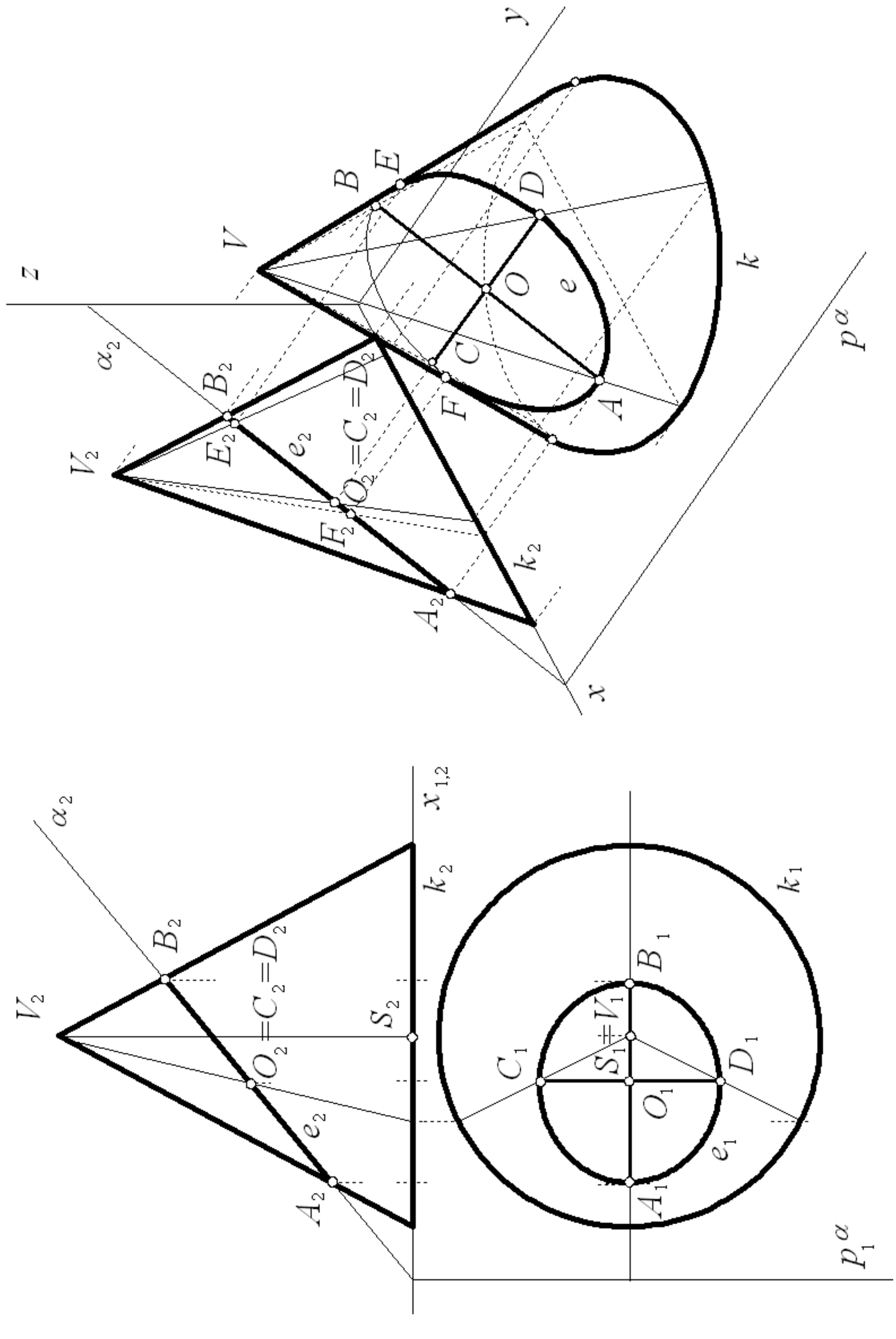


Figure 3. 8 Views of elliptic intersection on circular conical surface.

Parabolic intersection on circular conical surface of revolution can be achieved with intersection plane α parallel to one of the surface generating lines. Views of parabolic intersection on the circular conical surface by arbitrary plane not passing through the surface vertex generated in software Maple is in Fig 3. 9.

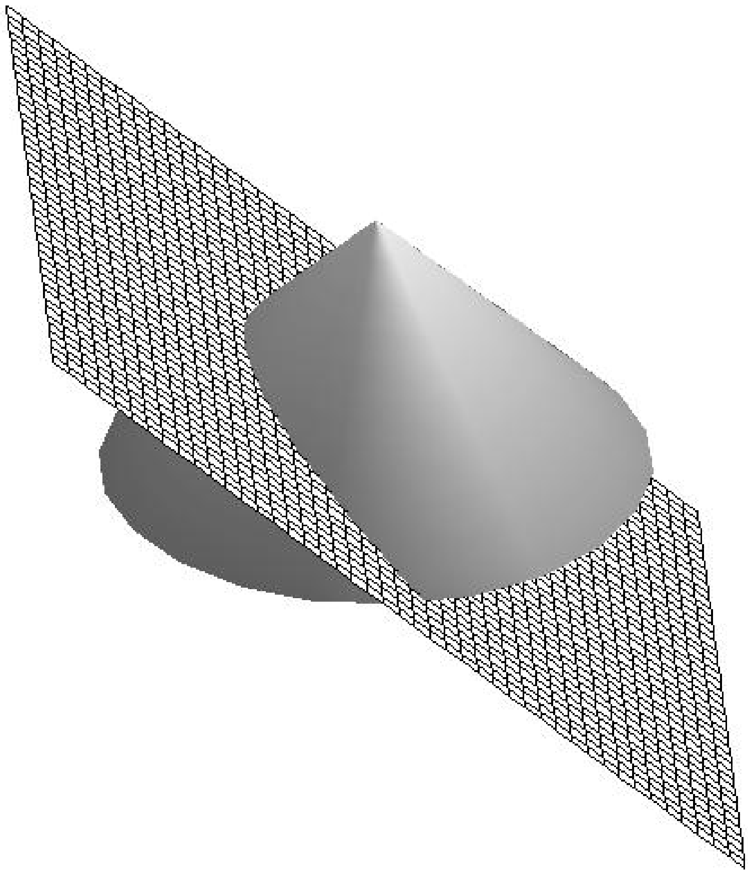


Figure 3. 9 Parabolic intersection on circular conical surface.

Construction of the parabolic intersection on the conical surface of revolution is presented in the Fig. 3. 10, where the parabolic intersection is mapped in Monge method on the left, and in the axonometric view on the right. Intersection plane α perpendicular to the frontal image plane appears in the edge view as a line parallel to one of the conical surface outlines in the front view. In the Monge method, intersection parabola can be determined directly by axis *o*, vertex *M* and two points *K* and *L* on the conical surface basic circle *k*, with attached tangent lines in these points. Axonometric view of this parabolic intersection is determined by 3 points *K*, *L*, *M* with tangent lines to parabola in them, and by the point *E* of the change of visibility. This point is the intersection of the plane α and the surface generating line forming the outline in the axonometric view, and can be found in the axonometric front view.

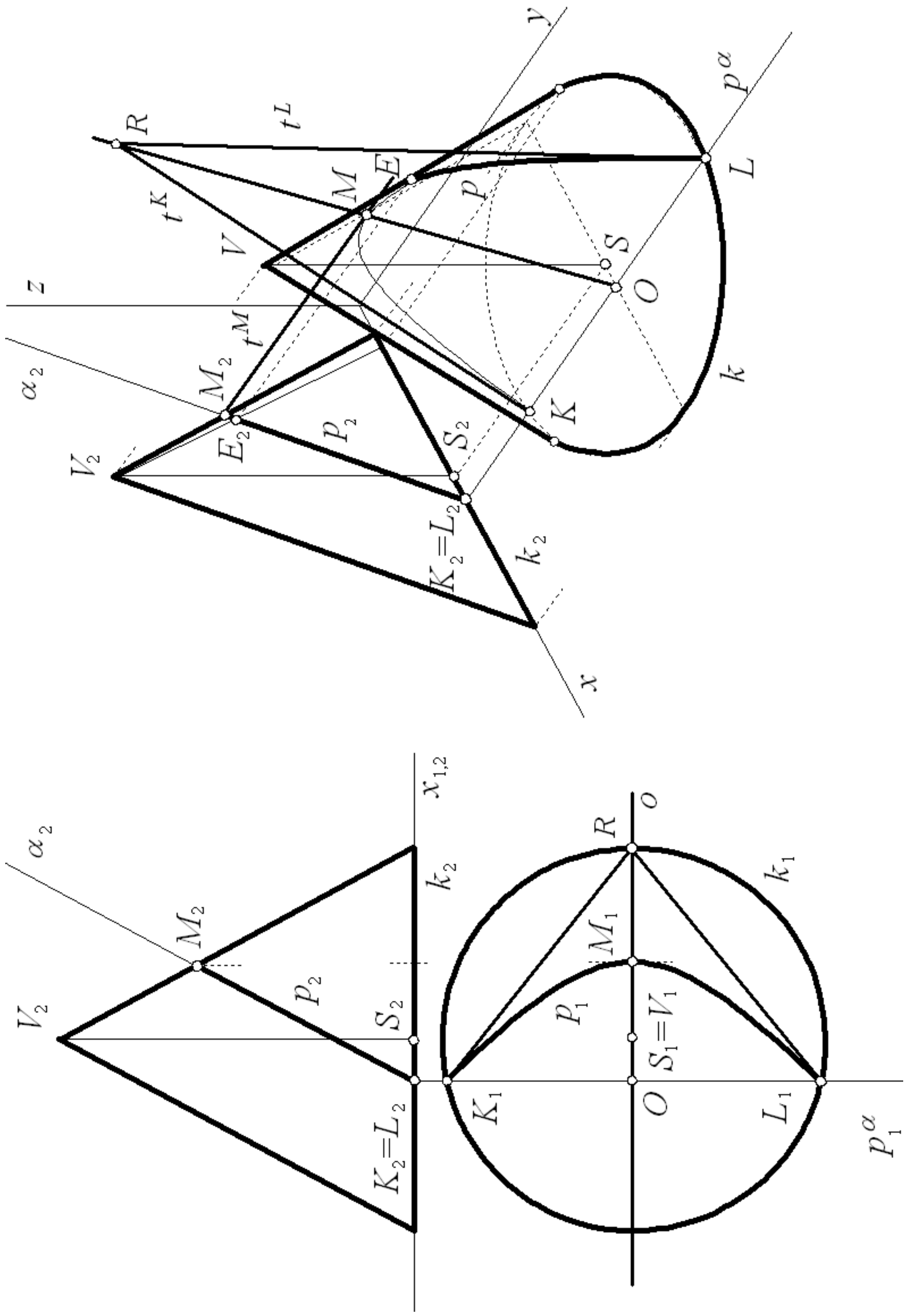


Figure 3. 10 Views of parabolic intersection on circular conical surface.

Hyperbola is a planar intersection of circular conical surface of revolution by plane α that is parallel to two different generating lines on the surface, while it is not passing through the conical surface vertex. View of the intersection generated in software Maple is in Fig 3. 11.

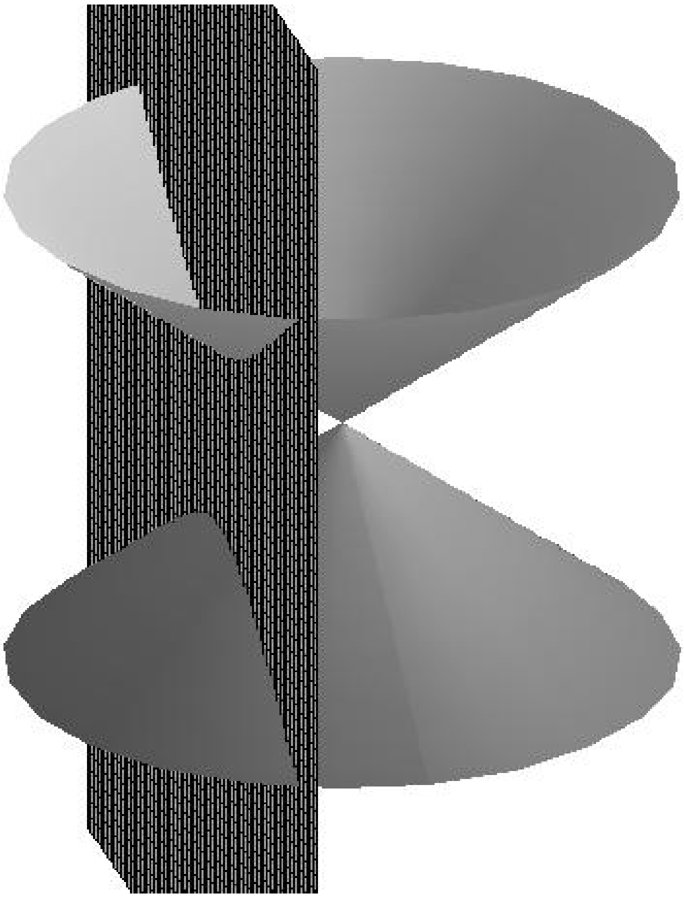


Figure 3. 11 Hyperbolic intersection on circular conical surface.

Construction is illustrated in Fig. 3. 12, in the Monge related views and in axonometric view. The intersection plane α is perpendicular to the frontal image plane and appears as line α_2 in the front view. In the Monge method, intersection hyperbola is determined directly by its major axis ${}^1o$ with major vertices M_1, M'_1. Its asymptotes ${}^1a$, ${}^2a$ are lines passing in direction of top views of generating lines *PV* and *TV* through the hyperbola centre *O*, which is the centre of the line segment *MM´*. These lines are intersections of the conical surface and the plane σ passing through the vertex *V* and parallel to the plane α. The end points *K*, *L*, *K´*, *L´* of the hyperbolic arcs are points on the conical surface basic circle *k* and a symmetric circle *k´* with respect to the surface vertex *V*. Both circles' top views coincide.

Similarly as in the previous constructions, the axonometric view of the intersection hyperbola can be found using axonometric front views of the conical surface generating lines. The upper arc is determined by points *K´M´L´* and the bottom arc by points *KML*. Here, axonometric views of intersection hyperbola points *M* and *M´* are not major vertices of the hyperbola axonometric view, but form its diameter passing through the hyperbola centre *O* and parallel to the frontal image plane. Visibility of the intersection hyperbola axonometric view changes in the point *E* on the conical surface outline, and it can be uniquely determined from the axonometric front view.

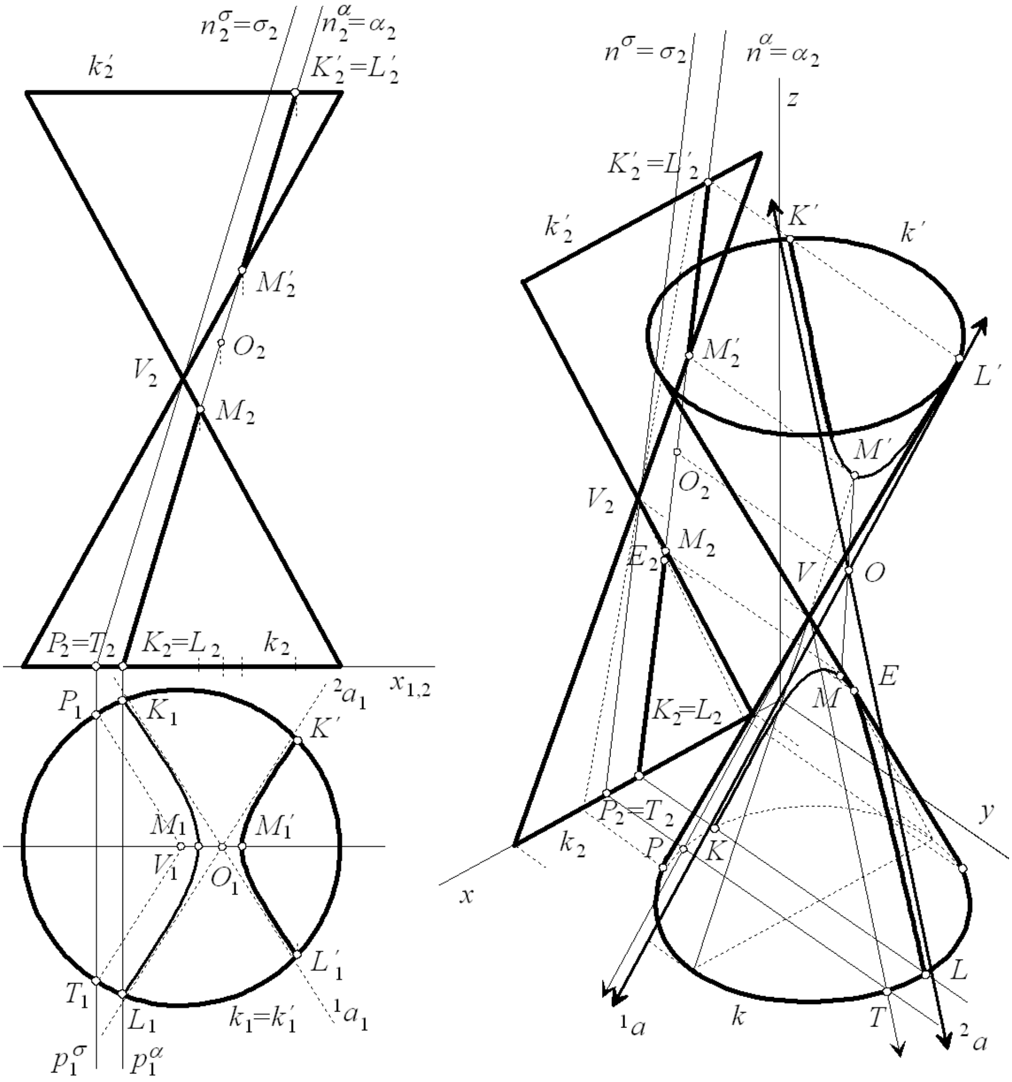


Figure 3. 12 Views of hyperbolic intersection on circular conical surface.

4 QUADRATIC SURFACES OF REVOLUTION

Surfaces of revolution are surfaces generated by revolution of an arbitrary generating curve about a fixed axis in the space. **Quadratic surfaces** (surfaces of the second degree) are all surfaces intersecting an arbitrary line in not more then two different points. We can distinguish between **singular** (cylindrical and conical) and **regular** (spherical surface, ellipsoids, paraboloids and hyperboloids) quadratic surfaces.
By revolving line *a* about axis *o* we can determine:

cylindrical surface of revolution	$a \parallel o$
conical surface of revolution	$a \times o$
one-sheet hyperboloid of revolution	a / o.

Revolving circle $k(S, r)$ about the axis *o* in the plane of circle, while point *S* is on the axis of revolution, $S \in o$, a **spherical surface (or sphere)** is determined.
Regular quadratic surface of revolution can be determined by revolution of a conic section about its axis. Revolving an ellipse about its major axis ^1o an ellipsoid of revolution called **prolate (or elongate) ellipsoid** can be created, while an **oblate (or flat) ellipsoid** of revolution is determined by revolution about its minor axis ^2o. Parabola revolving about its axis *o* generates a **paraboloid** of revolution. Revolving circle, ellipse or parabola form a system of **meridians** of the respective surface of revolution, while trajectories of their points form a system of surface **parallel circles**. **Equator** is the parallel circle with the largest radius, **neck** with the minimal radius. Both ellipsoids and paraboloid of revolution with their net of meridian ellipses or parabolas and parallel circles mapped in Maple can be seen in Fig. 4.1.

