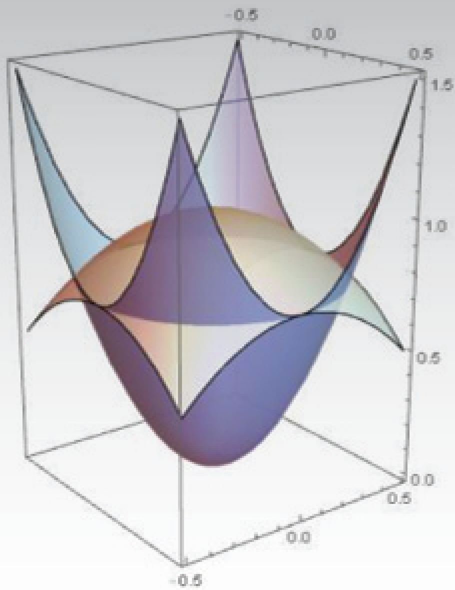
MATHEMATICS II



$$V(S) = 4\iiint_{S_4} 1\,dx\,dy\,dz = 4\int_0^{\frac{1}{2}}\int_0^{\sqrt{\frac{1}{4}-x^2}}\int_{3x^2+3y^2}^{1-(x^2+y^2)} 1\,dz\,dy\,dx = 4\int_0^{\frac{1}{2}}\int_0^{\sqrt{\frac{1}{4}-x^2}}\left(1-4x^2-4y^2\right)dy\,dx =$$

$$= 4\int_0^{\frac{1}{2}}\left[\left(1-4x^2\right)y - \frac{4y^3}{3}\right]_0^{\sqrt{\frac{1}{4}-x^2}} dx = \frac{4}{3}\int_0^{\frac{1}{2}}\left(1-4x^2\right)\sqrt{1-4x^2}\,dx = \left|\begin{array}{c} x = \frac{1}{2}\sin t \\ dx = \frac{1}{2}\cos t\,dt \\ x = 0 \Rightarrow t = 0 \\ x = \frac{1}{2} \Rightarrow t = \frac{\pi}{2} \end{array}\right| =$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}}\left(1-\sin^2 t\right)\cos^2 t\,dt = \frac{2}{3}\int_0^{\frac{\pi}{2}}\cos^4 t\,dt = \left[\frac{t}{4} + \frac{\sin 2t}{6} + \frac{\sin 4t}{48}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}$$

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Učebnica obsahuje výber tém z analytickej geometrie euklidovského priestoru a diferenciálneho a integrálneho počtu funkcií viac premenných, ktoré tvoria obsah základného kurzu matematiky určeného študentom všetkých bakalárskych študijných programov technických univerzít. Určená je najmä zahraničným študentom, ktorí študujú v anglickom jazyku na Strojníckej fakulte Slovenskej technickej univerzity v Bratislave, môže však slúžiť aj ako pomôcka pre všetkých študentov a záujemcov o štúdium matematiky a jej odbornú anglickú terminológiu.

This textbook contains selected topics from the coordinate geometry of the Euclidean space and from the differential and integral calculus of functions of more variables, which form the background contents of basic mathematics courses designed for students in all bachelor study programmes at technical universities. It is intended namely for international students who study in English at the Faculty of Mechanical Engineering at the Slovak University of Technology in Bratislava. Nevertheless, it can also serve as a study material for all students and other parties interested in the study of Mathematics and its special English terminology.

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FOREWORD

Mathematics should be as simple as possible,
but not simpler.

Albert Einstein

1 Analytic geometry

1.1 Introduction

Analytic or coordinate geometry is the study of geometric properties of figures determined by algebraic representations and manipulation of equations describing their positions, configurations, and properties. It can be explained simply as being concerned with defining geometrical shapes in a numerical way and extracting numerical information from that representation. Investigation of geometric objects is performed by means of algebraic operations upon symbols defined in terms of a coordinate system. René Descartes (1595 – 1650), a well-known French philosopher and mathematician was the first to apply algebra to geometry, so it is also known as Cartesian geometry. It is based on the idea that any point in a two-dimensional space can be represented by two numbers determining its position with respect to a coordinate system.

The most commonly used is the Cartesian coordinate system, a fixed origin at the point O, and two perpendicular lines, coordinate axes x and y, meeting at this point. Any point in a three-dimensional space can be analogously determined by three numbers, and so on; therefore points in the n-dimensional space are described as n-tuples of real numbers called Cartesian coordinates.

Other coordinate systems are possible, in a plane the most common alternative is polar coordinates, while in three dimensions common alternative coordinate systems include cylindrical and spherical coordinates. Because lines, circles, spheres, and other geometric figures can be regarded as collections of points in a plane or space that satisfy certain equations, they can be explored in an analytic way, via their equations and formulas, in addition to their synthetic representations by graphs. Generally, most of analytic geometry deals with measuring distances, angles and with the investigation of the position of basic geometric objects – points, lines and planes, conic sections, or quadratic surfaces.

1.2 Three-dimensional Euclidean space

Let us consider rectangular right-handed Cartesian coordinate system $Oxyz$, where O is origin of coordinates, lines x, y, z are coordinate axes, and planes xy, xz, yz are coordinate planes. Each point in the 3-dimensional space can be identified with an ordered triple of real numbers $[x, y, z]$, its Cartesian coordinates, whereas coordinates of the system origin are $O = [0, 0, 0]$.

Distance of two arbitrary points $A = [x_A, y_A, z_A]$, $B = [x_B, y_B, z_B]$ can then be defined by the Euclidean distance formula

$$d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}\ .$$

Thus the 3-dimensional space is called the Euclidean (metric) space with the Euclidean metric and it is denoted $\mathbf{E}^3$.

The Cartesian coordinates of an arbitrary point $M = [x_M, y_M, z_M]$ in the space determine its distances from the coordinate planes, $z_M = d(M, xy)$, $y_M = d(M, xz)$, and $x_M = d(M, yz)$, as illustrated in Fig. 1.1.

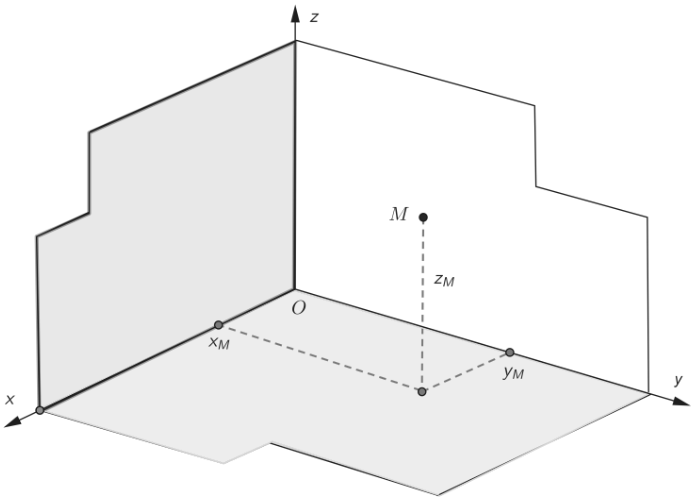


Fig. 1.1. Cartesian coordinates of point *M*.

Oriented line segment *AB*, with the initial (start) point *A* and terminal (end) point *B* determines a vector (direction) in the space, which can be represented by an ordered triple of real numbers

$$\mathbf{a} = (x_B - x_A, y_B - y_A, z_B - z_A),$$

where $x_B - x_A$, $y_B - y_A$, $z_B - z_A$ are the components of **a**, as shown in Fig. 1.2. Components of a vector do not depend on its location.

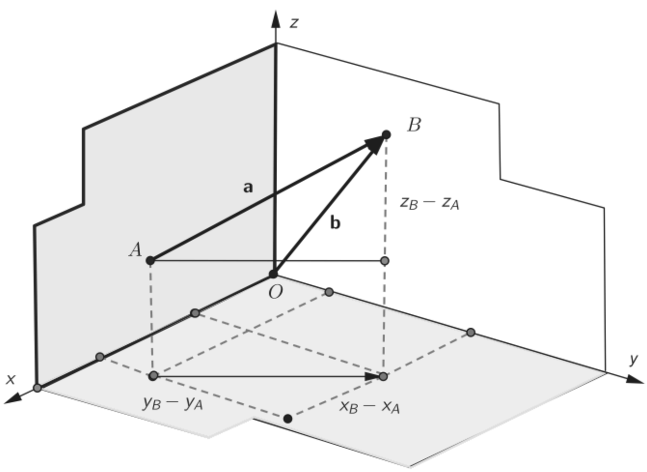


Fig. 1.2. Components of vectors **a** = **AB, b** = **OB**.

If $A = O$, then vector $\mathbf{b} = \mathbf{OB} = (x_B, y_B, z_B)$ is said to be the position vector of point *B*.

The zero vector is denoted $\mathbf{0} = (0, 0, 0)$, and $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ are unit vectors in the directions of each of the coordinate axes.
If $\mathbf{a} = (a_1, a_2, a_3)$, then the non-negative real number

$$|\mathbf{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$$

is called the length (magnitude, norm) of the vector **a**.

Examples

1. $|\mathbf{0}| = \sqrt{0^2 + 0^2 + 0^2} = 0$
2. $|\mathbf{i}| = \sqrt{1^2 + 0^2 + 0^2} = 1, |\mathbf{j}| = 0\sqrt{1^2 + 1^2 + 0^2} = 1, |\mathbf{k}| = \sqrt{0^2 + 0^2 + 1^2} = 1$

If $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and λ is a real number, then the sum of vectors **a** and **b** is the vector

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

and the scalar multiple of vector **b** by a scalar λ is the vector

$$\lambda\mathbf{b} = (\lambda b_1, \lambda b_2, \lambda b_3).$$

See Fig 1.3 for a visual representation of both the sum of two vectors and the scalar multiple of a vector.

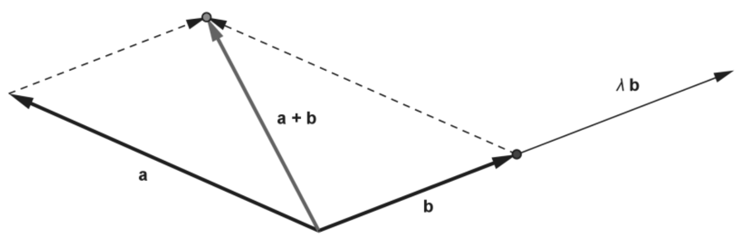


Fig. 1.3. Sum of vectors, scalar multiple of vector.

Two vectors **a** and **b** are said to be collinear (parallel), if and only if a nonzero real number $\lambda \in \boldsymbol{R}$ exists, such that $\mathbf{b} = \lambda.\mathbf{a}$.
Three vectors **a**, **b**, **c** are linearly independent, if a linear combination of these vectors $k\mathbf{a} + l\mathbf{b} + m\mathbf{c} = \mathbf{0}$ exists, such that at least one of coefficients k, l, and m is a nonzero real number, i.e. $k^2 + l^2 + m^2 \neq 0$. This means that at least one from vectors **a**, **b**, **c** is a linear combination of the two others.
Unit vectors **i**, **j**, **k** are linearly independent vectors in $\mathbf{E}^3$ forming the ortho-normal basis of the 3-dimensional Euclidean space. Consequently, any vector $\mathbf{a} = (a_1, a_2, a_3)$ can be represented as a linear combination of the unit vectors **i**, **j**, **k** with scalars equal to the vector coordinates in this basis,

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}.$$

In $\mathbf{E}^3$ we distinguish two kinds of vector products, the scalar (dot) product and the vector (cross) product.

The scalar product of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the real number determined by the formula

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 .$$

Remark. It can be proved that another possible definition of the scalar product is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\alpha,$$

where α is the smaller angle formed by vectors **a** and **b**. Hence, two nonzero vectors **a** and **b** are perpendicular if and only if their scalar product equals zero.

Examples

1. Vectors $\mathbf{a} = (2, -3, 1)$ and $\mathbf{b} = (1, 1, 1)$ are perpendicular, because they are nonzero vectors, and $\mathbf{a}\cdot\mathbf{b} = 2 - 3 + 1 = 0$, while $|\mathbf{a}| = \sqrt{14}, |\mathbf{b}| = \sqrt{3}$.
2. The length of vector $\mathbf{a} = (4, 3, 12)$ is $|\mathbf{a}| = \sqrt{16+9+144} = \sqrt{169} = 13.$

The vector product of two nonzero vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with the following properties:

1. $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\alpha,$
2. **c** is perpendicular to both vectors **a** and **b**,
3. vectors **a**, **b**, **c** in this order form what is referred to as right-handed system.

Remark. Property 1. can be geometrically interpreted in the following way: length of vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ equals to the area of a parallelogram formed by vectors **a** and **b**.

Two vectors **a** and **b** are parallel if and only if their vector product is a zero vector. Some of the basic properties of the vector product are the following:

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}),$
2. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j},$
3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \times \mathbf{a}) \times \mathbf{b}.$

The relationship between the components of the vector product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ and the components of both vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ can be derived by expanding the determinant

$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

The Lagrange formula is the identity involving both, the scalar and the vector product of three vectors

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b}).$$

The Cauchy–Schwarz inequality states that for any two vectors **a** and **b** it holds that

$$(\mathbf{a}\cdot\mathbf{b})^2 \leq |\,\mathbf{a}\,||\,\mathbf{b}\,|\,.$$

The scalar triple product of three vectors **a**, **b**, **c** is a real number determined as the value of the determinant

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Three nonzero vectors **a**, **b**, **c** are said to be coplanar if their scalar triple product equals to zero. The scalar triple product of 3 nonzero vectors **a**, **b**, **c** is a nonzero number equal to the volume of the parallelepiped formed by the respective vectors, $V = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$, as shown in Fig 1.4.

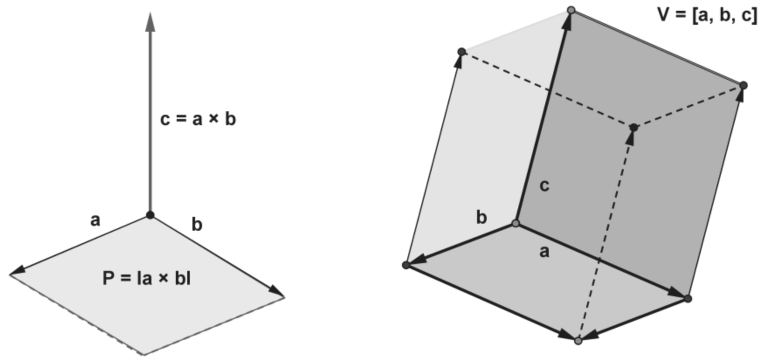


Fig. 1.4. Geometric interpretation of vector product and mixed scalar product.