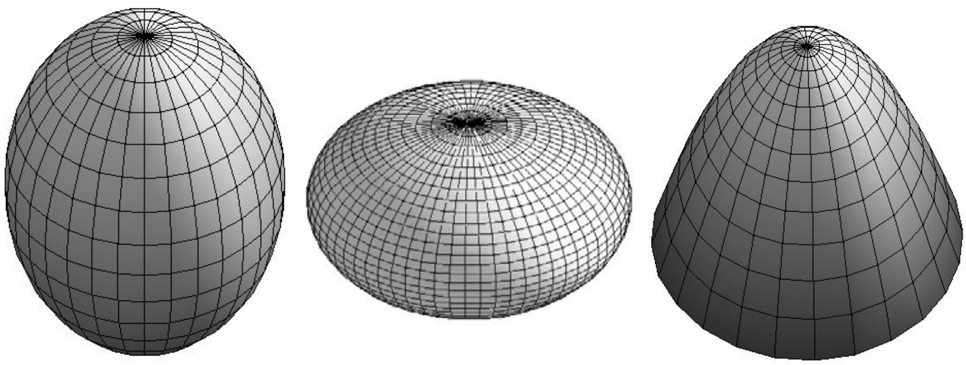


Figure 4. 1 Prolate (left), oblate (middle) ellipsoid of revolution, paraboloid of revolution.

Revolution of a hyperbola about its minor axis ^2o determines a **1-sheet hyperboloid** of revolution, whereas revolution of a hyperbola about its major axis ^1o generates a **hyperboloid of revolution of two sheets**. Both hyperboloids are mapped with their net of meridian hyperbolas and paralel circles in Fig. 4.2.

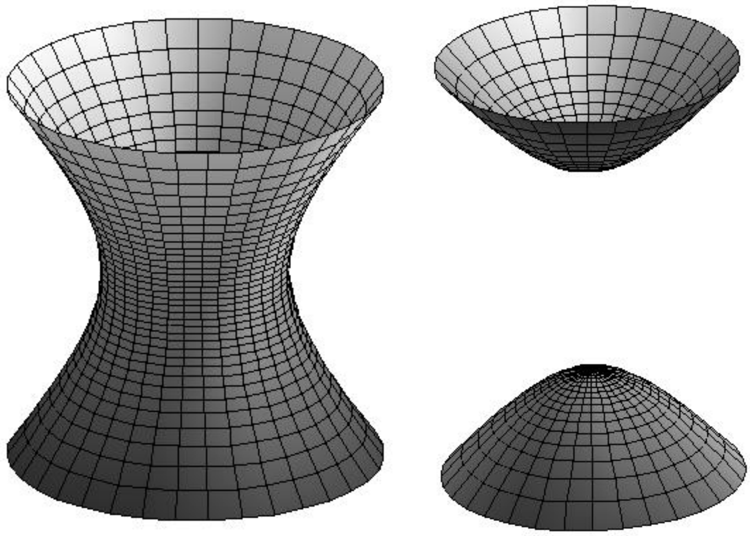


Figure 4. 2 Hyperboloid of revolution: of 1-sheet (left), of 2-sheets (right).

4.1 SPHERE

Sphere can be determined as a set of all points in the same distance $r \neq 0$ from a given fixed point S, which is the centre of the sphere $G = (S, r)$ with radius r.

Any line o passing through the centre S is the axis of the sphere G. Planes passing through the axis o of a sphere are **meridian planes** μ, these intersect the sphere in circles that are meridians with centres in the point S and radius equal to the radius r of the sphere, $m = (S, r)$. Sphere G can be therefore generated by revolution of meridian m about axis o.

Circles, which are intersections of the sphere G and a system of parallel planes α_i perpendicular to the axis o are parallel circles k_i with centres K_i located on axis o and radii r_i from the interval $\langle 0, r \rangle$. Any point T on the sphere G is the point located on one meridian m^T and one parallel circle k, as their common point. Tangent plane τ to the sphere G in the point T is determined by tangent lines t^m and t^k in this point to the respective circles m^T and k located in planes μ^T and α^T. Normal to the sphere in the point T is a line n passing through this point perpendicularly to the tangent plane τ^T of the sphere. All normals to the sphere G are meeting in one common point, centre S of the sphere. In Fig. 4. 3 the construction of a tangent plane and normal to the sphere in the point T is presented in the Monge method.

Any plane ρ that is intersecting sphere $G = (S, r)$, regardless of its position, intersects it in a circle l. Centre O of this circle is located on line passing through the sphere centre S and perpendicular to the intersection plane ρ. Radius p of the intersection circle $l, p = \sqrt{r^2 - d^2}$, can be derived from triangle LSO with a right angle at the vertex O, where L is an arbitrary point of the intersection circle l, therefore $|SL| = r$, and distance of the plane ρ from the sphere centre S is $|SO| = d$.

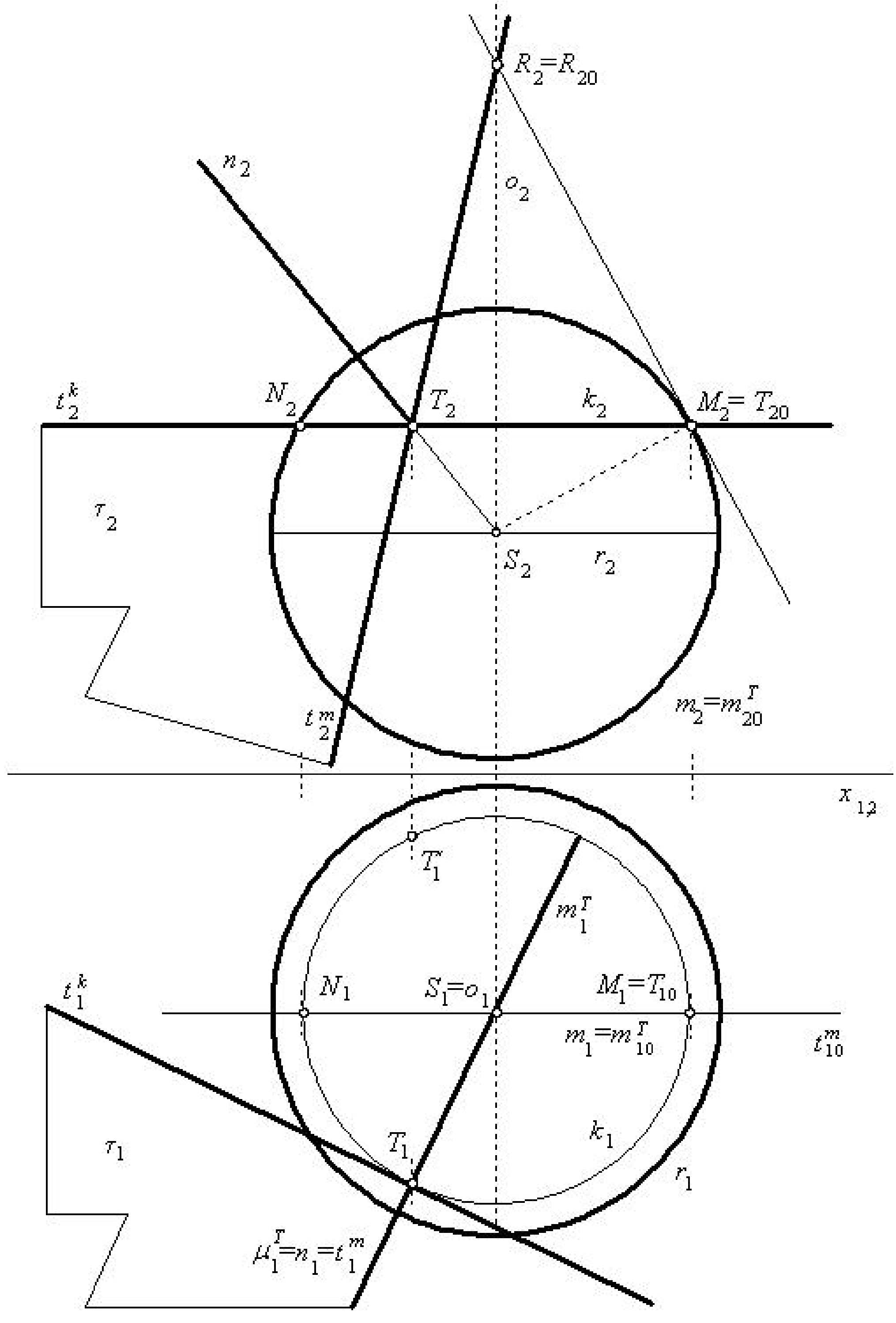


Figure 4. 3 Related views of tangent plane and normal to the sphere

Choosing the intersection plane ρ in a position perpendicular to the frontal image plane, construction of the intersection circle l = (O, p) is provided in Fig. 4. 4. Intersection circle is mapped in the front view to the line segment l_2. Top view appears in the form of an ellipse l_1. Its centre O is on the line h passing through the sphere centre S perpendicularly to the intersection plane ρ. In the front view centre O coincides with views of points A and B, which are mapped in the top view to the major vertices of l_1 and can be found on sphere parallel circle k. End points C_2 and D_2 of l_2 are mapped in the top views as minor vertices of ellipse l_1. Visibility in the top view is determined by the sphere equator r, whose common points E an F with the circle l are mapped in the top views to the points of change of visibility of the ellipse l_1. Part of circle l over the equator r is visible in the top view.

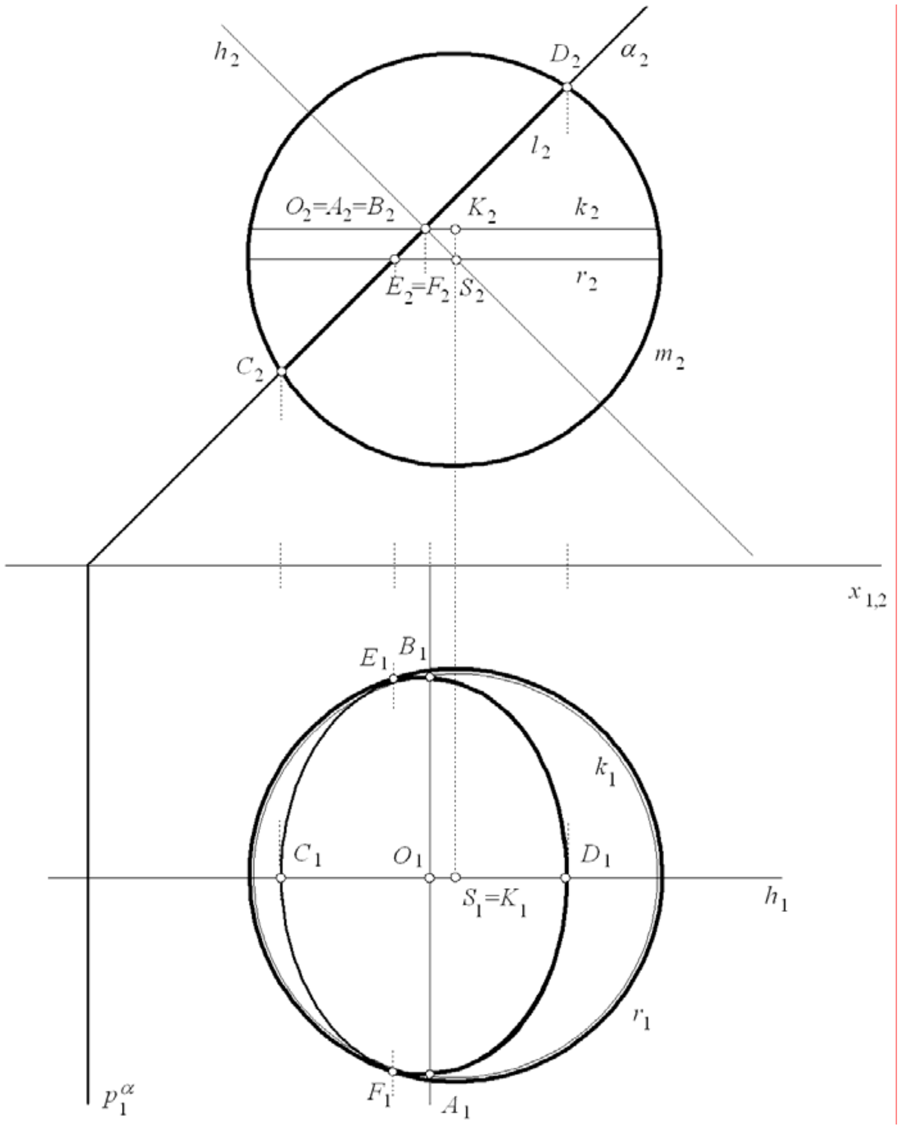


Figure 4. 4 Related views of the planar intersection on the sphere

If the sphere $G(S, r)$ is placed in the Euclidean space $\mathbf{E}^3$ with the coordinate system $\{O, x, y, z\}$ and the centre is point $S = [m, n, p]$, the analytic equation of the sphere is

$$(x-m)^2 + (y-n)^2 + (z-p)^2 = r^2$$

4.2 ELLIPSOIDS

Ellipsoid of revolution **E** determined by revolution of an ellipse with centre at point $S = [m, n, p]$ and axes parallel to the coordinate axis x and z with semi-axis a and b, whose axis is parallel to coordinate axis z has the analytic equation

$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{a^2} + \frac{(z-p)^2}{b^2} = 1,$$

while for $a < b$ a prolate, and for $a > b$ an oblate ellipsoid of revolution is generated. Planar intersection of an ellipsoid of revolution can be:

a) a circle, if the intersection plane is perpendicular to the ellipsoid axis,

b) an ellipse in all other cases.

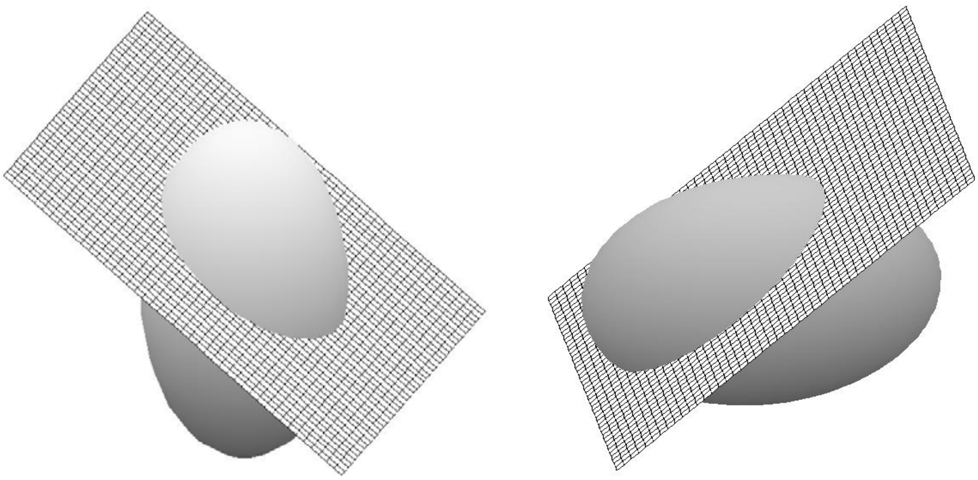


Figure 4. 4 Views of elliptic intersection on ellipsoids of revolution.

Intersection of an oblate ellipsoid (with axis o and with meridian ellipse m) by plane ρ perpendicular to the ground image plane is an ellipse e (in Fig. 4. 5) mapped as line segment e_1 in the top view and as an ellipse e_2 in the front view. End points A_1, B_1 of the top view mapped to the major vertices A_2, B_2 of the frontal view e_2 are points on the ellipsoid equator r. Centre O of the ellipse e coincides in the top view with top views of points C, D, which are mapped to the minor vertices of the intersection frontal view are on parallel circles k and l of the ellipsoid. Visibility in the frontal view is determined by meridian m, its common points E and F with the intersection ellipse e can be determined from the top view. The front view e_2 of the intersection ellipse e is tangent to the ellipsoid outline m_2 in the points E_2 and F_2.

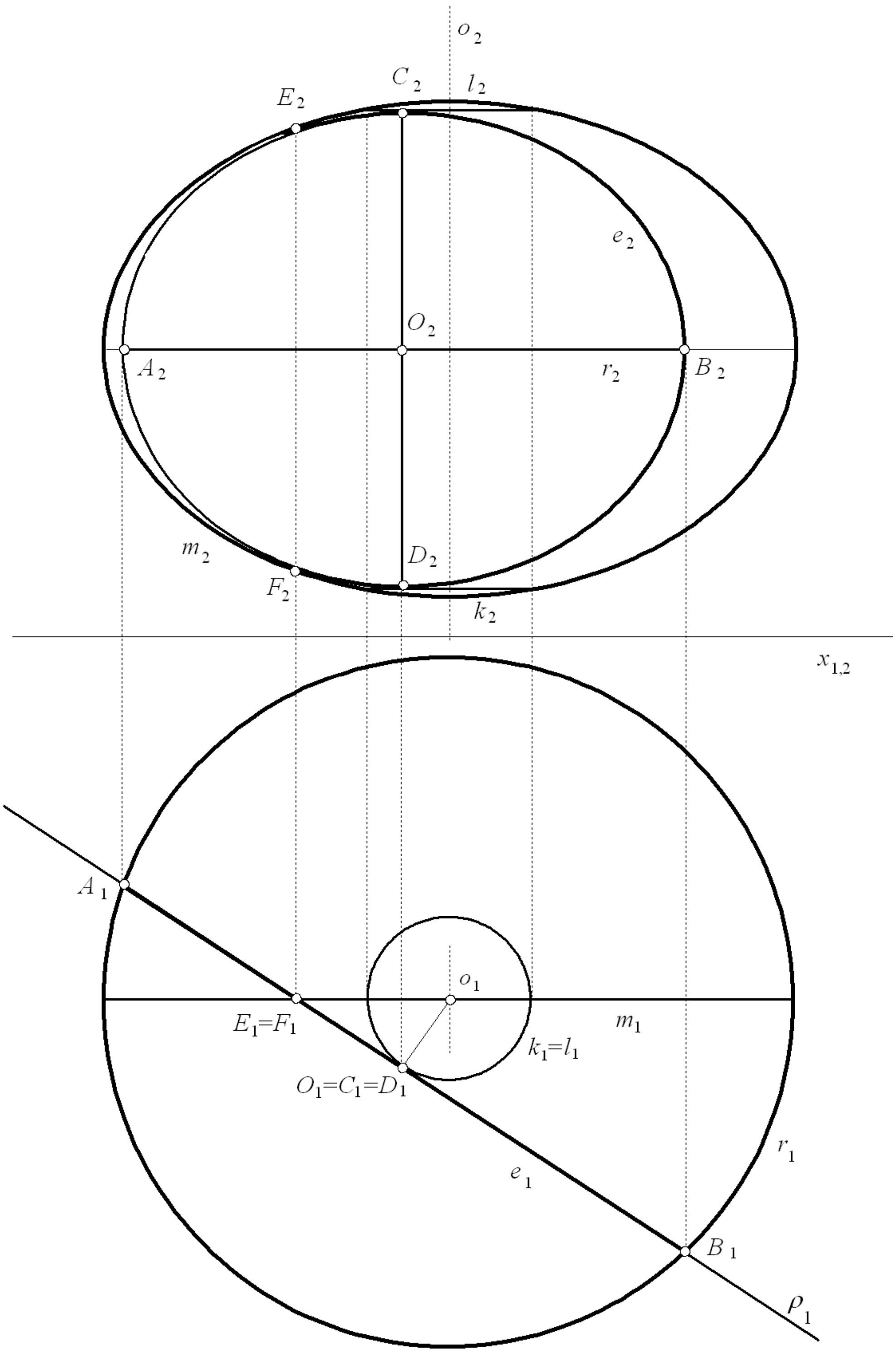


Figure 4. 5 Related views of planar intersection on oblate ellipsoid of revolution.