Examples

1. The vector product of two vectors $\mathbf{a} = (2, -3, 1)$ and $\mathbf{b} = (1, 1, 1)$ is the vector $\mathbf{c} = (-4, -1, 5)$ perpendicular to both **a** and **b**, because $\mathbf{a}\cdot\mathbf{c} = -8 + 3 + 5 = 0$, and $\mathbf{b}\cdot\mathbf{c} = -4 - 1 + 5 = 0$, $|\mathbf{c}| = \sqrt{16+1+25} = \sqrt{42}$.
2. Three vectors $\mathbf{a} = (1, 3, -1)$, $\mathbf{b} = (-2, 2, -4)$ and $\mathbf{c} = (1, -1, 2)$ are coplanar, because their triple scalar product [**a**, **b**, **c**] equals to zero

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 2 & -4 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} -2 & -4 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = 0\,.$$

An angle formed by two nonzero vectors can be determined as

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}, \alpha \in \langle 0, \pi \rangle.$$

The angle of two collinear equally oriented vectors is $\alpha = 0$, the angle of two collinear vectors with opposite orientation is $\alpha = \pi$. Perpendicular nonzero vectors **a** and **b** form angle $\alpha = \pi/2$, and because $\cos(\pi/2) = 0$, from the above formula for their scalar product it holds that $\mathbf{a} \cdot \mathbf{b} = 0$.

1.3 Linear objects in space

Planes and lines are geometric figures that can be analytically determined by linear expressions representing relationships between coordinates of their points. Two types of equations are distinguished in general, implicit and explicit equations. Relations holding for triples of Cartesian coordinates of object points are called object implicit equations. Formulas for the evaluation of Cartesian coordinates x, y and z of the object points are called parametric equations of an object, and they depend on the value of one or two real parameters. The analytic equation of a plane will be derived from its geometric definition. A plane can be uniquely determined by any one of the following:

1. three non-collinear points
2. two intersecting lines
3. two different parallel lines
4. a line and a point not on this line
5. a point and a direction (perpendicular to the plane).

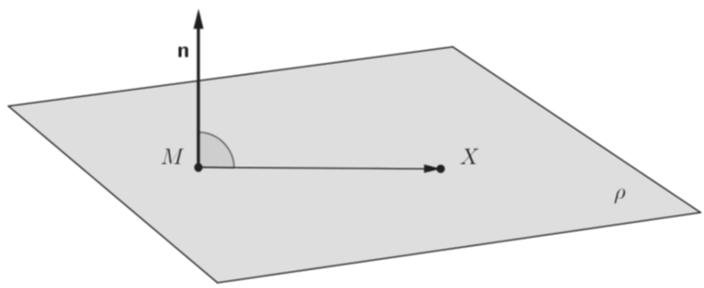


Fig. 1.5. Plane ρ determined by point M and direction $\mathbf{n}$ perpendicular to ρ.

Any non-zero vector $\mathbf{n} = (a, b, c)$ perpendicular to the plane ρ is called a normal vector to the plane ρ. Let $M = [x_M, y_M, z_M]$ be a point in the plane ρ, as can be seen in Fig 1.5. Point M and an arbitrary point $X = [x, y, z]$ in the plane ρ form the vector perpendicular to the plane normal vector $\mathbf{n}$, $\mathbf{MX} = (x - x_M, y - y_M, z - z_M)$, thus the implicit equation of plane ρ can be determined from their scalar product as

$$a(x - x_M) + b(y - y_M) + c(z - z_M) = 0,\ a^2 + b^2 + c^2 \neq 0,$$

which can be rewritten in the following form, called the general equation of a plane

$$ax + by + cz + d = 0, \text{ where } d = -ax_M - by_M - cz_M.$$

Constant $d = 0$ in the equation of the plane passing through the origin O.
The equation of the plane intersecting coordinate axes in the points $P = [p, 0, 0]$, $Q = [0, q, 0, 0]$ and $R = [0, 0, r]$ can be reduced to the intercept form

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$

Such plane intersects coordinate plane *xy* in line *PQ*, coordinate plane *xz* in line *PR* and coordinate plane *yz* in line *Q*R. View of this plane sketched in the axonometric projection method is presented in Fig. 1.6, as triangle *PQR* visible in the rectangular trihedron with vertex in the origin *O*, edges in the positive semi-axis *x*+, *y*+, *z*+, and faces as parts of coordinate planes *xy*, *xz*, *yz*.

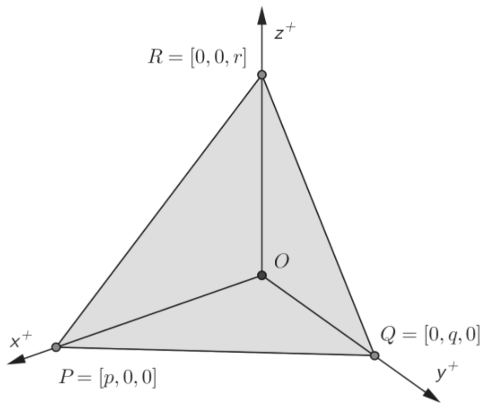


Fig. 1.6. Plane intersecting coordinate axes.

Planes in special position to the coordinate planes and coordinate axes are determined by equations of special forms. Views and equations of planes parallel to one from coordinate axes, therefore perpendicular to coordinate planes determined by the two other coordinate axes are shown in Fig. 1.7.

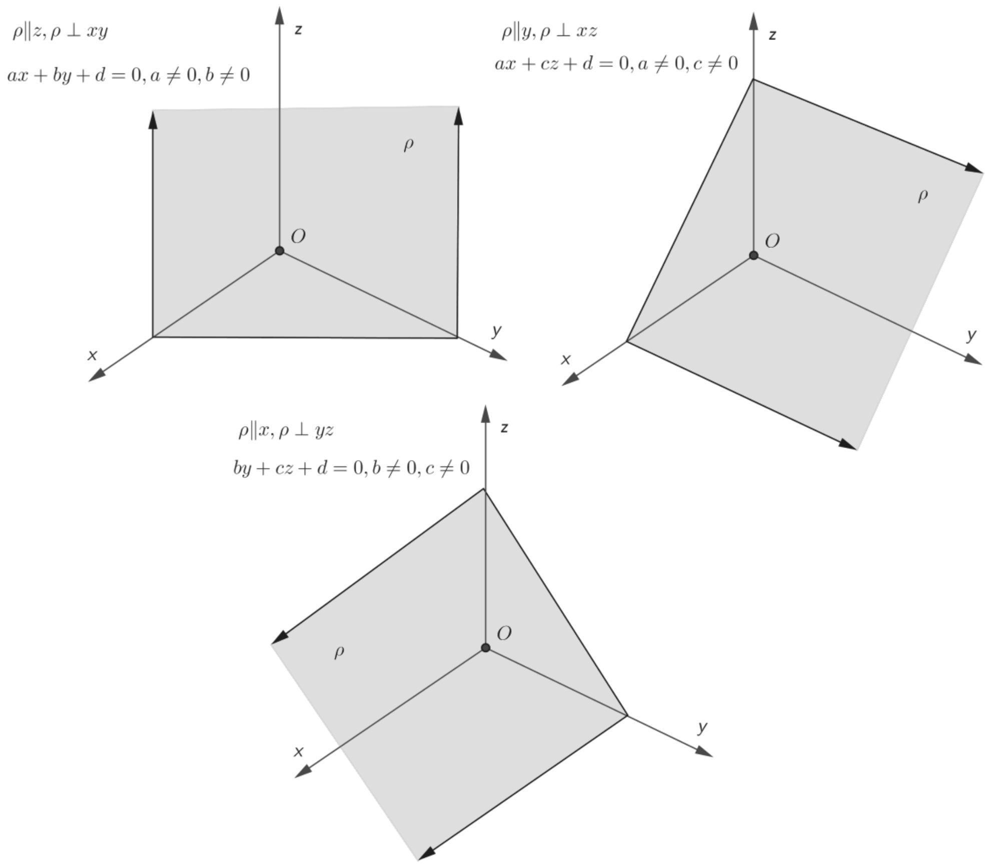


Fig. 1.7. Planes parallel to coordinate axes.

Planes parallel to one coordinate plane, therefore perpendicular to the coordinate axis that is not in the respective plane, are viewed with their equations in Fig. 1.8.