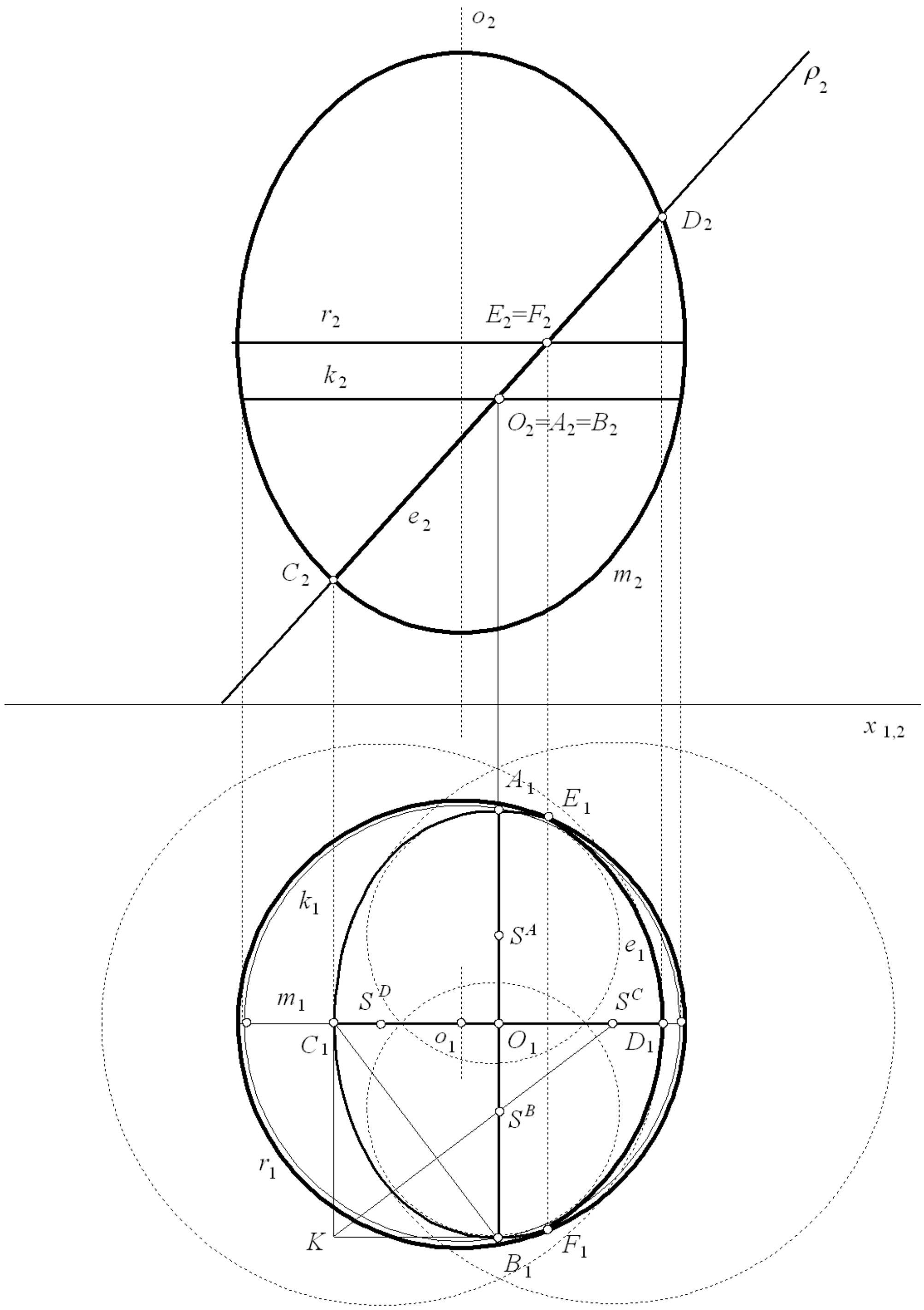


Figure 4. 6 Related views of planar intersection on prolate ellipsoid of revolution.

Construction of elliptic intersection on a prolate ellipsoid with axis *o* perpendicular to the ground image plane and with meridian ellipse *m* in the plane parallel to the frontal image plane is presented in Fig 4. 6. Intersection plane ρ is perpendicular to the frontal image plane. Intersection ellipse *e* appears as a line segment e_2 in the front view, with end points C_2 and D_2, while points *C* and *D* are points on the ellipsoid meridian *m*. They are mapped in the top views as minor vertices C_1 and D_1 of ellipse e_1 that is the intersection ellipse *e* top view. Major vertices A_1 and B_1 of the ellipse e_1 can be determined from the front views A_2 and B_2, coinciding with the front view of the intersection ellipse centre O_2. Points *A* and *B* are on one parallel circle *k* of the ellipsoid. Visibility of the intersection top view is determined by equator *r*, its common points *E* and *F* with the intersection ellipse *e* can be directly determined from the front view, where their front views coincide. Ellipse e_1 can be constructed by means of its four hyper-osculating circles with centres S^A, S^B, and S^C, S^D, symmetric in pairs of major and minor vertices.

4.3 PARABOLOID

If paraboloid of revolution **P** is placed in Euclidean space $\mathbf{E}^3$ with coordinates system $\{O, x, y, z\}$, and it is determined by parabola with vertex at the point $V = [m, n, p]$, parameter *p*, and axis parallel to the coordinate axis *z*, then the analytic equation of the sphere can be written as

$$\frac{(x-m)^2}{p} + \frac{(y-n)^2}{p} = -2(z-p)$$

Planar intersection of a paraboloid of revolution can be:

a) a circle, if the intersection plane is perpendicular to the paraboloid axis,

b) a parabola, if the intersection plane is parallel to the paraboloid axis,

b) an ellipse in all other cases.

Elliptic and parabolic intersection on paraboloid of revolution mapped in Maple is presented in Fig. 4. 7.

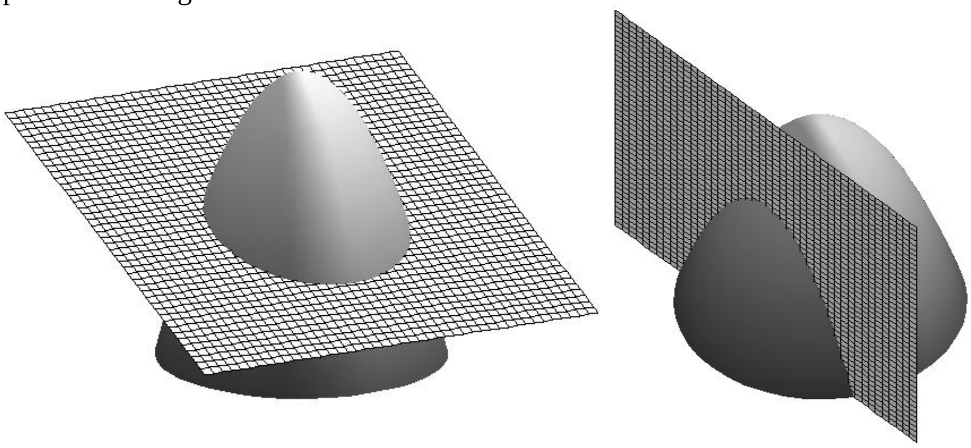


Figure 4. 7 Elliptic and parabolic intersection on paraboloid of revolution.

Theorem 1. Orthographic view of any ellipse on a paraboloid of revolution in a plane perpendicular to the paraboloid axis is a circle.

Construction of an elliptic intersection on a paraboloid of revolution is presented in the Monge method in Fig. 4. 8. Paraboloid of revolution is determined by axis o perpendicular to the ground image plane and meridian parabola m in the plane parallel to the frontal image plane. Intersection plane ρ is perpendicular to the frontal image plane, therefore the intersection ellipse e appears in the front view as a line segment e_2 with end points A_2 and B_2. Points A and B are points on the paraboloid meridian m. Top views A_1 and B_1 form one diameter of the circle e_1, which is the top view of the intersection ellipse e, see the Theorem 1. Centre O of the ellipse e is mapped to the centre O_2 of the line segment e_2 and to the centre O_1 of the circle e_1 in the top view. Intersection is entirely visible in the top view.

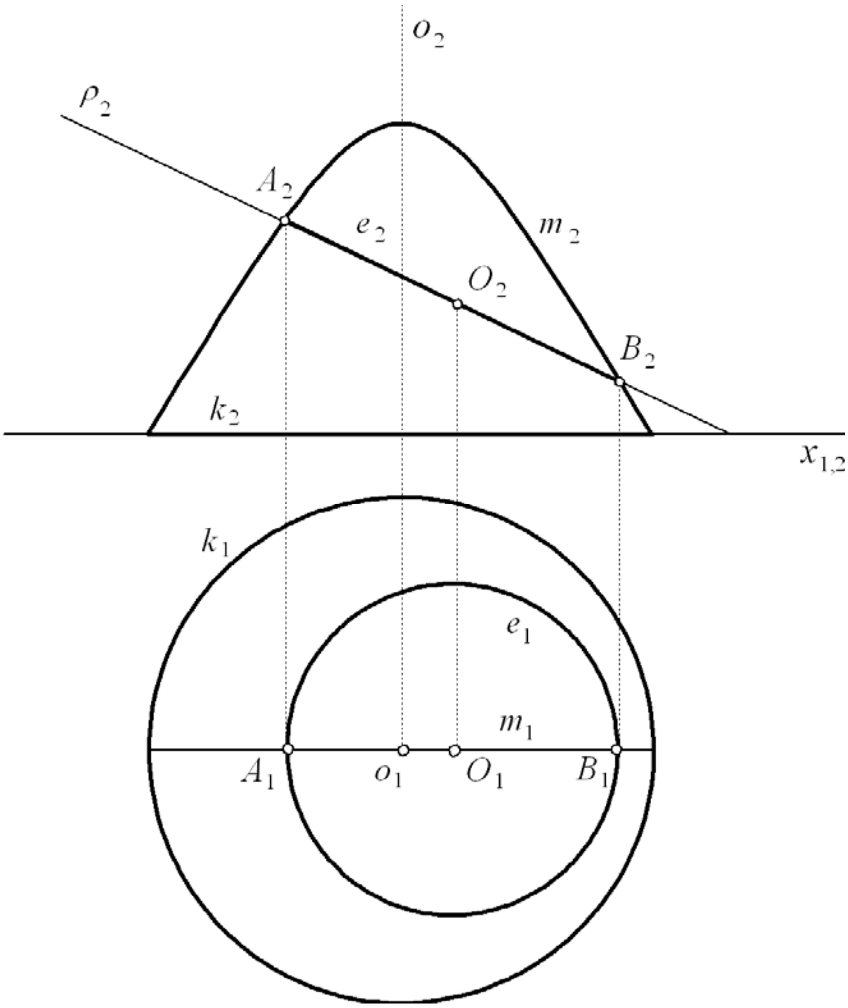


Figure 4. 8 Elliptic intersection on paraboloid of revolution in Monge method.

Plane ρ parallel to the axis o of the paraboloid of revolution is intersecting it in a parabola. In Fig. 4. 9 this construction is illustrated in the Monge method. Intersection plane appears in the edge view ρ_1, so the respective parabolic arc p_1 of the intersection is mapped as line segment A_1B_1. Points A and B are on one parallel circle k, in which ground image plane π intersects paraboloid of revolution, therefore their front views A_2, B_2 are on the coordinate axis x. Axis o' of the intersection parabola is parallel to the axis o of the paraboloid, while its vertex V is mapped in the top view to centre V_1 of line segment A_1B_1. Front view V_2 can be determined by means of a parallel circle h, passing through the point V on the paraboloid of revolution. Applying the properties of a subnormal and subtangent to the intersection parabola front view p_2, tangent lines t_2^A, t_2^B in the points A_2, B_2 and hyper-osculating circle in the vertex V can be attached. Visibility is determined by the meridian parabola m, its common point with the intersection parabola p can be derived from the top view. Parabolic arc AVE in front of the meridian m is visible.

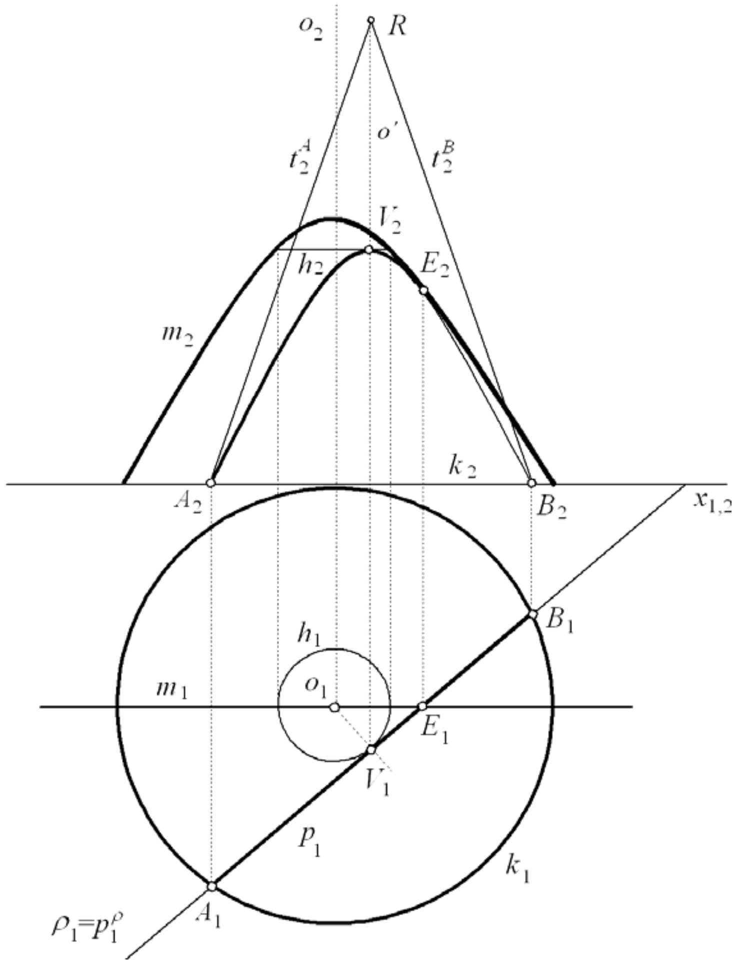


Figure 4. 9. Related views of parabolic intersection on paraboloid of revolution.

4.4 HYPERBOLOIDS

Hyperbola revolving about one of its axes generates a hyperboloid, whereas its asymptotes, as lines meeting in one common point (centre S of the hyperbola) at the axis of revolution, generate a conical surface of revolution with the main vertex S called **asymptotic conical surface** of a hyperboloid. Asymptotic conical surface is inside the hyperboloid of one sheet, while it is outside the hyperboloid of two sheets.
Hyperboloid of revolution **H** determined by a hyperbola with centre at point $S = [m, n, p]$ and axes parallel to the coordinate axis x and z with semi-axis a and b, whose axis is parallel to the coordinate axis z has the analytic equation

$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{a^2} - \frac{(z-p)^2}{b^2} = \pm 1,$$

while a 1-sheet hyperboloid is generated for +1, and a 2-sheet hyperboloid for −1. Planar intersection of a hyperboloid of revolution can be:

a) a circle, if the intersection plane is perpendicular to the hyperboloid axis,

b) a conic section of the same type, as the intersection of the plane and asymptotic conical surfaces of the hyperboloid.

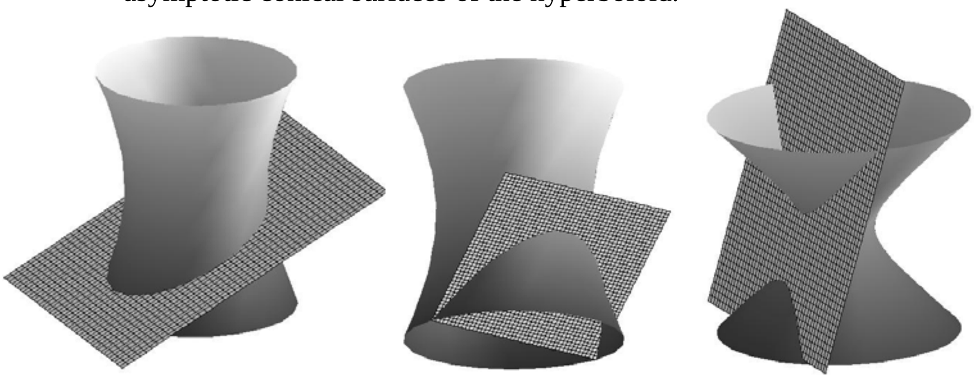


Figure 4. 10 Conic sections on a 1-sheet hyperboloid of revolution.

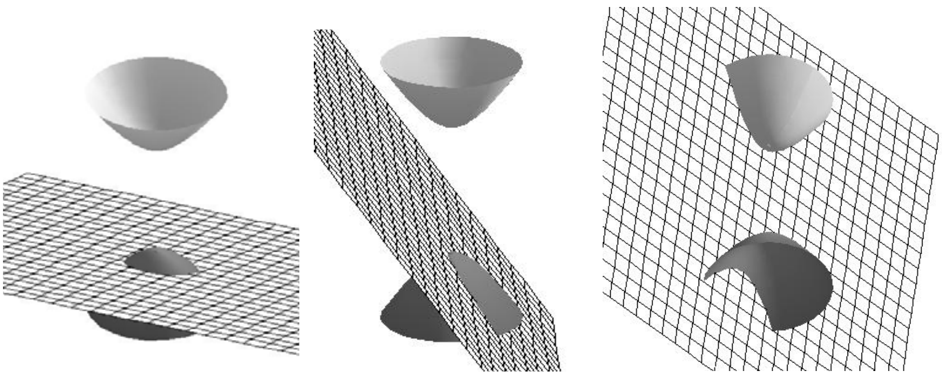


Figure 4. 11 Conic sections on a 2-sheet hyperboloid of revolution.

Construction of the elliptic intersection on the 1-sheet hyperboloid of revolution in the Monge method is presented in Fig. 4. 12. Intersection plane ρ perpendicular to the frontal image plane intersects all lines on the hyperboloid asymptotic conical surface, and intersection ellipse on the hyperboloid appears as a line segment e_2 in the front view. Top view in the form of an ellipse e_1 can be constructed in the same way as the elliptic intersection on the ellipsoid of revolution.

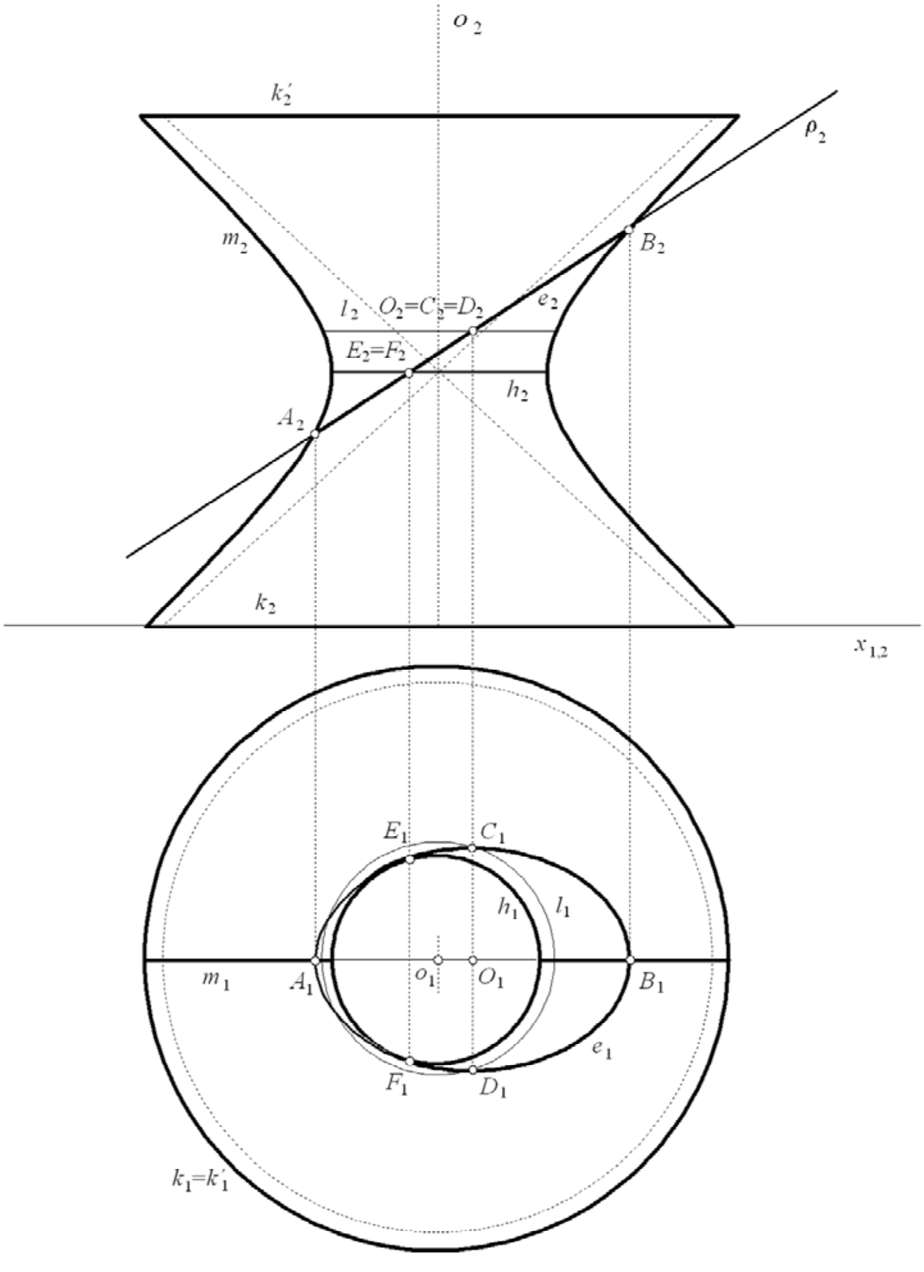


Figure 4. 12 Views of elliptic intersection on 1-sheet hyperboloid of revolution.

Parabolic intersection on the 2-sheet hyperboloid of revolution in the Monge method is presented in Fig. 4. 13. Intersection plane ρ perpendicular to the frontal image plane is parallel to the outline of the asymptotic conical surface of the hyperboloid, therefore it intersects all but one line on this surface, and intersection conic is a parabola on both, asymptotic conical surface and hyperboloid of revolution. Segment of the intersection parabola p on the hyperboloid patch appears as a line segment p_2 in the front view, top view is in the form of the parabolic arc p_1. End points A_1 and B_1 of p_1 are mapped to one end point $A_2 = B_2$ of the line segment p_2 , while the other endpoint V_2 is mapped in the top view as the vertex V_1 of the parabolic arc p_1. Axis o, focus F, hyper-osculating circle $h^V(S^V, p)$ and tangent lines t_1^A, t_1^B in endpoints A_1 and B_1 to the parabolic arc p_1 can be constructed by means of known constructions.
Hyperbola is the intersection of a hyperboloid of revolution by the plane which also intersects the surface asymptotic conical surface in a hyperbola; therefore it is parallel to two generating lines on this asymptotic conical surface. Construction of a hyperbolic intersection on a 1-sheet hyperboloid of revolution is presented in Fig. 4. 14, while how to construct this intersection on a 2-sheet hyperboloid of revolution can be seen in Fig. 2. 15.
Intersection plane in Fig. 4. 14 is perpendicular to the ground image plane, therefore the segment of the intersection hyperbola on the mapped patch of the hyperboloid of revolution appears as line segment $h_1 = P_1R_1$ in the top view. Its end points determine endpoints of the 2 branches of the intersection hyperbola arcs in the top view, and these points are located on the hyperboloid parallel circles k and k' symmetric with respect to the neck circle l. Centre of h_1 is the top view of the intersection hyperbola centre O coinciding with the top views of its major vertices A and B. Their front views can be attached using views of hyperboloid parallel circles. Major axis is line $o_2 = A_2B_2$. Asymptotes ${}^1a$, ${}^2a$ of the intersection hyperbola are lines in direction of asymptotic conical surface generating lines, which are intersections of asymptotic conical surface and plane σ parallel to the intersection plane ρ and passing through its vertex V. The intersection hyperbola is entirely visible in the front view, as it has no common points with the surface meridian m and is located in front of it. Hyperbola can be constructed more precisely when the hyper-osculating circles are attached in its major vertices.
Hyperbolic intersection on the 2-sheet hyperboloid of revolution in Fig. 4.15 is in the plane perpendicular to the frontal image plane. Segment of the intersection hyperbola h on the hyperboloid patch appears as a pair of line segments $P_2'A_2$ and B_2P_2, top view is in the form of the hyperbolic arc h_1 with two branches. End points P_1' and R_1' of one branch of h_1 are views of one end point $P_2' = R_2'$ of one line segment, while the other endpoint A_2 is mapped in the top view A_1 as one vertex of the hyperbolic arc h_1. End points of the other line segment B_2P_2 determine the other branch end points P_1 and R_1 and other major vertex B_1. Major axis is $o_1 = A_1B_1$. Asymptotes ${}^1a$, ${}^2a$ are as in the previous example lines parallel to those generating lines, in which plane σ parallel to the intersection plane ρ and passing through the vertex V intersects the asymptotic conical surface. Branch $P'AR'$ is visible, while PBR is not visible in the intersection hyperbola top view.

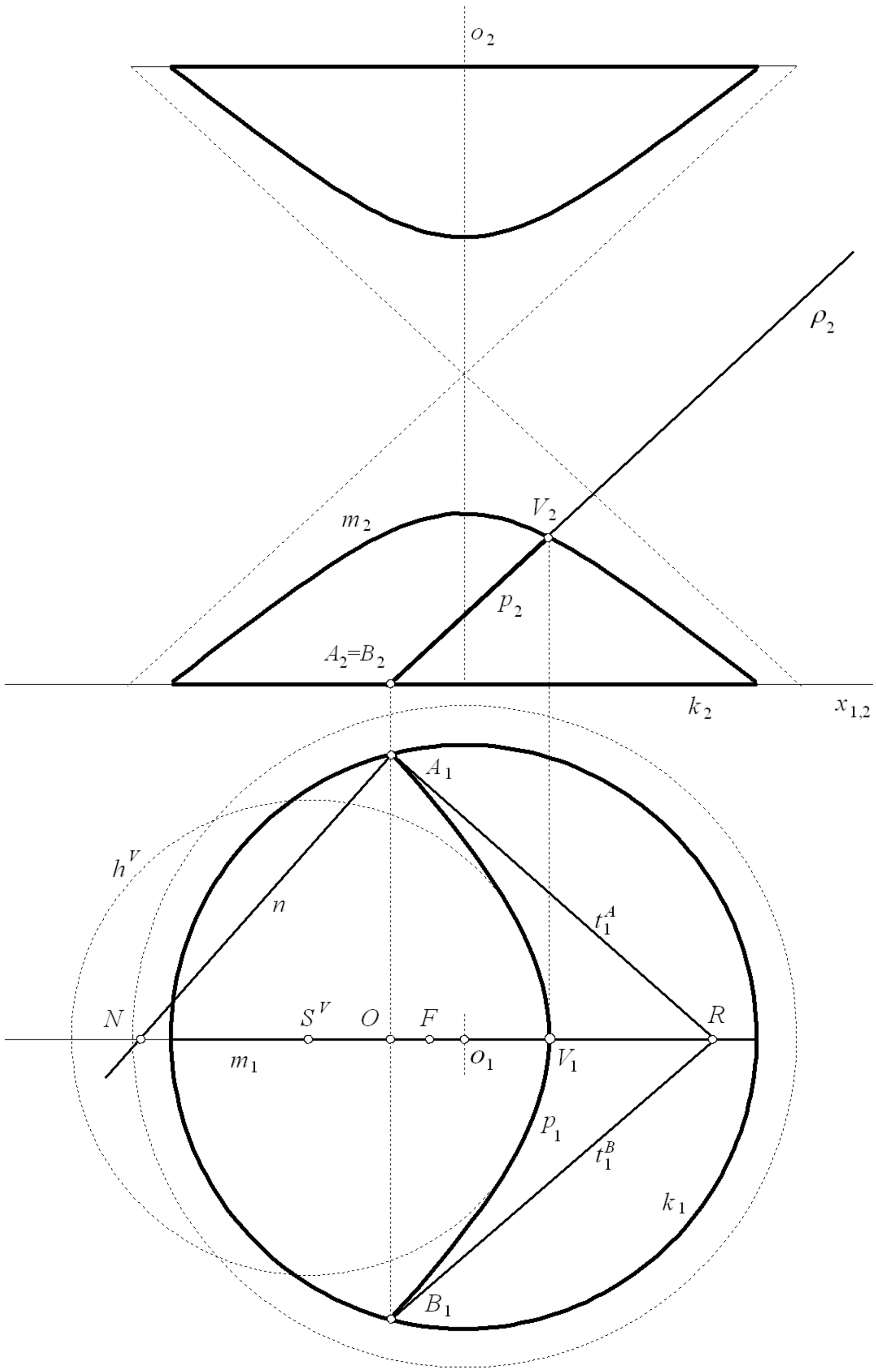


Figure 4. 13 Views of parabolic intersection on 2-sheet hyperboloid of revolution.

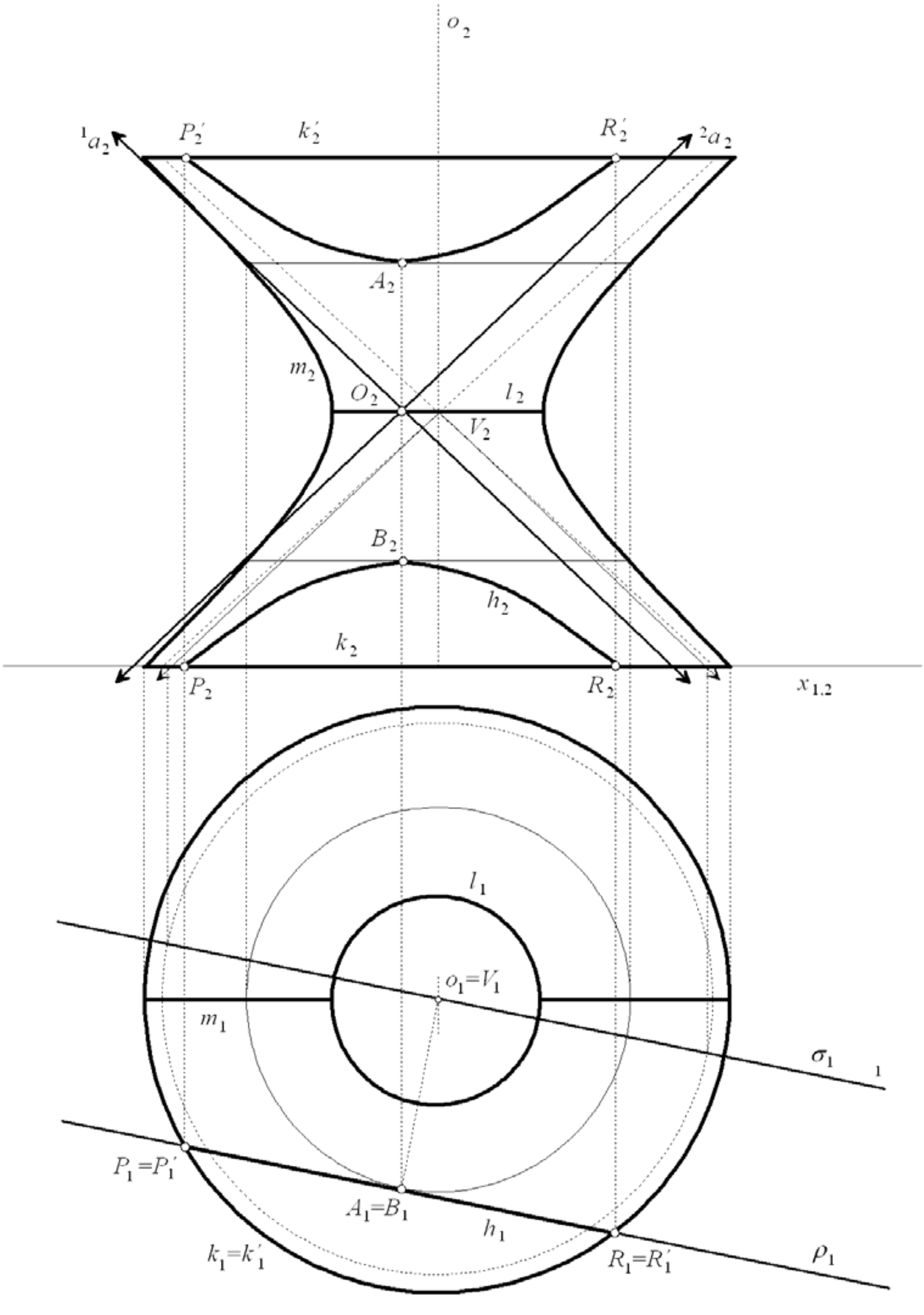


Figure 4. 14 Views of hyperbolic intersection on 1-sheet hyperboloid of revolution.

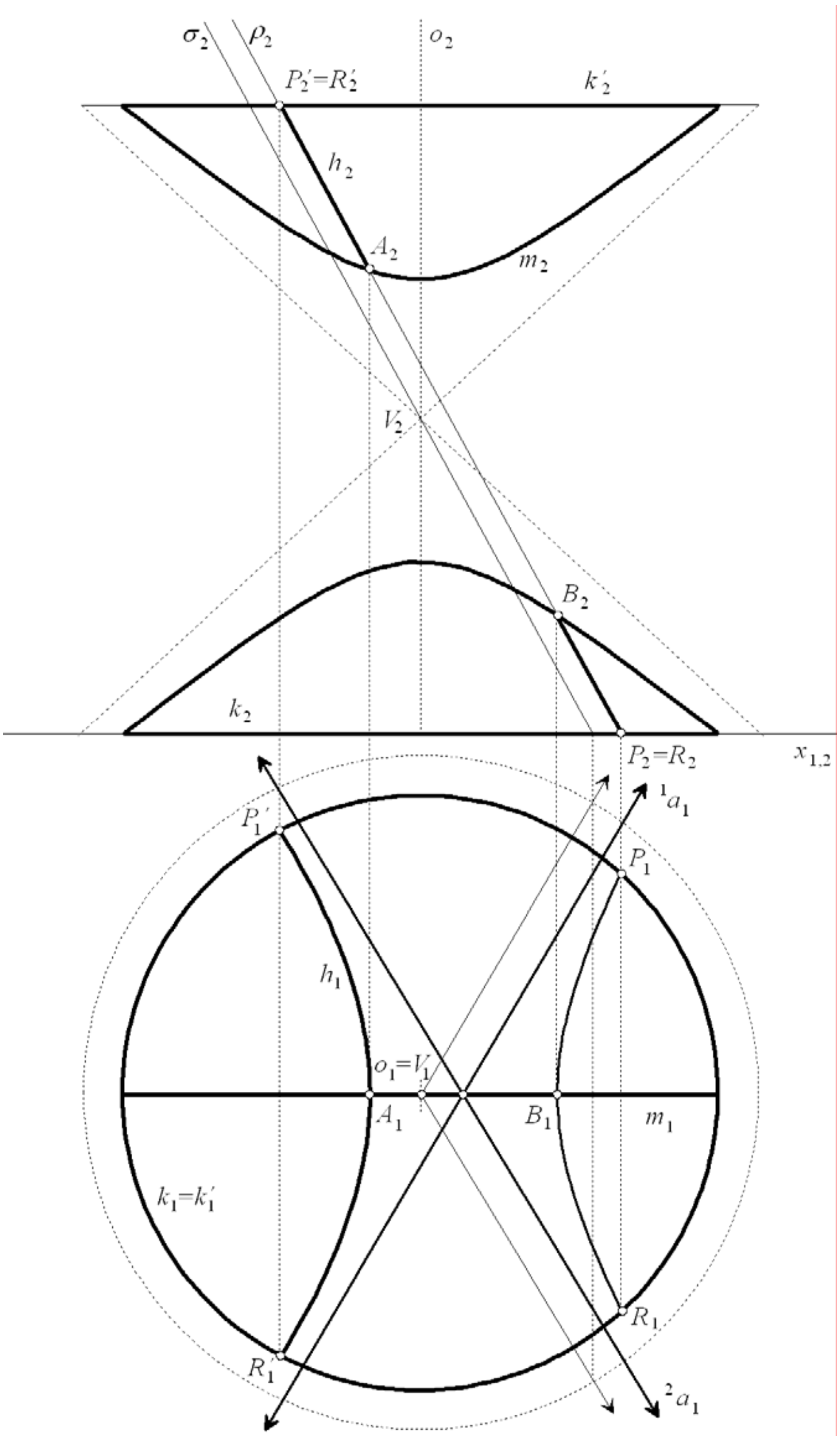
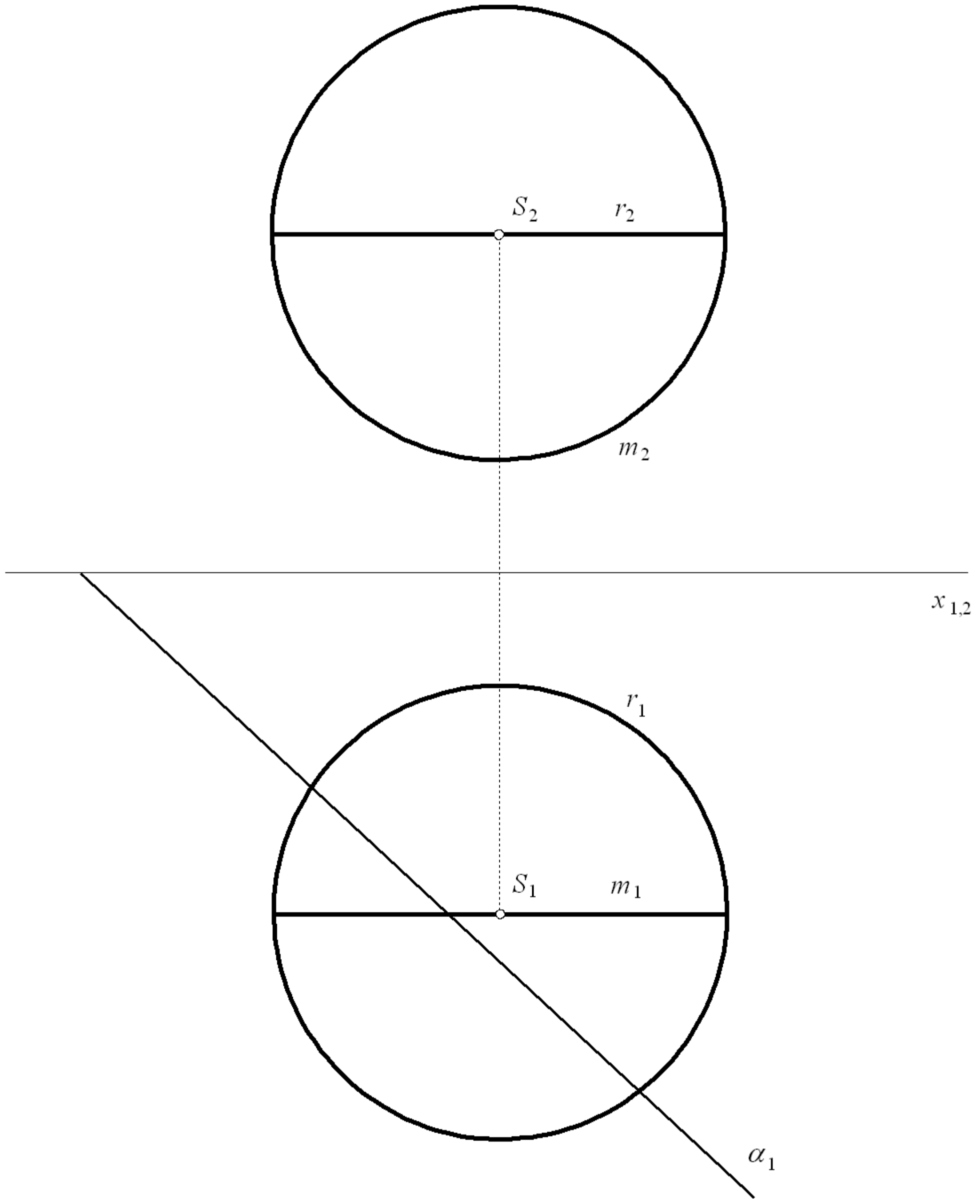