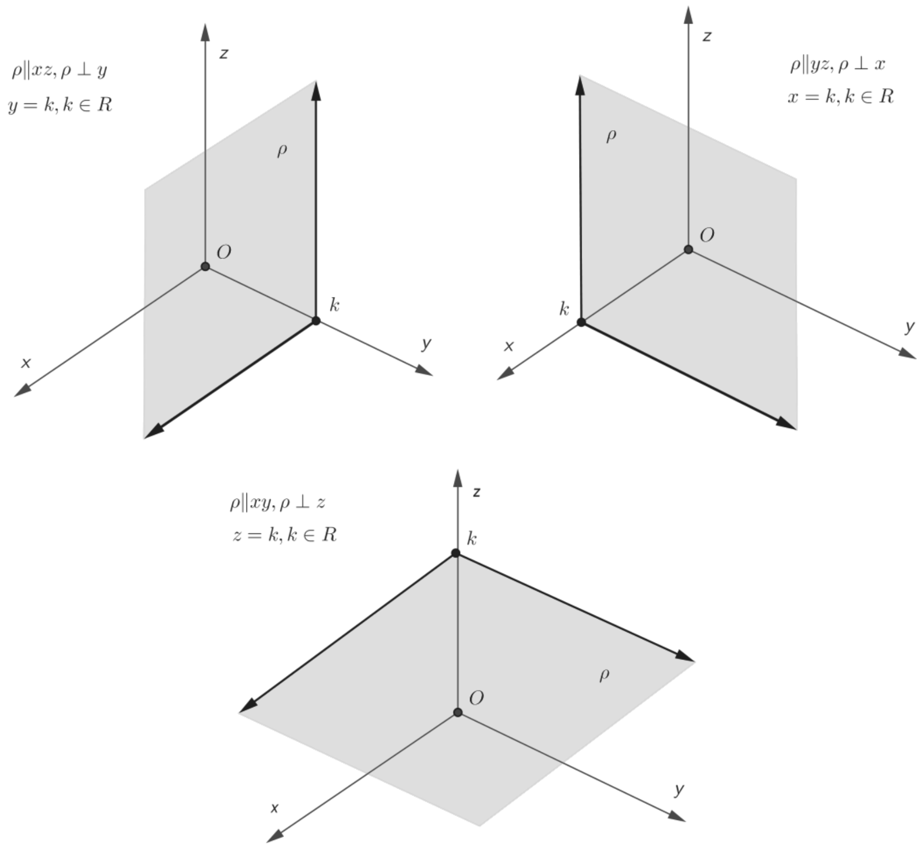


Fig. 1.8. Planes parallel to coordinate planes.

Examples

1. The general equation of a plane passing through the point $A = [2, -3, 1]$ and perpendicular to the vector $\mathbf{n} = (1, 1, 1)$ is $x + y + z = 0$. This plane passes through the origin of coordinates, as obviously $d = -2 + 3 - 1 = 0$.
2. Plane with the general equation $5x - 10y + 4z - 20 = 0$ is passing through points $P = [4, 0, 0]$, $Q = [0, -2, 0]$ and $R = [0, 0, 5]$ on the coordinate axes.
3. The equations of the coordinate planes *xy*, *xz* and *yz* are $z = 0$, $y = 0$, and $x = 0$, respectively.
4. The plane defined by the equation $x - z = 1$ passes in direction of coordinate axis *y* through the point $P = [1, 0, 0]$ on coordinate axis *x* and point $R = [0, 0, -1]$ on coordinate axis *z*. This plane intersects coordinate plane *xy* in a line parallel to coordinate axis *y* and passing through point *P*, and coordinate plane *yz* in a line also parallel to coordinate axis *y* but passing through the point *Q*. This plane is perpendicular to coordinate plane *xz*, and intersects this coordinate plane in a line determined by points *P* and *Q*.

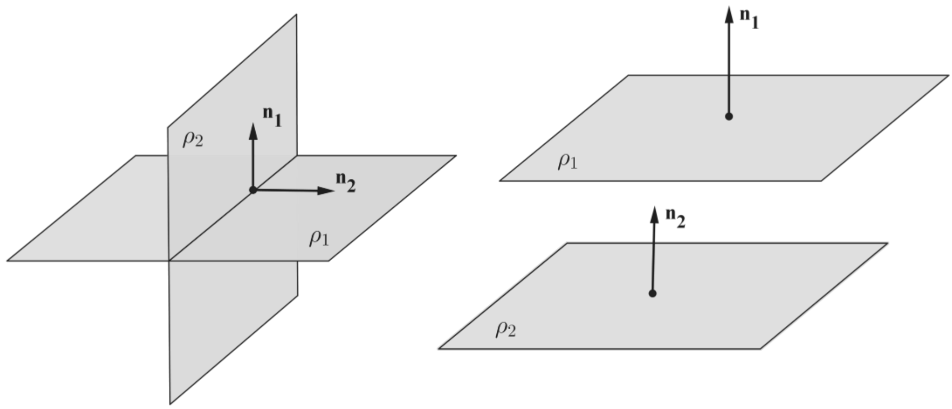


Fig. 1.9. Position of two planes.

Let two planes ρ_1, ρ_2 with normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ be defined by the general equations

$$a_1x + b_1y + c_1z + d_1 = 0, \quad \mathbf{n}_1 = (a_1, b_1, c_1),$$
$$a_2x + b_2y + c_2z + d_2 = 0, \quad \mathbf{n}_2 = (a_2, b_2, c_2).$$

Planes ρ_1, ρ_2 are perpendicular, if their normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ are perpendicular, which yields that $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
Planes ρ_1, ρ_2 are parallel, if their normal vectors $\mathbf{n}_1$, $\mathbf{n}_2$ are collinear, which means, there exists a real number λ such that $(a_1, b_1, c_1) = \lambda(a_2, b_2, c_2)$. Parallel planes either coincide, i.e. they have all points in common, or they have no common points, and we speak about two different parallel planes. Planes that are not parallel intersect in a common line, called intersection (or pierce) line of the two planes.

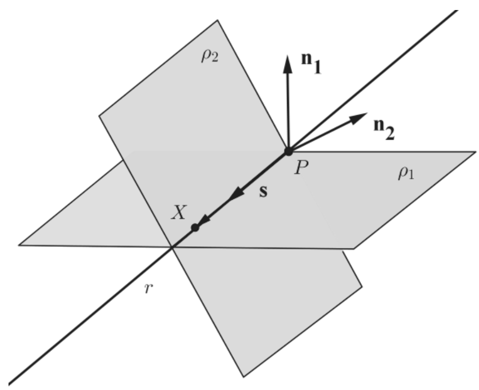


Fig. 1.10. Intersection line of two planes.

Any line r in the space $\mathbf{E}^3$ can be determined as the intersection of two non-parallel planes ρ_1, ρ_2. Therefore, the pair of their general equations forms the general equation of this pierce line r. Vector $\mathbf{s} = \mathbf{n}_1 \times \mathbf{n}_2$, parallel to pierce line r, is called the direction vector of line r. The line can be unambiguously defined by one point $P = [x_P, y_P, z_P]$ and direction vector $\mathbf{s} = (s_x, s_y, s_z)$, and represented by the vector equation

$\mathbf{PX} = t.\mathbf{s},\ t \in \boldsymbol{R},$

which stands for three parametric equations in the coordinate form

$$x = x_P + t\, s_x, \quad y = y_P + t\, s_y, \quad z = z_P + t\, s_z, \quad t \in \boldsymbol{R},$$

where variable t is called a parameter. Specific value of parameter t determines the Cartesian coordinates of one point on the line and the position of this point on the line with respect to the given fixed point P.