Figure 4. 15 Views of hyperbolic intersection on 2-sheet hyperboloid of revolution.

5 PROBLEMS

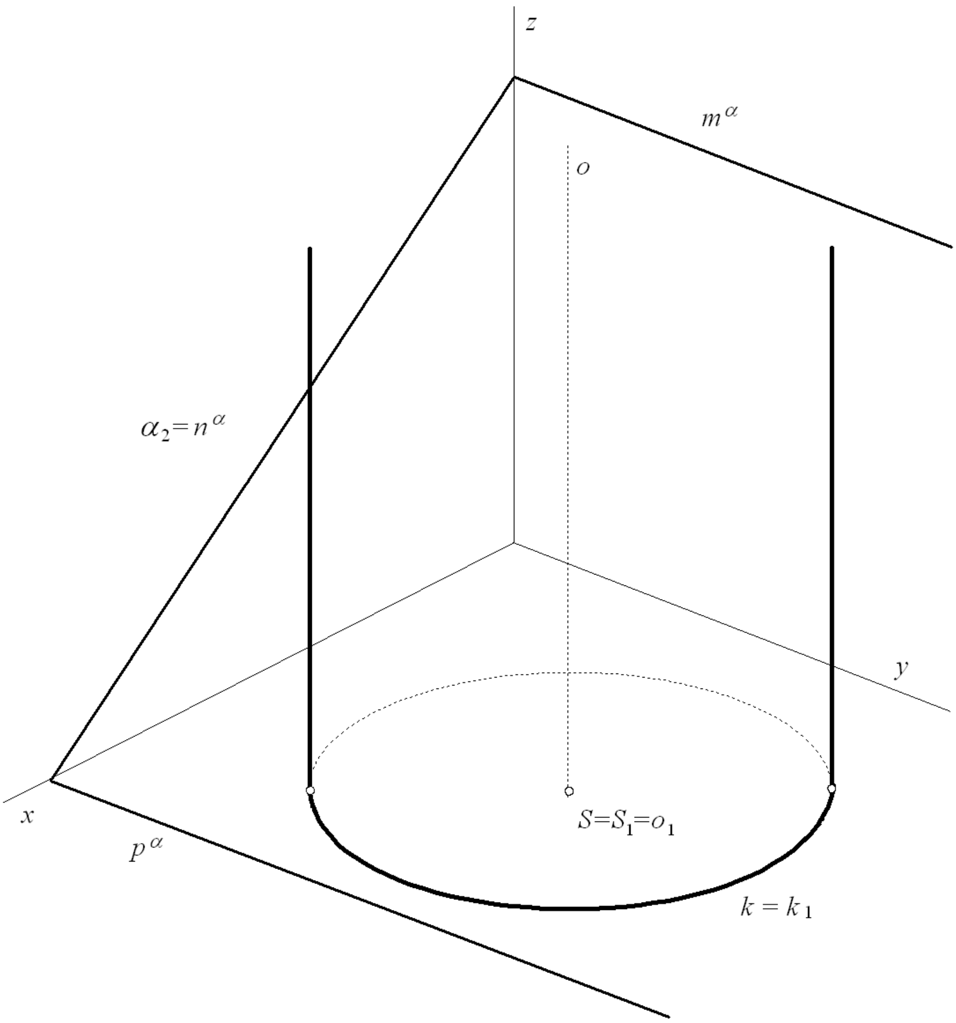
5.1 PLANAR INTERSECTIONS ON SPHERE

1. Find related views of the intersection on the sphere $G(S, r)$ by plane α perpendicular to the ground image plane.

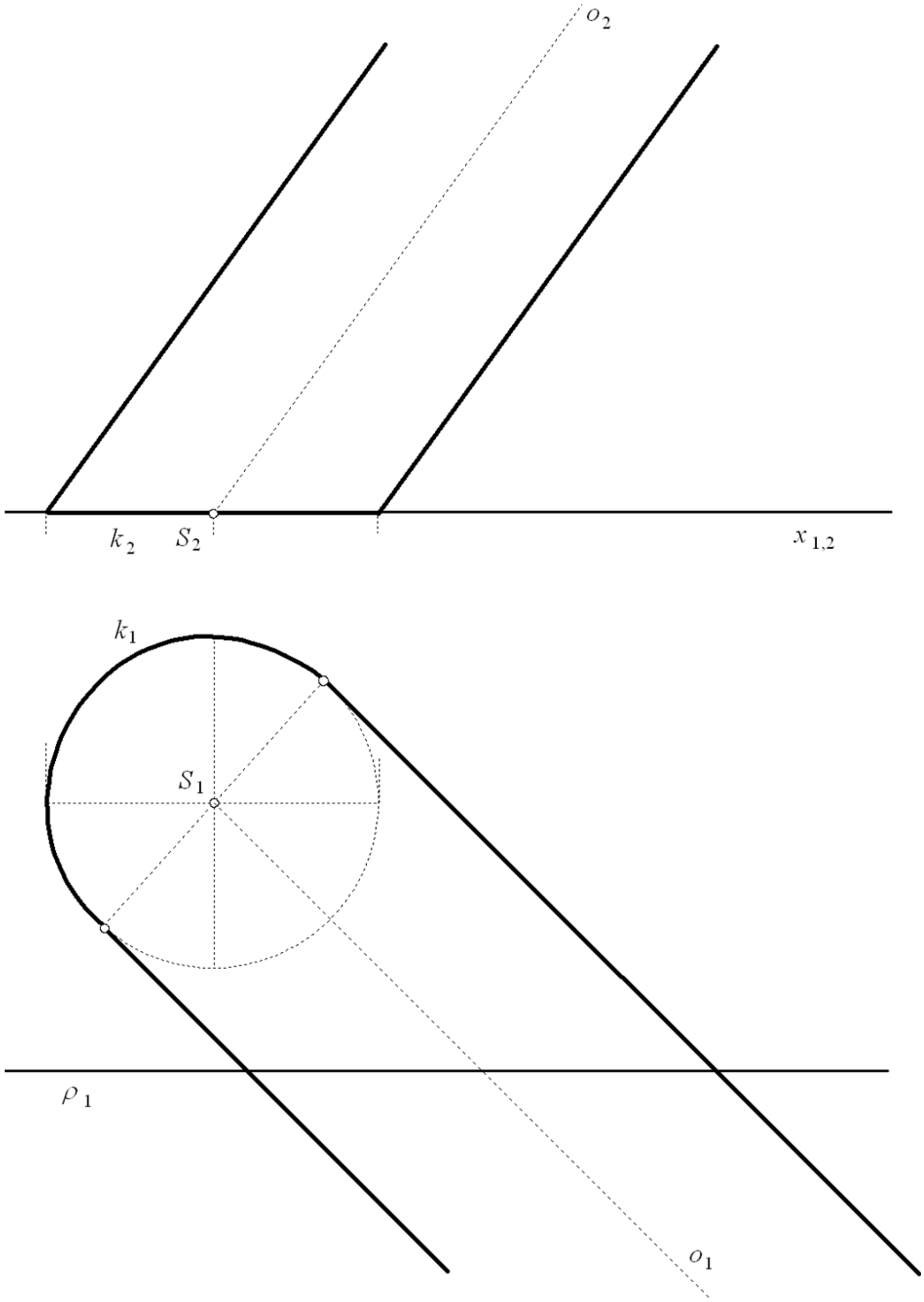


5.2 PLANAR INTERSECTIONS ON CYLINDRICAL SURFACES

2. Find axonometric view of the planar intersection of a cylindrical surface of revolution with the basic circle $k(S, r)$ in the ground image plane and axis o by the plane α perpendicular to the frontal image plane.

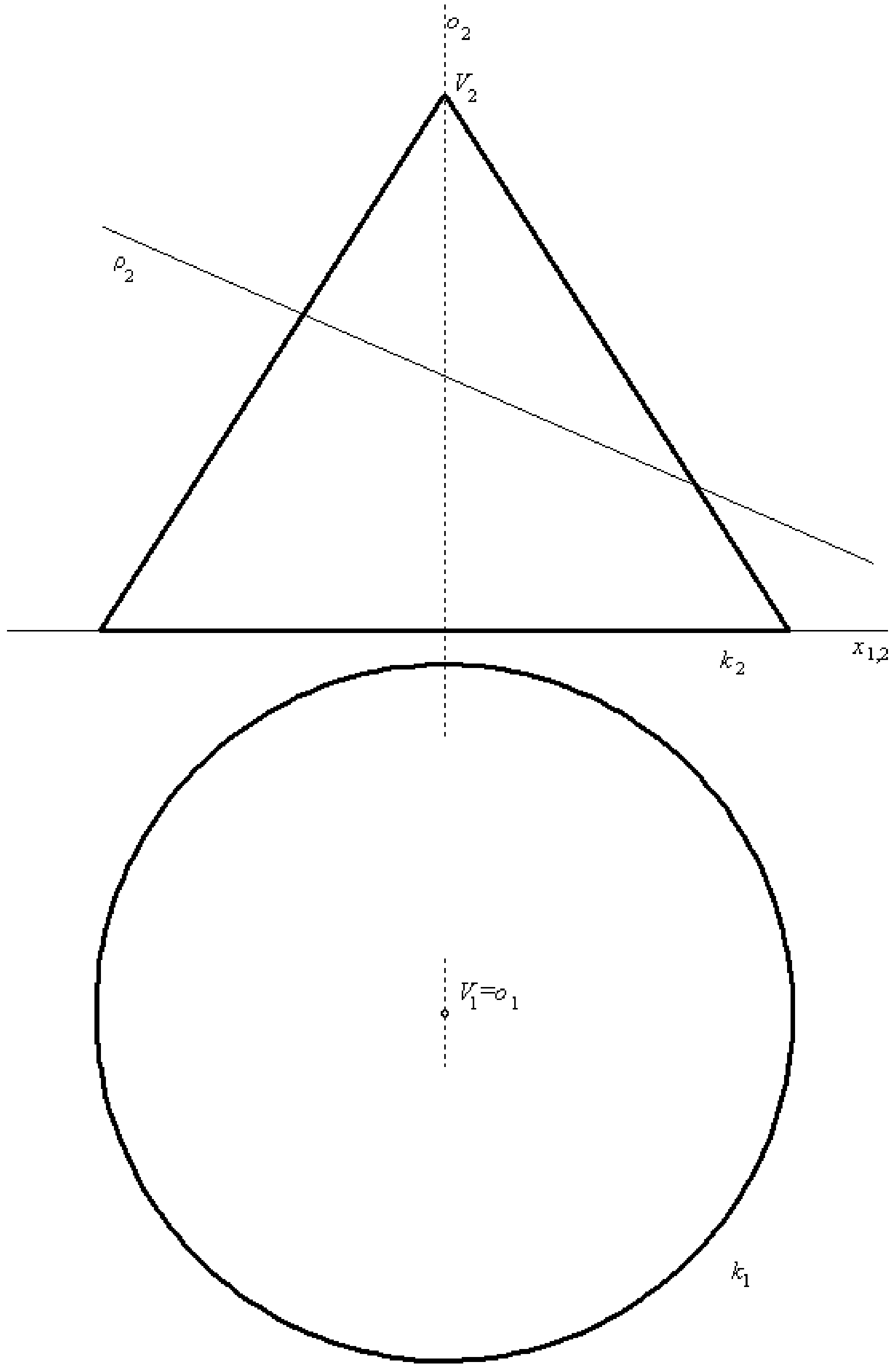


3. Find related views of the planar intersection of a circular cylindrical surface with the basic circle $k(S, r)$ in the ground image plane and axis o by the plane ρ parallel to the frontal image plane.

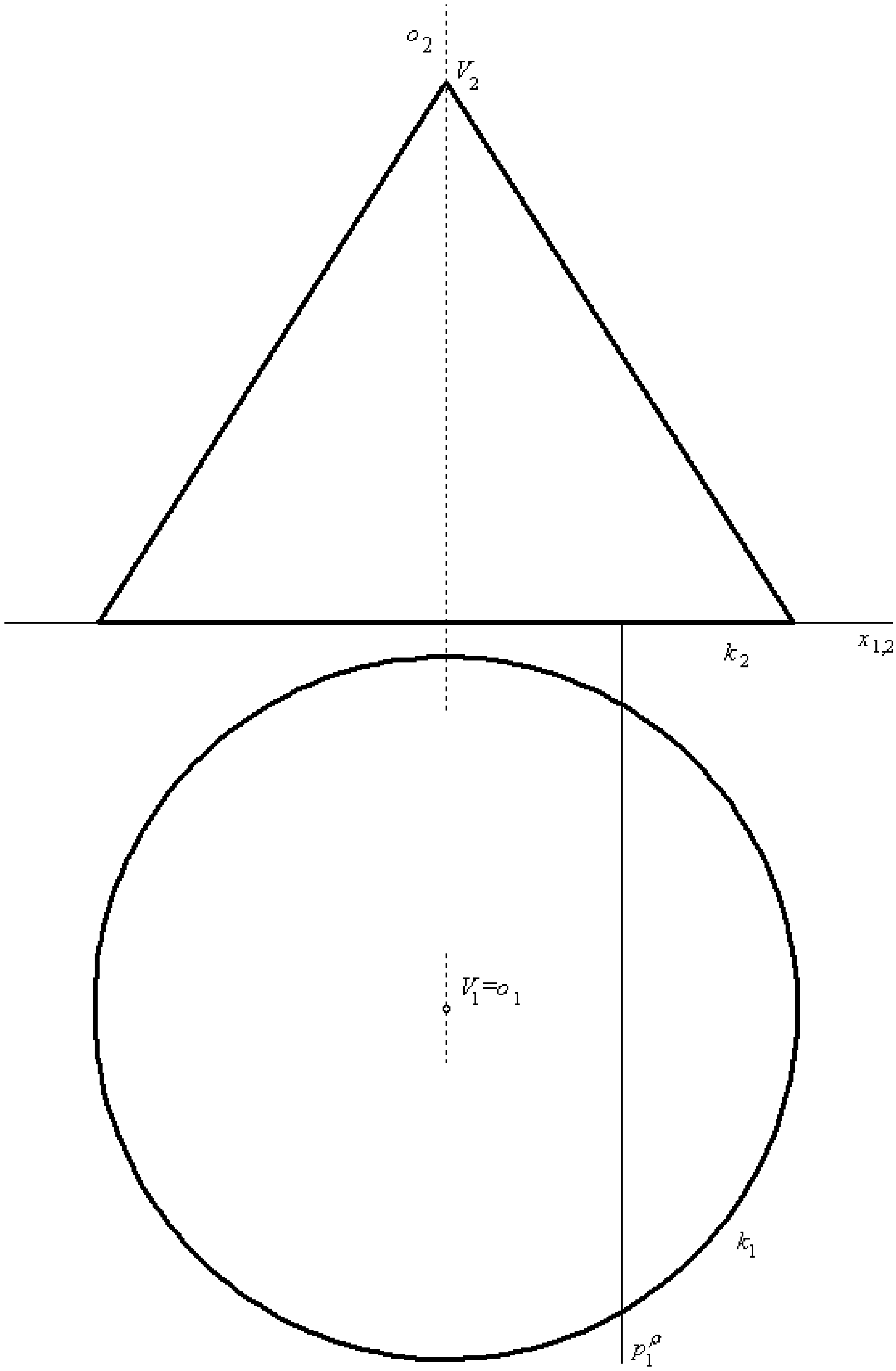


5.3 PLANAR INTERSECTIONS ON CONICAL SURFACES

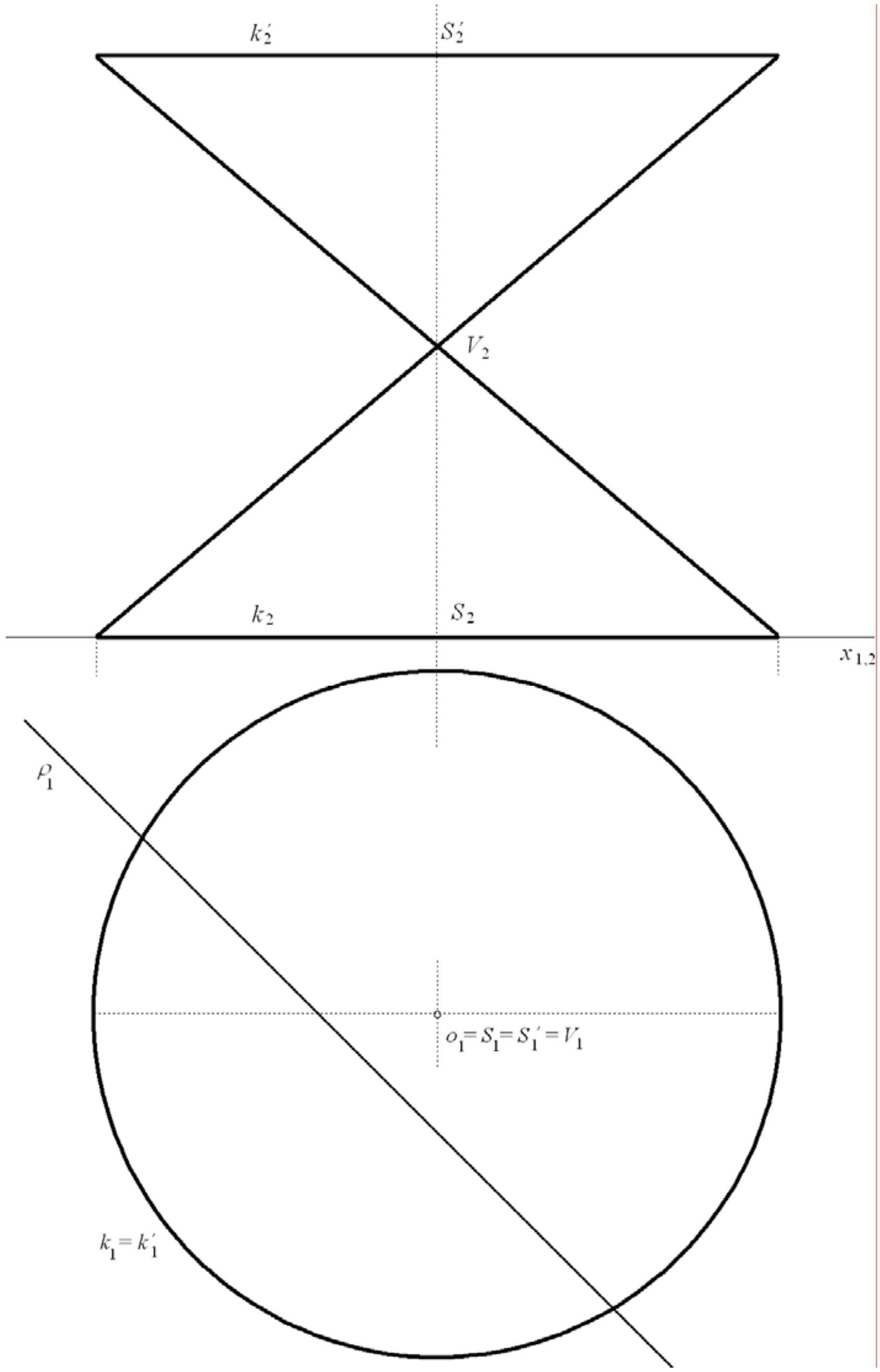
4. Find related views of the elliptic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane by the plane ρ perpendicular to the frontal image plane.



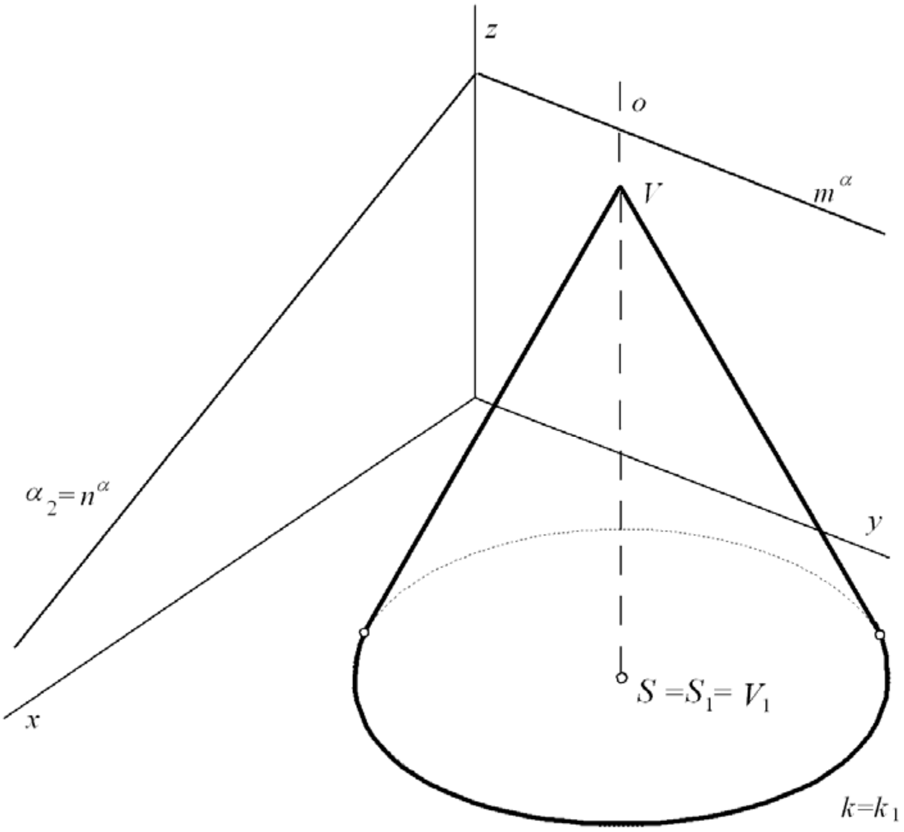
5. Find related views of the parabolic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane by the plane ρ perpendicular to the frontal image plane and determined by the trace p^{ρ}.



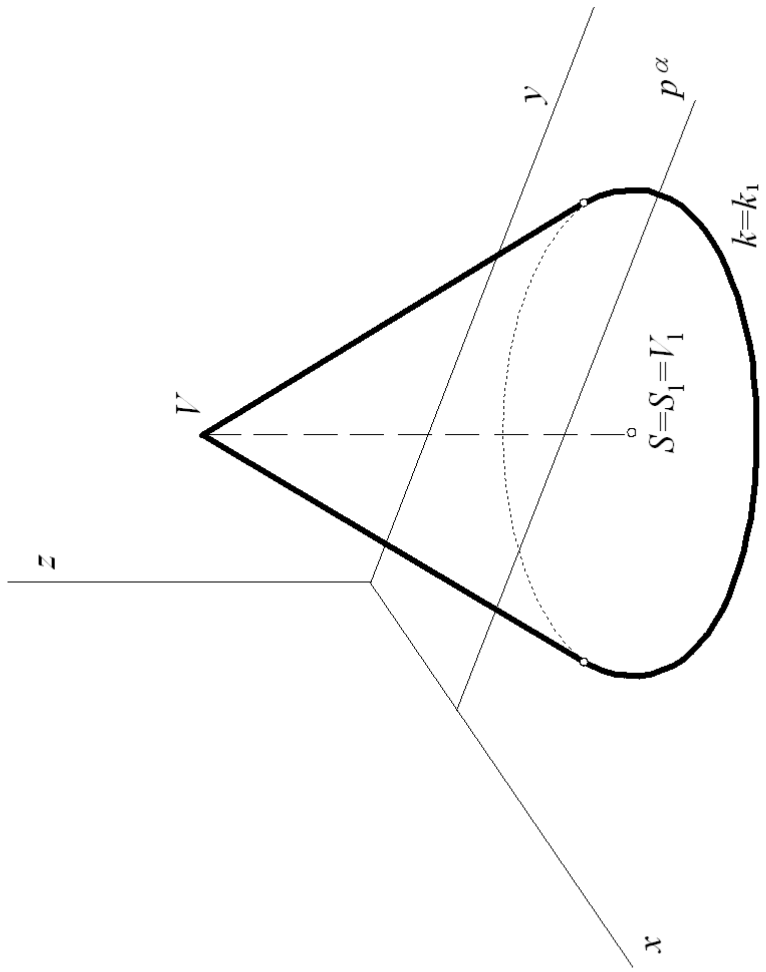
6. Find related views of the hyperbolic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane by plane ρ perpendicular to the ground image plane.



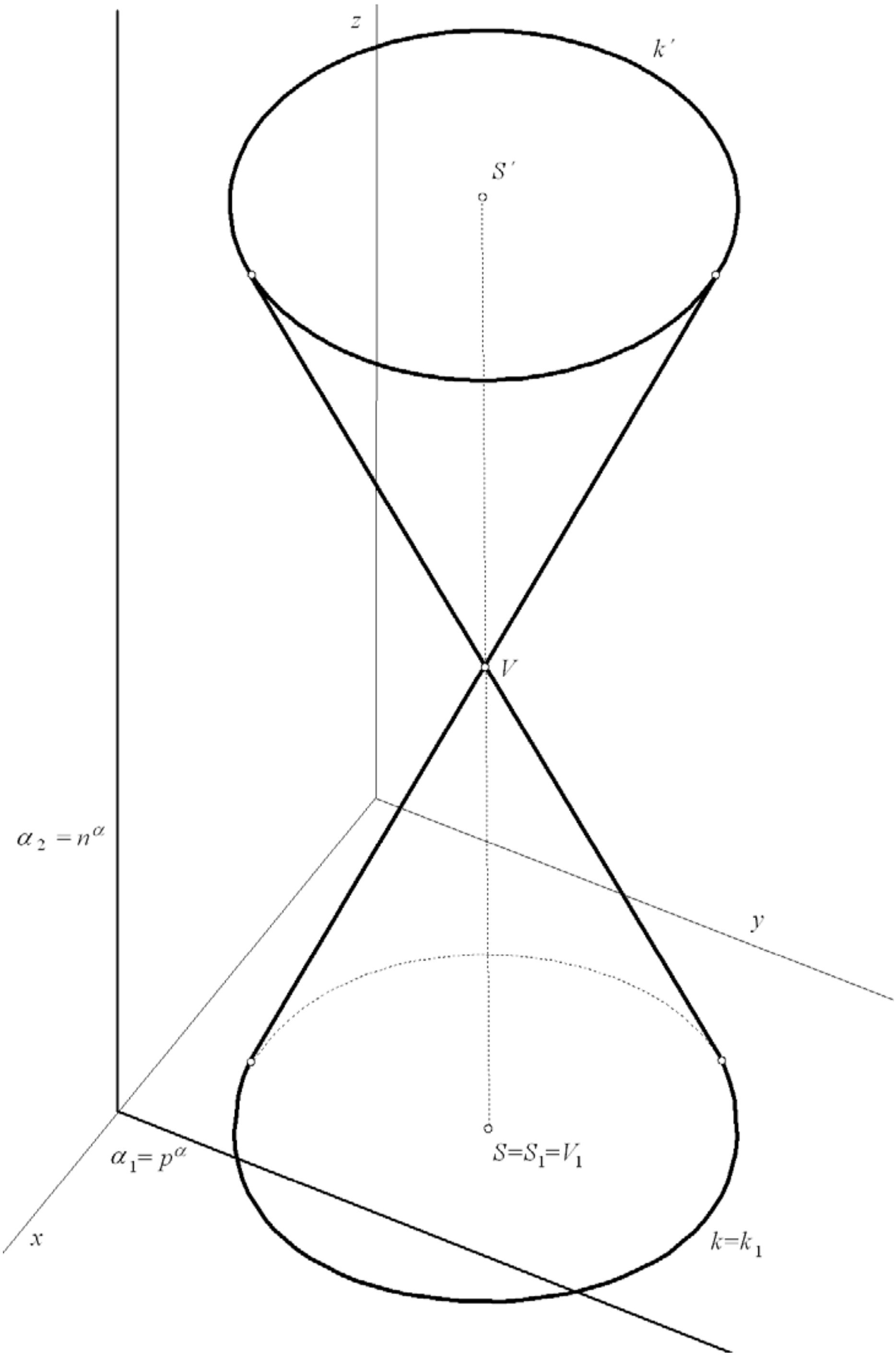
7. Find axonometric views of the elliptic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane and axis o by the plane α perpendicular to the frontal image plane.



8. Find axonometric views of the parabolic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane by the plane α perpendicular to the frontal image plane and determined by the trace p^{α}.

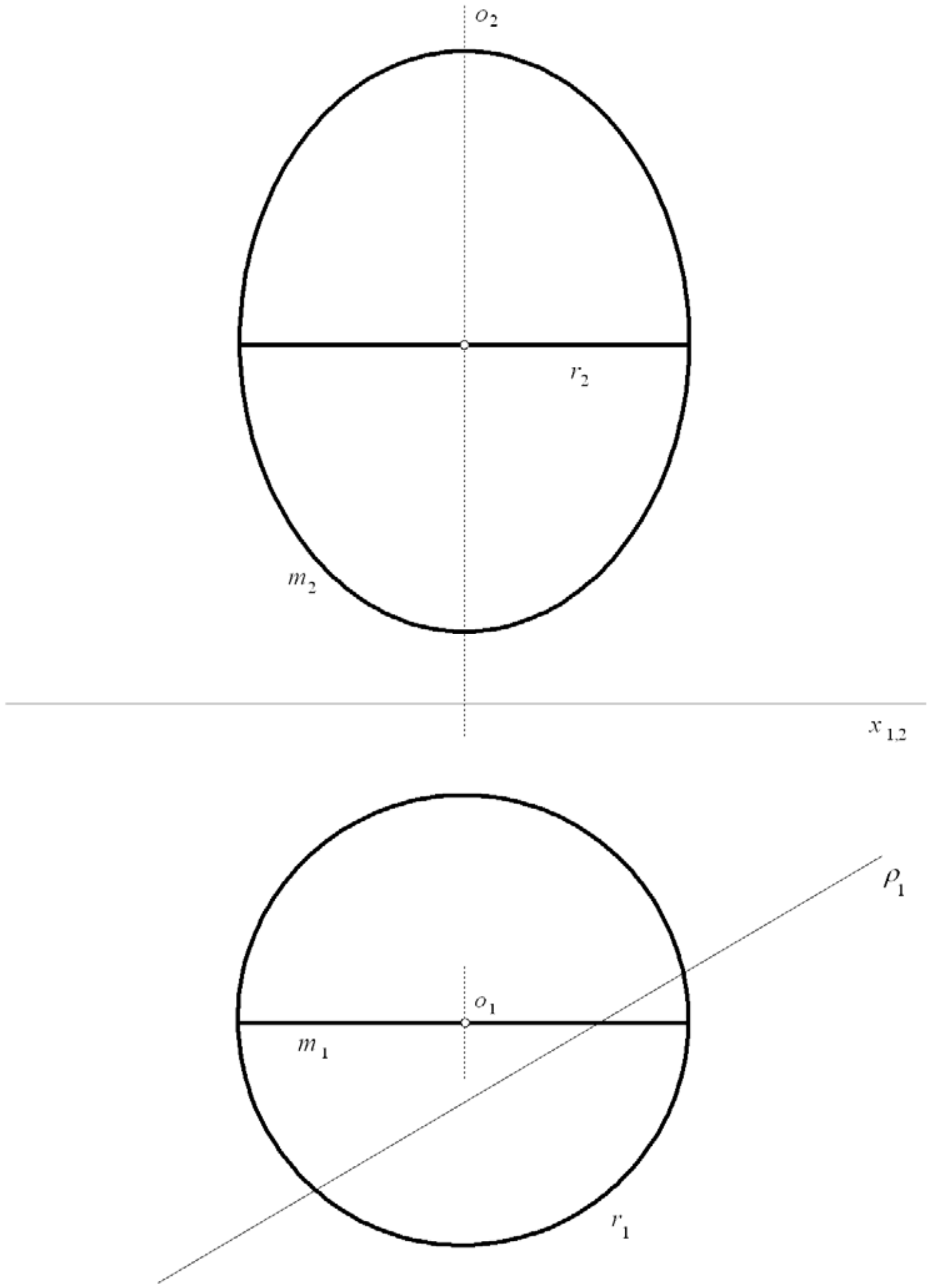


9. Find axonometric views of the hyperbolic intersection on the conical surface of revolution with the basic circle $k(S, r)$ in the ground image plane by the plane α parallel to the side image plane.

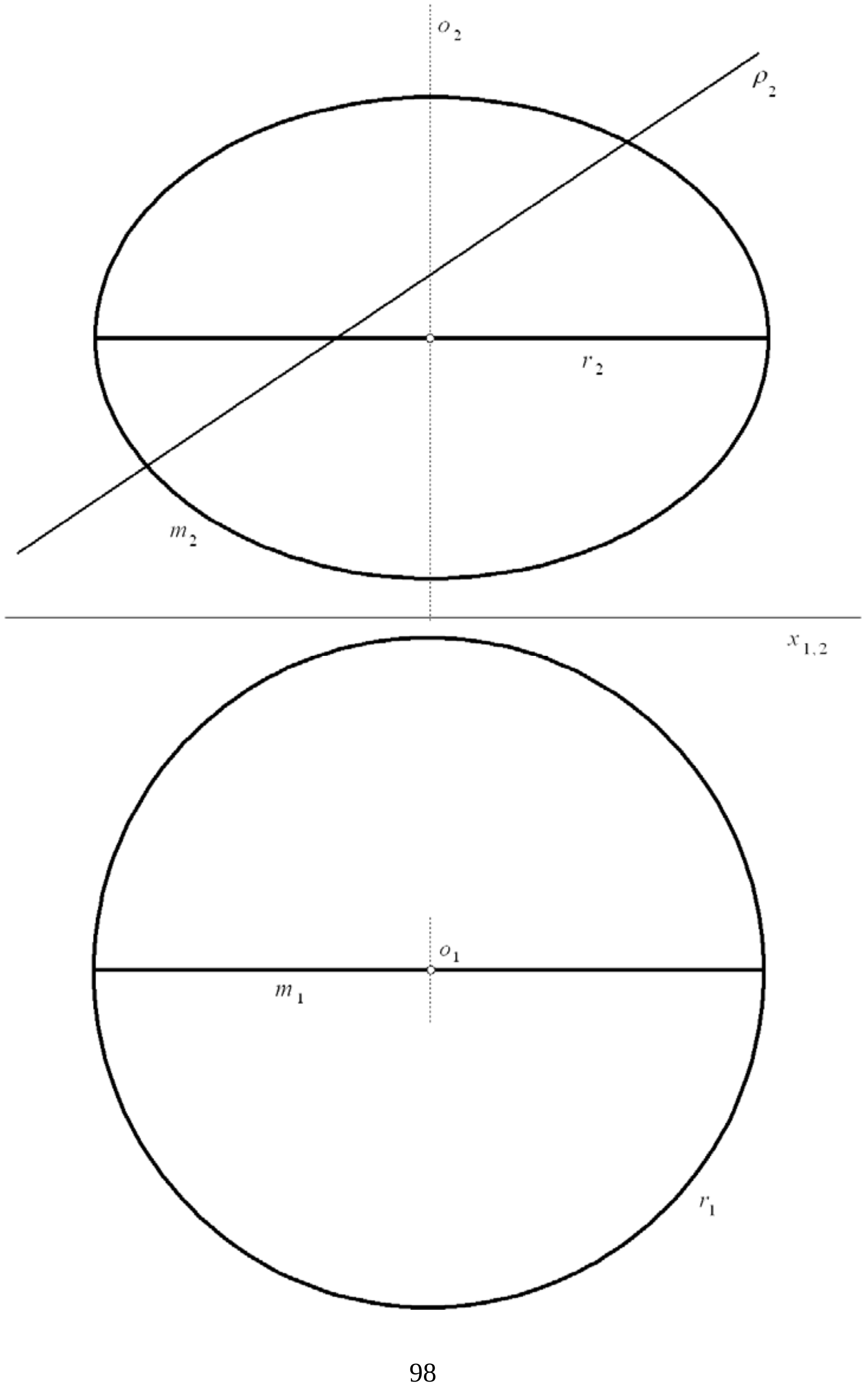


5.4 PLANAR INTERSECTIONS ON ELLIPSOIDS

10. Find related views of the planar intersection on the ellipsoid of revolution with axis o by the plane ρ perpendicular to the ground image plane.