Let two lines be represented by the parametric equations

$$p\colon x = x_P + t\, x_1, \quad y = y_P + t\, y_1, \quad z = z_P + t\, z_1, \quad t \in \boldsymbol{R}, \quad \mathbf{s}_1 = (x_1, y_1, z_1),$$

$$q\colon x = x_Q + u\, x_2, \quad y = y_Q + u\, y_2, \quad z = z_Q + u\, z_2, \quad u \in \boldsymbol{R}, \quad \mathbf{s}_2 = (x_2, y_2, z_2).$$

Lines p and q are parallel, if their direction vectors $\mathbf{s}_1$, $\mathbf{s}_2$ are collinear, which means there exists a real number λ such that $(x_1, y_1, z_1) = \lambda\,(x_2, y_2, z_2)$, as seen in Fig. 1.11.

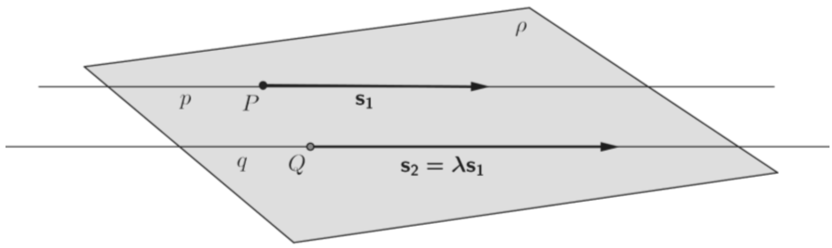


Fig. 1.11. Parallel lines.

Lines p and q are intersecting, if they have exactly one common point, as shown in Fig. 1.12. It means, there exist real numbers t_0 and u_0 such that

$$x_P + t_0\, x_1 = x_Q + u_0\, x_2\,, \quad y_P + t_0\, y_1 = y_Q + u_0\, y_2\,, \quad z_P + t_0\, z_1 = z_Q + u_0\, z_2\,.$$

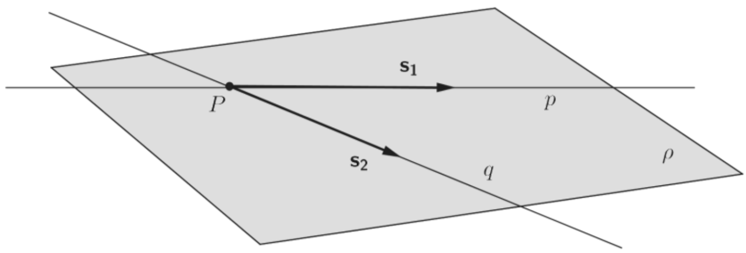


Fig. 1.12. Intersecting lines.

Lines p and q are skew, if they are neither parallel nor intersecting. Skew lines have no common points and their direction vectors are non-collinear.
Lines p and q are perpendicular, if their direction vectors $\mathbf{s}_1$, $\mathbf{s}_2$ are perpendicular, therefore, if $x_1 x_2 + y_1 y_2 + z_1 z_2 = 0$.

Examples

1. Parametric equations of the coordinate axes are:
   x: $x = t, y = 0, z = 0$, y: $x = 0, y = t, z = 0$, z: $x = 0, y = 0, z = t, t \in \boldsymbol{R}$.
2. Planes with general equations ρ_1: $x + 2y - z = 3$ and ρ_2: $2x - y + 3z = 1$ are intersecting planes, their normal vectors $\mathbf{n}_1 = (1, 2, -1)$ and $\mathbf{n}_2 = (2, -1, 3)$ determine the direction vector of their common line, $\mathbf{s} = (5, -5, -5)$. One common point of planes, i. e. one point on their intersection line, can be determined as a point, in which the line intersects coordinate plane xy with coordinate $z = 0$. The other two coordinates can be determined from the equations of these planes, as a solution of the system of two equations: $x + 2y = 3$, $2x - y = 1$. Its unique solution is $x = 1$, $y = 1$. The parametric equations of the intersection line are
   $x = 1 + 5u, y = 1 - 5u, z = 5u, u \in \boldsymbol{R}$.
3. Lines p, q determined by equations p: $x = -1 + t, y = 18 + 9t, z = 10 + 5t, t \in \boldsymbol{R}$ and q: $x + y - 2z + 3 = 0$, $3x - 2y + 3z + 9 = 0$ are identical, because the direction vector of line q is vector $\mathbf{u} = (1, 1, -2) \times (3, -2, 3) = (-1, -9, -5)$ collinear with the direction vector $(1, 9, 5)$ of line p. A common point of lines p and q is point $P = [-1, 18, 10]$ on line p, whose coordinates satisfy equations of line q.
4. Parametric equations of line l passing through point $L = [1, 0, 0]$ and parallel to the line given by the pair of general equations $x - y + z - 5 = 0$, $x + 2y - 7 = 0$ can be derived by means of its direction vector $\mathbf{s}$. This vector is a cross product of planes normal vectors defined by equations, $\mathbf{n}_1 = (1, -1, 1)$, $\mathbf{n}_2 = (1, 2, 0)$, and $\mathbf{s} = (-2, 1, 3)$. The parametric equations of line l are: $x = 1 - 2t, y = t, z = 3t, t \in \boldsymbol{R}$.

Let plane $\rho: ax + by + cz + d = 0$ with normal vector $\mathbf{n} = (a, b, c)$ be given and line p: $x = x_P + t\,x_s, y = y_P + t\,y_s, z = z_P + t\,z_s, t \in \boldsymbol{R}$, with direction vector $\mathbf{s} = (x_s, y_s, z_s)$.
Plane ρ and line p are parallel, if normal vector $\mathbf{n}$ and direction vector $\mathbf{s}$ are perpendicular, i.e. if $ax_s + by_s + cz_s = 0$.
Plane ρ and line p are perpendicular, if normal vector $\mathbf{n}$ and direction vector $\mathbf{s}$ are collinear, i.e. if there exists $\lambda \in \boldsymbol{R}$ such that $(a, b, c) = \lambda\,(x_s, y_s, z_s)$.
Line p lies in the plane ρ if

$$ax_P + by_P + cz_P + d = 0, \text{ and } ax_s + by_s + cz_s = 0.$$

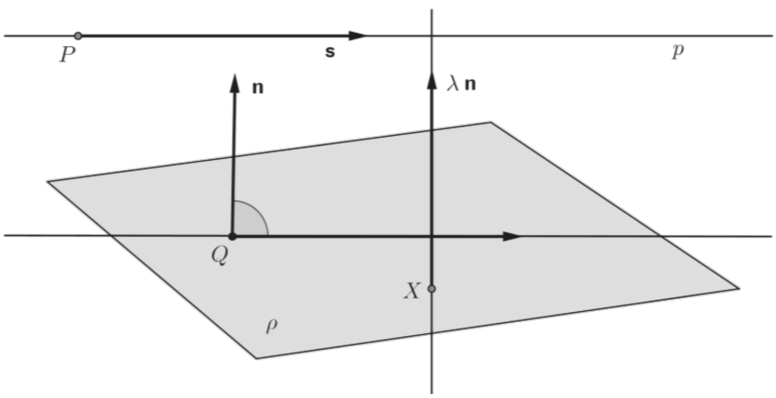


Fig. 1.13. Position of plane and line.