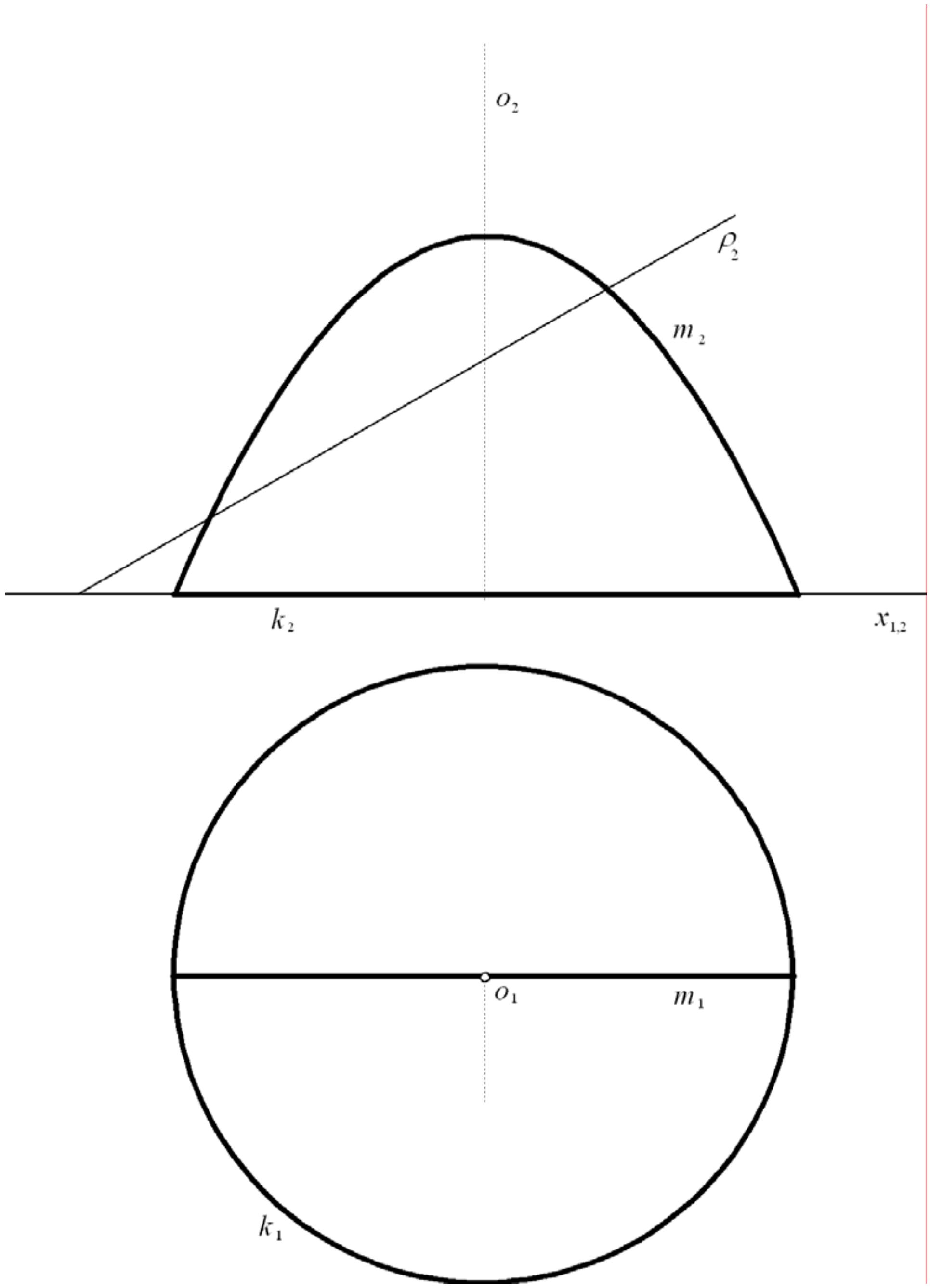


11. Find related views of the planar intersection on the ellipsoid of revolution with axis o by the plane ρ perpendicular to the frontal image plane.

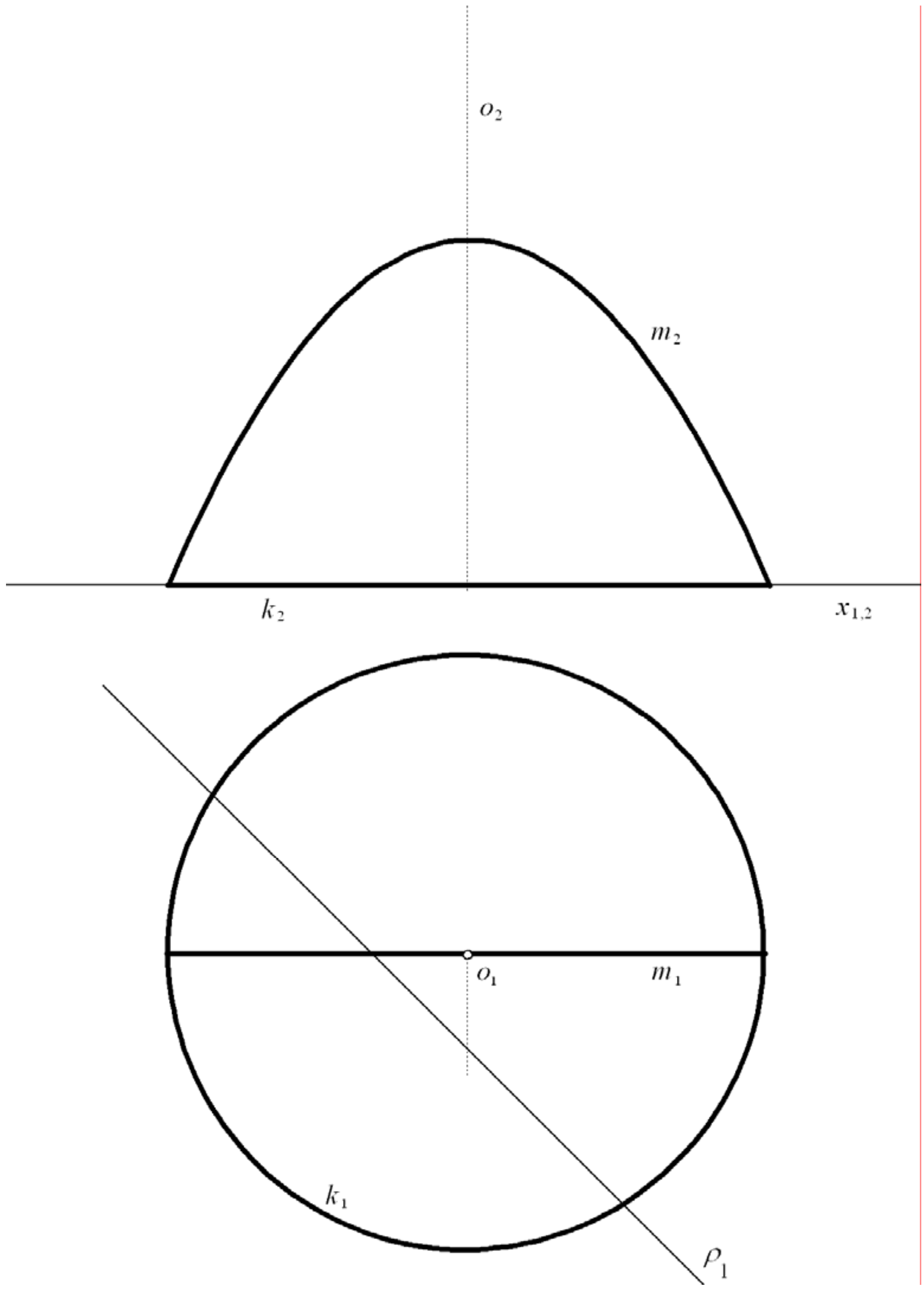


5.5 PLANAR INTERSECTIONS ON PARABOLOIDS

12. Find related views of the elliptic intersection on the paraboloid of revolution with the axis o by the plane ρ perpendicular to the frontal image plane.

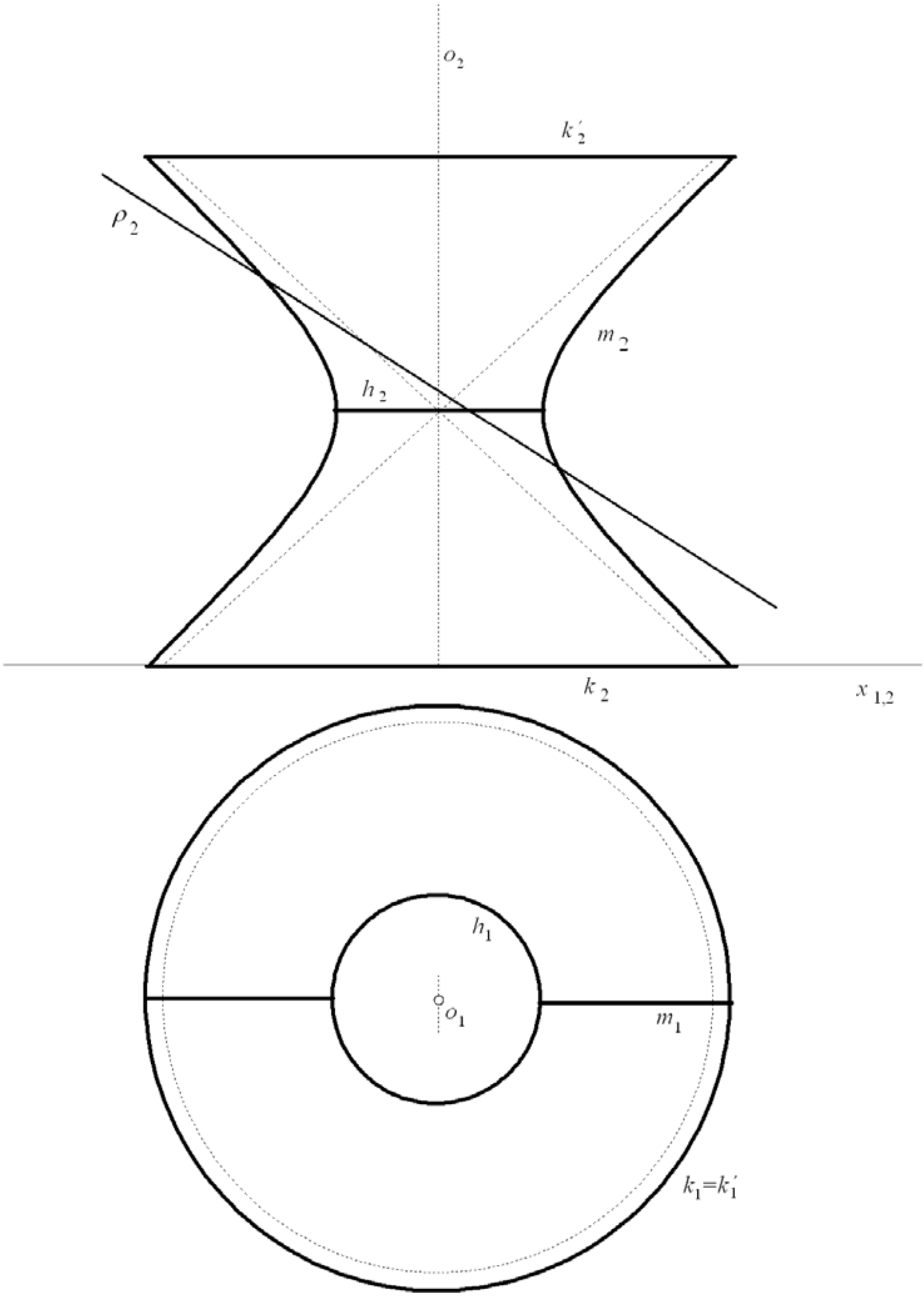


13. Find related views of the planar intersection on the paraboloid of revolution with axis o by the plane ρ perpendicular to the ground image plane.

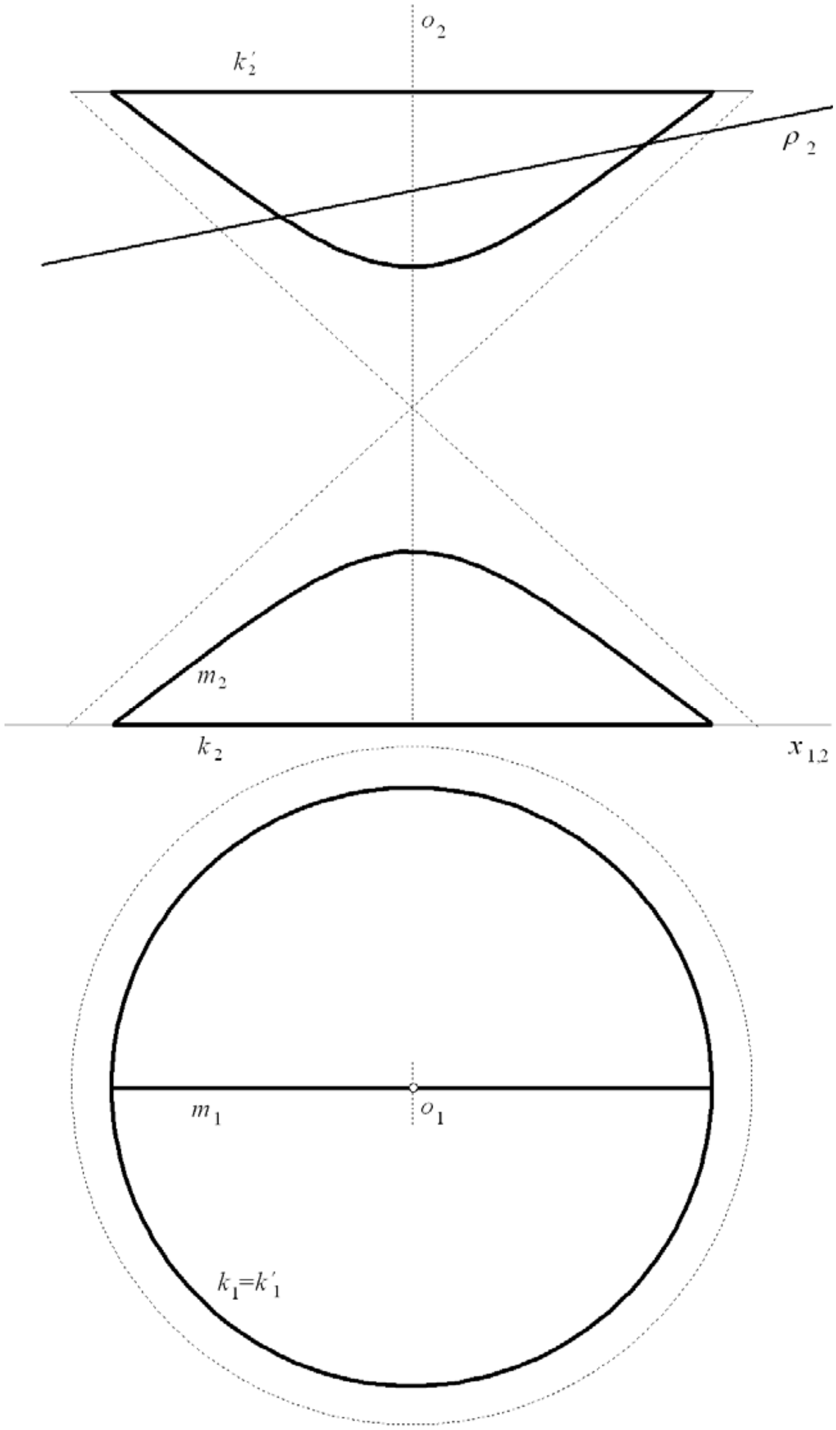


5.6 PLANAR INTERSECTIONS ON HYPERBOLOIDS

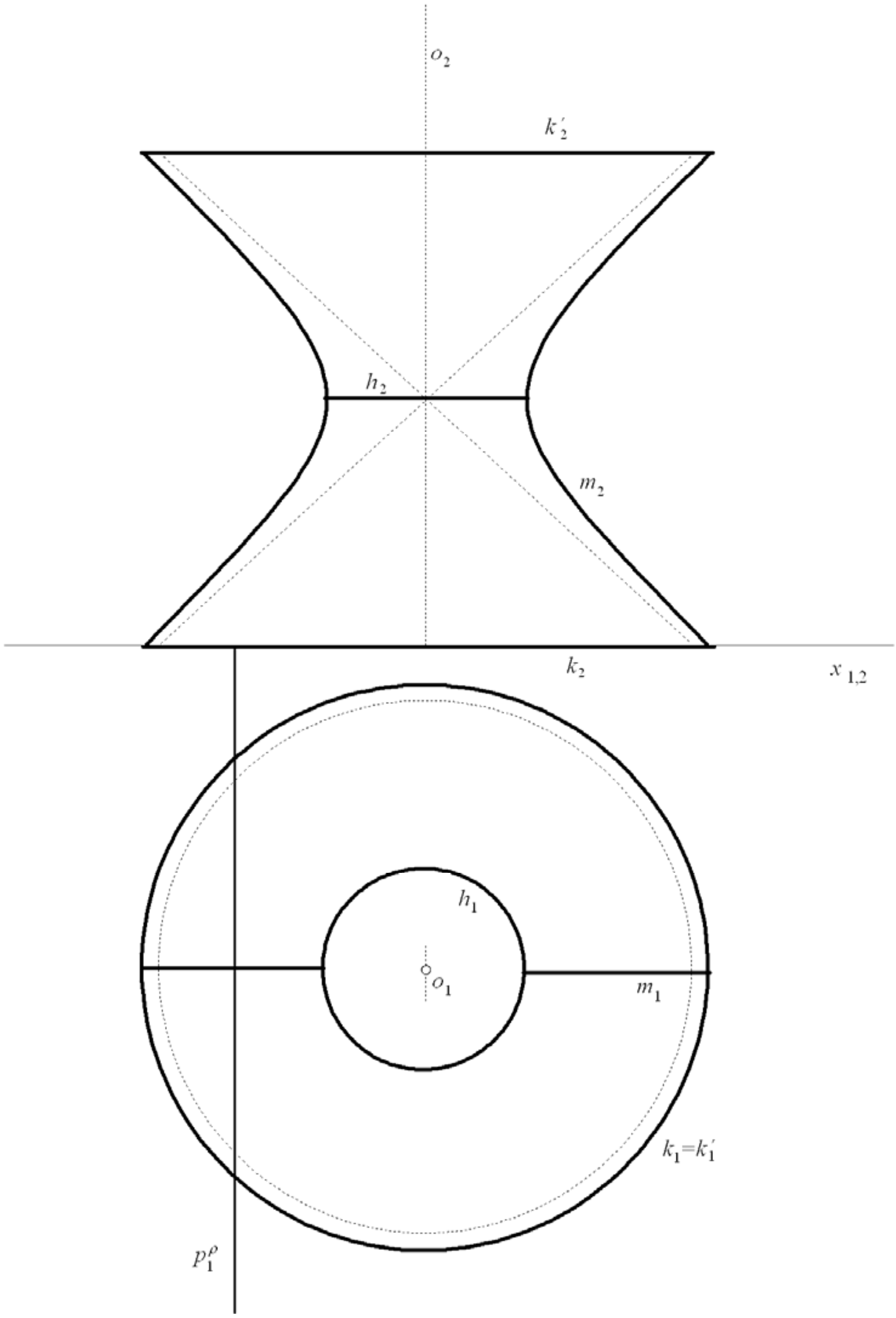
14. Find related views of the elliptic intersection on the 1-sheet hyperboloid of revolution with axis o by the plane ρ perpendicular to the frontal image plane.



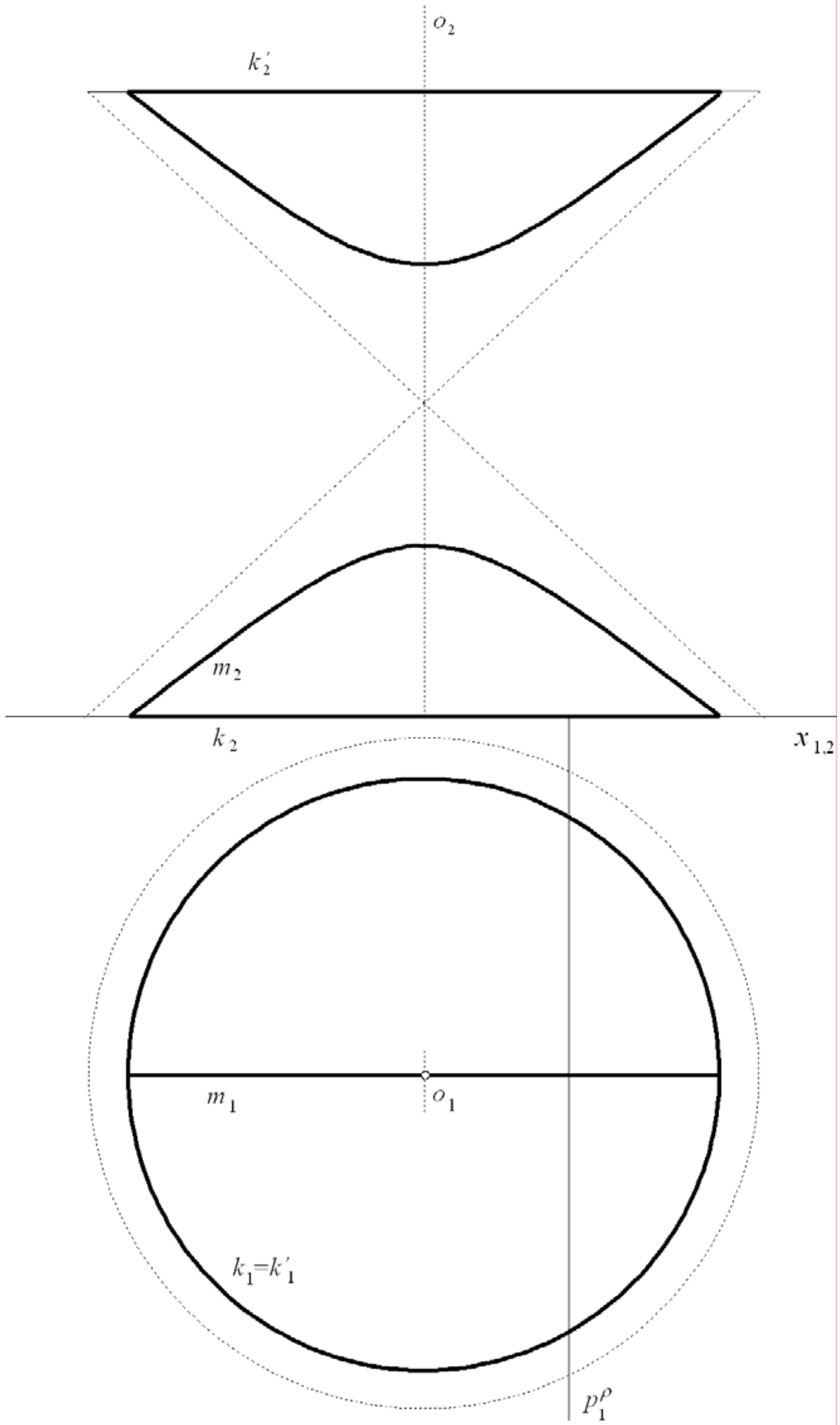
15. Find related views of the elliptic intersection on the 2-sheet hyperboloid of revolution with axis o by the plane ρ perpendicular to the frontal image plane.



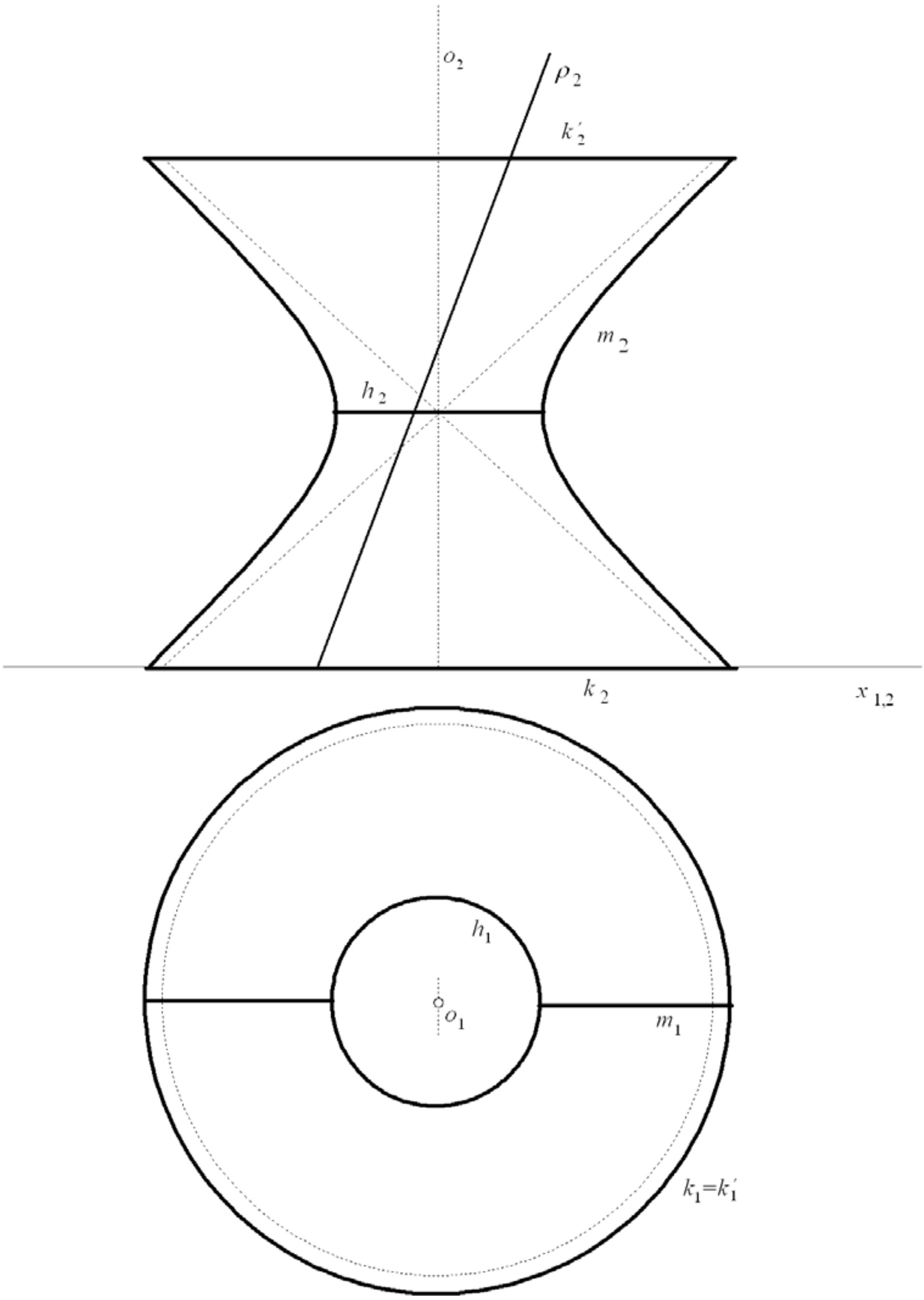
16. Find related views of the parabolic intersection on the 1-sheet hyperboloid of revolution with the axis o by the plane ρ passing through the trace p^{ρ} and perpendicular to the frontal image plane.



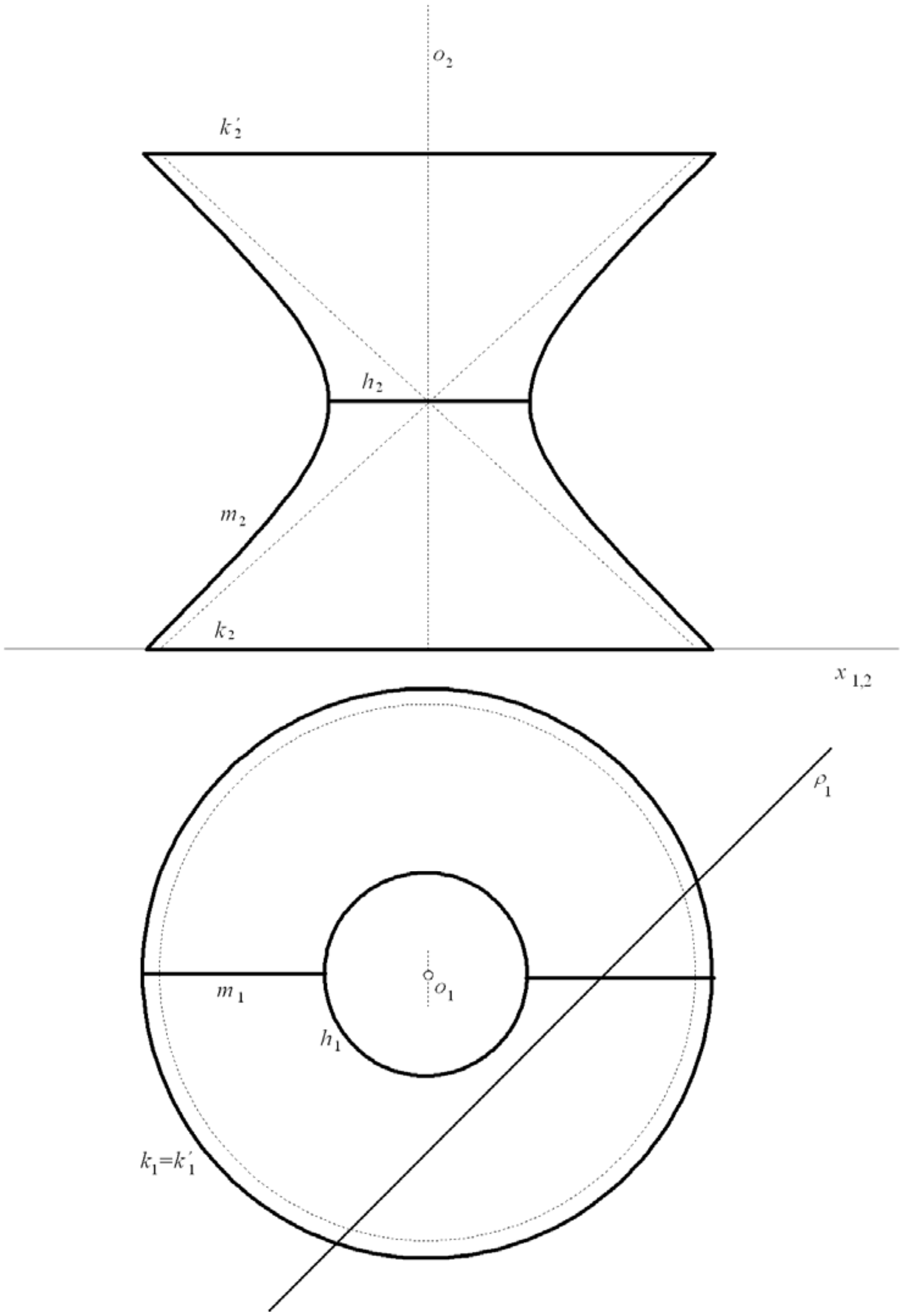
17. Find related views of the parabolic intersection on the 2-sheet hyperboloid of revolution with the axis o by the plane ρ passing through the trace p^{ρ}.



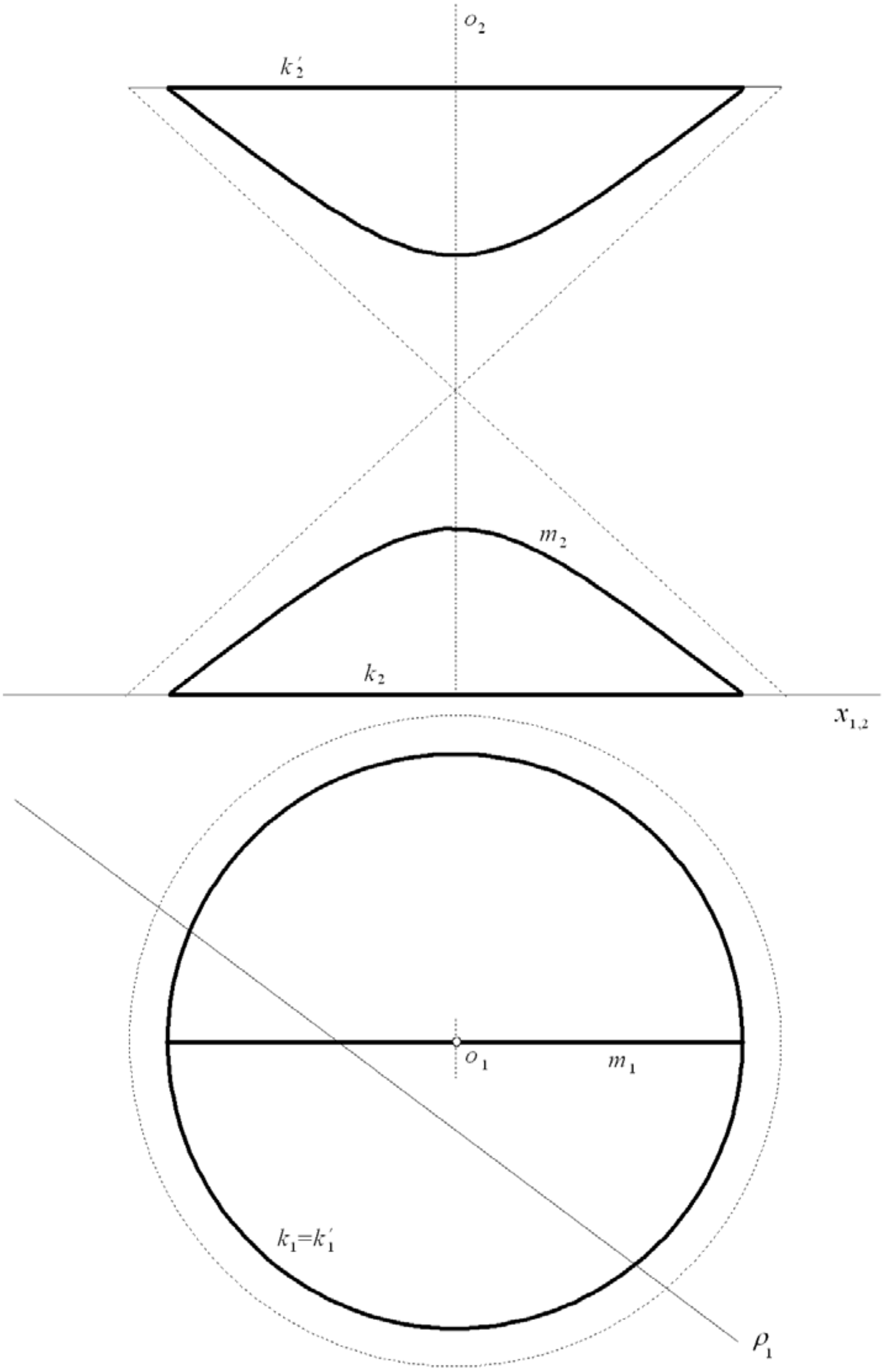
18. Find related views of the planar intersection on the 1-sheet hyperboloid of revolution with axis o by the plane ρ perpendicular to the frontal image plane.



19. Find related views of the planar intersection on the 2-sheet hyperboloid of revolution with axis o by the plane ρ perpendicular to the ground image plane.



20. Find related views of the planar intersection on the 2-sheet hyperboloid of revolution with axis o by the plane ρ perpendicular to the ground image plane.



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V roku 1974 ukončila štúdium na Prírodovedeckej fakulte Univerzity Komenského v odbore Matematika - Deskriptívna geometria. Od roku 1982 pôsobí na Strojníckej fakulte STU v Bratislave, v súčasnosti ako mimoriadna profesorka v odbore Aplikovaná matematika a vedúca Ústavu matematiky a fyziky. Okrem pedagogickej činnosti a výučby na všetkých troch stupňoch štúdia v predmetoch Matematika 1, Matematika 2, Konštrukčná geometria, Aplikovaná matematika a Optimalizačné metódy aktívne pracuje v oblasti matematického vzdelávania na európskych univerzitách ako členka riadiaceho výboru organizácie SEFI Mathematics Working Group a členka koordinačného výboru spoločnosti European Women in Mathematics. Vo vedeckej činnosti sa zameriava na výskum v oblastiach geometrické modelovanie, diferenciálna geometria kriviek a plôch, vizualizácie a zobrazovacie metódy v *n*-rozmerných priestoroch. Jej publikačná činnosť zahŕňa viac ako 150 vedeckých článkov v časopisoch a zborníkoch medzinárodných konferencií a 10 knižných publikácií. Pôsobí ako predsedníčka Slovenskej spoločnosti pre Geometriu a Grafiku a je koordinátorkou a riešiteľkou mnohých domácich a európskych výskumných projektov.

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