Two intersecting lines define a single plane as do two parallel lines, see Fig. 1.11 and Fig. 1.12. The plane can be determined by point $P = [x_P, y_P, z_P]$ and two non-collinear direction vectors $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and it is represented by a parametric equation

$$\mathbf{PX} = t.\mathbf{u} + s.\mathbf{v},\ t, s \in \boldsymbol{R},$$

which is a symbolic form of three parametric equations

$$x = x_P + t\,u_x + s\,v_x,\ \ y = y_P + t\,u_y + s\,v_y,\ \ z = z_P + t\,u_z, + s\,v_z,\ \ t, s \in \boldsymbol{R},$$

where variables t and s are real parameters. The values of parameters t and s determine the Cartesian coordinates of the points in the plane and their position with respect to the fixed point P and scalar multiples of direction vectors **u** and **v**.

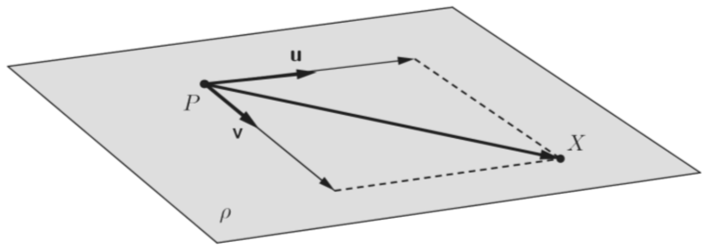


Fig. 1.14. Parametric representation of plane.

The general equation of plane determined by three non-collinear points $A = [x_A, y_A, z_A]$, $B = [x_B, y_B, z_B]$, $C = [x_C, y_C, z_C]$ can be derived from the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_C & y_C & z_C & 1 \end{vmatrix} = 0\,.$$

Parametric equations can be composed from one point, e.g. point A, and plane direction vectors $\mathbf{u} = \mathbf{AB}$ and $\mathbf{v} = \mathbf{AC}$. The normal vector to a plane is the vector product of vectors **u** and **v**, $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, which is perpendicular to vector **AX** determined by the point A and an arbitrary point X in the plane, therefore their scalar product is zero, and the equation of the plane can be represented also as $\mathbf{n} \cdot \mathbf{AX} = 0$.

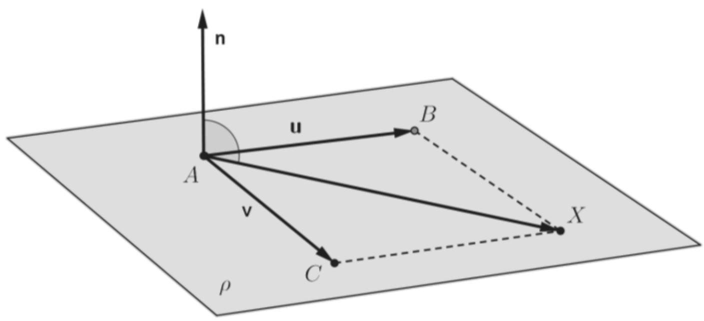


Fig. 1.15. Plane determined by three non-collinear points.

Equations of planes determined by two parallel or intersecting lines or by a line and a point not on this line can be derived in a similar way.

Examples

1. The parametric equations of line k passing through the point $M = [2, 3, -1]$ and perpendicular to the plane ρ: $x + 2y - z = 0$ are
 $x = 2 + t$, $y = 3 + 2t$, $z = -1 - t$, $t \in \boldsymbol{R}$.
2. Line l: $x = 2$, $y = 1 + 3t$, $z = -3 + 2t$, $t \in \boldsymbol{R}$, and plane ρ: $x - 2y + 3z - 1 = 0$ are parallel, as direction vector $\mathbf{s} = (0, 3, 2)$ of line l and normal vector $\mathbf{n} = (1, -2, 3)$ of plane ρ are perpendicular, because their scalar product is 0.
3. The general equation of a plane ρ passing in the direction of coordinate axis x and perpendicular to line q: $x + y - z + 3 = 0$, $x - y + z + 9 = 0$ can be determined from the unit vector (1, 0, 0) in the direction of axis x and direction vector of line q, which is the normal vector $\mathbf{n}$ of the plane ρ. Vector $\mathbf{n}$ is the cross product of normal vectors $\mathbf{n}_1 = (1, 1, -1)$ and $\mathbf{n}_2 = (1, -1, 1)$, $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 = (0, -2, -2)$. The general equation of plane ρ is $-2y - 2z + d = 0$.
4. The general equation of a plane determined by the line $p = AB$, $A = [3, 1, 1]$, $B = [1, 4, 2]$, and point $C = [0, 0, 4]$ is $10x + 3y + 11z - 44 = 0$, normal vector $\mathbf{n} = (10, 3, 11)$ are determined as the cross product of vector $\mathbf{u} = \mathbf{AB} = (-2, 3, 1)$ and vector $\mathbf{v} = \mathbf{AC} = (-3, -1, 3)$, while the value of coefficient $d = -44$ can be achieved from coordinates of any of the points A, B, C.

1.4 Distances and angles

The distance of two points measured by Euclidean formula can be used for measuring distances of any two objects sharing no common points, e.g. the distance of a point and a line, a point and a plane, or the distance of two parallel lines or planes.
Let $P = [x_P, y_P, z_P]$ be a point not in the plane $\rho: ax + by + cz + d = 0$.

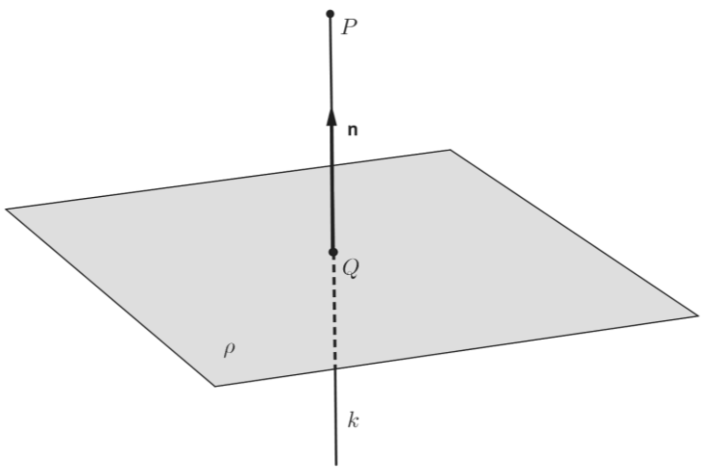


Fig. 1.16. Distance of point and plane.

The distance of the point P from the plane ρ can be measured as the distance of points P and Q, where Q is the intersection point of the plane ρ and perpendicular line k passing through the point P. This distance can be calculated from the formula

$$d(P,\rho) = d(PQ) = \frac{|ax_P + by_P + cz_P + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

The distance of the point P from the line p: $X = A + t\,\mathbf{s}$, $t \in \boldsymbol{R}$, where $A = [x_A, y_A, z_A]$, $\mathbf{s} = (a, b, c)$, can be determined as the distance of points P and Q, where Q is the pierce point of line p and plane ρ passing through point P perpendicularly to line p.

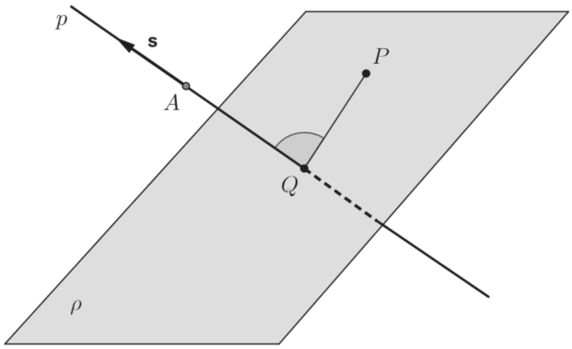


Fig. 1.17. Distance of point and line.

The distance of two parallel planes $\rho: ax + by + cz + d_1 = 0, \pi: ax + by + cz + d_2 = 0$ equals to the distance of two intersection points of these planes and line k perpendicular to both of them, and it can be calculated by the formula

$$d(\pi,\rho) = d(PR) = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

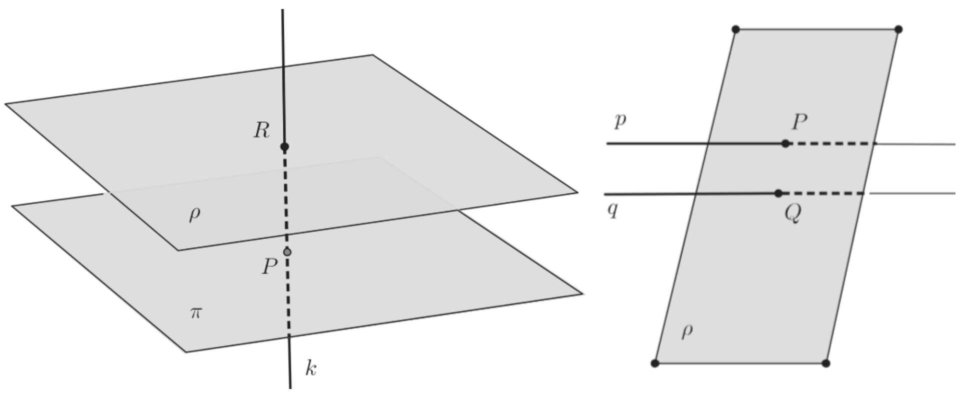


Fig. 1.18. Distance of parallel planes and parallel lines.

Similarly, distance of two parallel lines $p \,||\, q$ is the distance of the pierce points P and Q of these lines with the plane perpendicular to both parallels p and q.

Examples

1. Distance of point $M = [1, 3, -1]$ and plane ρ: $x + 2y - 2z = 0$ is
$$d(M,\rho) = \frac{|1+6+2|}{\sqrt{1^2+2^2+(-2)^2}} = \frac{9}{\sqrt{9}} = 3.$$
2. Line l: $x = 2$, $y = 1 + 2t$, $z = -3 + 2t$, $t \in \boldsymbol{R}$, and point $A = [1, 3, -1]$ are at the distance, which can be measured in the plane ρ passing through point A perpendicularly to the line l. The normal vector of this plane is the direction vector of line l, therefore plane equation is $2y + 2z - 4 = 0$. The intersection point of line l and plane ρ is point $B = [2, 3, -1]$, and the distance of points A and B equals to the distance of point A and line l, $d(A,l) = d(AB) = \sqrt{1+0+0} = 1$.
3. The distance of two parallel planes $\pi: x+y+z+4=0$, $\rho: x+y+z+7=0$ equals to $d(\pi,\rho) = \dfrac{|4-7|}{\sqrt{3}} = \sqrt{3}$.
4. Two parallel lines, $p = PR$, $P = [2, 1, 3]$, $R = [1, 2, -1]$ and q passing through the point $Q = [1, -1, 3]$, are at a distance, which can be measured in the plane ρ perpendicular to both lines and passing, for instance, through the point Q, and defined by equation $x - y + 4z - 14 = 0$. Plane ρ intersects line p at the point $T = [37/18, 17/18, 50/18]$ and distance of points Q and T equals to the distance of lines p and q, which is $d(QT) = \dfrac{1}{6}\sqrt{178}$.

The angle formed by two lines equals to the angle of their direction vectors, therefore parallel lines form angles 0 or π.
A line with direction vector $\mathbf{u}$ and a plane with normal vector $\mathbf{n}$ determine the angle that can be calculated by the formula

$$\sin\varphi = \frac{\mathbf{u}\cdot\mathbf{n}}{|\mathbf{u}||\mathbf{n}|}.$$

A plane and a line parallel to it therefore form angles $\varphi = 0$, or $\varphi = \pi$, depending on the orientation of vectors $\mathbf{u}$ and $\mathbf{n}$.

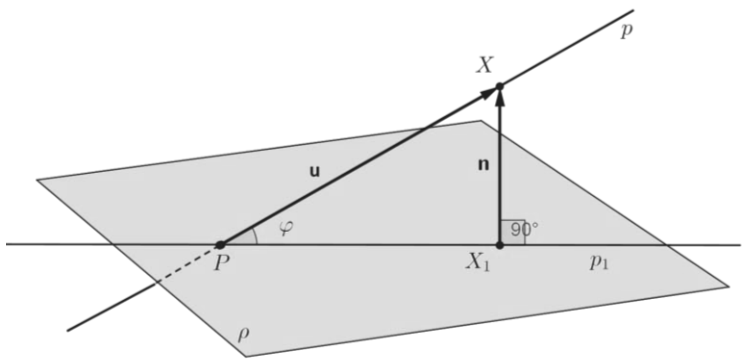


Fig. 1.19. Angle of line and plane.

The angle of two planes is determined as the acute angle of their normal vectors,

$$\cos\varphi = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}, \quad \varphi \in \left\langle 0, \frac{\pi}{2} \right\rangle.$$

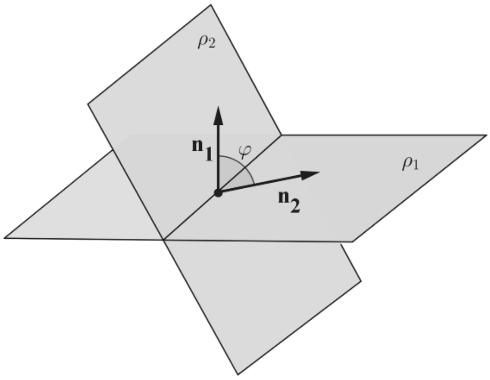


Fig. 1.20. Angle of two planes.

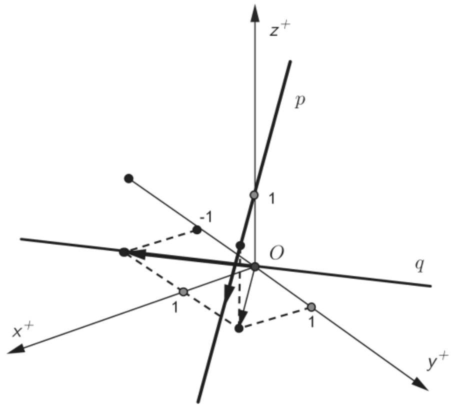


Fig. 1.21. Angle of two lines (example 4).

Examples

1. Lines p: $2x - y + z + 1 = 0$, $x + 2y - 2z + 2 = 0$ and q: $y - z = 0$, $x + 2y - 2z + 2 = 0$ are parallel, because they form an angle determined by their direction vectors $\mathbf{s}_p = (2, -1, 1) \times (1, 2, -2) = (0, 5, 5)$, and $\mathbf{s}_q = (0, 1, -1) \times (1, 2, -2) = (0, -1, 1)$, therefore $\cos\varphi = \frac{(0,5,5)\cdot(0,-1,1)}{\sqrt{0+25+25}.\sqrt{1+1}} = \frac{-10}{10} = -1$, and $\varphi = \pi$.
2. Planes ρ: $2x + y - 3z + 2 = 0$ and τ: $x - 2y + 1 = 0$ are perpendicular, because their angle is defined by the formula $\cos\varphi = \frac{(2,1,-3)\cdot(1,-2,0)}{\sqrt{4+1+9}.\sqrt{1+4}} = 0$, and $\varphi = \pi/2$.
3. The size of the angle formed by line p: $x = 2 + 2t$, $y = 1 + t$, $z = -3 + 2t$, $t \in \boldsymbol{R}$, and plane ρ: $4x + 2y - 4z + 1 = 0$ can be calulated by
$$\sin\varphi = \frac{(2,1,2)\cdot(4,2,-4)}{\sqrt{4+1+4}.\sqrt{16+4+16}} = \frac{2}{3\sqrt{14}}, \ \varphi = \arcsin\frac{2}{3\sqrt{14}} = 0.179131.$$
4. Skew lines p: $x = 1 + u$, $y = 1 + u$, $z = 1$ and q: $x = v$, $y = -v$, $z = 0$, see Fig. 1.21, are perpendicular, because the scalar product of their direction vectors equals to zero, $(1, 1, 0)\cdot(1, -1, 0) = 1 - 1 = 0$.
5. Planes $x + 2z - 2 = 0$, $3x + 6z - 12 = 0$ with collinear normal vectors (1, 0, 2) and (3, 0, 6) are parallel, their distance can be calculated as $d = \frac{|-2-(-4)|}{\sqrt{5}} = \frac{2}{\sqrt{5}}$.
6. Line p: $z - 3 = 0$, $x + y - 2 = 0$ is parallel to plane ρ: $x + y + z - 4 = 0$, because direction vector $\mathbf{s} = (-1, 1, 0)$ of the line p that is the vector product of normal vectors (0, 0, 1) and (1, 1, 0), is perpendicular to the normal vector $\mathbf{n} = (1, 1, 1)$ of plane ρ, as their scalar product is 0. Distance of line p and plane ρ is $d = \frac{1}{\sqrt{3}}$.

1.5 Quadratic surfaces

A quadratic equation in three variables *x*, *y* and *z* in the general form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^3 + a_{12}xy + a_{13}xz + a_{23}yz + a_{14}x + a_{24}y + a_{34}z + a_{44} = 0\,,$$

where at least one of real coefficients $a_{11}, a_{22}, a_{33}, a_{12}, a_{13}, a_{23}$ is non-zero can represent one of the following point sets in the space $\mathbf{E}^3$:

1. an empty set, e.g. $x^2 + y^2 + 1 = 0$
2. a single point, e.g. $x^2 + y^2 + z^2 = 0$, or $x^2 + y^2 + z^2 - 6x + 9 = 0$
3. two planes, e.g. $xy = 0$, or $4x^2 - y^2 = 0$
4. a single plane, e.g. $x^2 = 0$
5. a single line, e.g. $x^2 + y^2 = 0$
6. a quadratic surface, e.g. singular (cylindrical or conical), or regular (ellipsoid, hyperboloid of one or two sheets, elliptic or hyperbolic paraboloid).

Any quadratic surface can be analytically represented by a quadratic equation, the form of which is determined by its position in the coordinate system. Quadratic surfaces in the basic position, i.e. with axes in one of the coordinate axes or in lines parallel to coordinate axes and with vertices located on coordinate axes are represented by equations in the simple canonical forms with real positive constants, *a*, *b*, *c* or *r*.

Cylindrical surfaces

A surface generated by all straight lines passing in a given direction **s** through points on a given curve *k* that is not in the plane in direction **s** is called cylindrical surface. Lines on the cylindrical surface are called generators or rulings, while curve *k* is called the generatrix (generating or basic curve), or also the directrix.

The analytic representations of elliptic (circular for $a = b = r$), hyperbolic and parabolic cylindrical surfaces with a generating ellipse (circle), hyperbola and parabola in the coordinate plane *xy* (with axes in the coordinate axes) and rulings in the direction of coordinate axis *z* are in the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x^2 + y^2 = r^2, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x^2 = \pm cy\,.$$

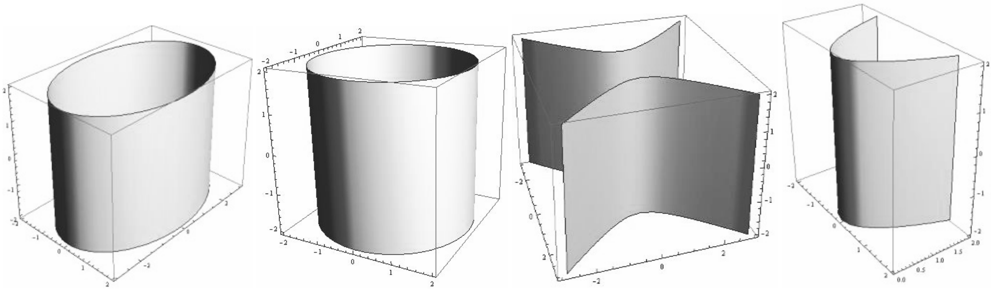


Fig. 1.22. Cylindrical surfaces.

Analogously, the equations of cylindrical surfaces with a generatrix in other coordinate planes *xz*, or *yz* and rulings in the direction of the respective orthogonal coordinate axes *y* or *x* can be derived. Cylindrical surfaces determined by an ellipse (a circle) or a hyperbola in a more general position, i.e. with its centre at the point $S = [m, n, 0]$ and its axes in the direction of one of the coordinate axes, or a parabola with its vertex at the point $V = [m, n, 0]$ and its axis in the direction of coordinate axis *y* can be represented in the form

$$\frac{(x-m)^2}{a^2}+\frac{(y-n)^2}{b^2}=1, \quad (x-m)^2+(y-n)^2=r^2,$$
$$\frac{(x-m)^2}{a^2}-\frac{(y-n)^2}{b^2}=\pm 1, \quad (x-m)^2=\pm c(y-n).$$

A circular cylindrical surface is called cylindrical surface of revolution as it can be generated by revolving one generator line in a given direction about an axis of the surface passing through the circle centre in the respective direction.

Conical surfaces

A surface generated by all straight lines passing through a given point *V* and intersecting a given curve *k* that is not in the same plane with the point *V* is called a conical surface. Lines on a conical surface are called generators (rulings), point *V* a vertex and curve *k* is called a generatrix (generating or basic curve), or also a directrix. The analytic representations of elliptic conical surfaces with vertex at the origin and generating ellipse in the plane parallel to one of the coordinate planes *xy*, *xz* or *yz* (with axes in the coordinate axes) are in the forms

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2=0, \quad \frac{x^2}{a^2}+\frac{z^2}{b^2}-y^2=0, \quad \frac{y^2}{a^2}+\frac{z^2}{b^2}-x^2=0.$$

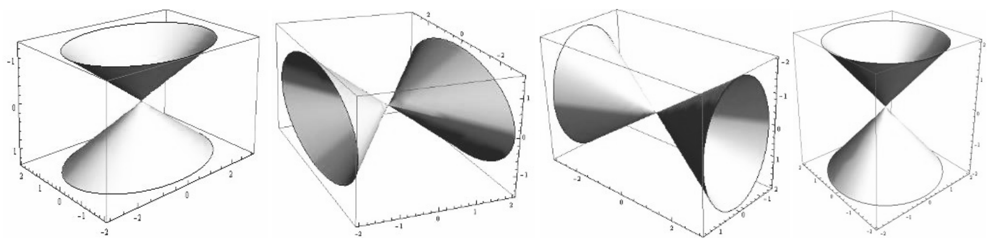


Fig. 1.23. Conical surfaces.

A conical surface of revolution can be created by generating circle *k* in a plane parallel to one coordinate plane with centre on the perpendicular coordinate axis that is the axis of the conical surface of revolution. Such a surface can be also generated by revolving a line passing through the origin about one of the coordinate axis.

A more general equation of a cylindrical conical surface with a vertex at the point on one of the coordinate axes can be obtained in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-(z-c)^2=0, \quad \frac{x^2}{a^2}+\frac{z^2}{b^2}-(y-c)^2=0, \quad \frac{y^2}{a^2}+\frac{z^2}{b^2}-(x-c)^2=0.$$

Ellipsoids

The canonical form of the equation of an ellipsoid with its centre at origin and axes in the coordinate axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The coordinate planes *xy*, *xz* and *yz* intersect ellipsoid in ellipses with pairs of semi-axes (a, b), (a, c) and (b, c). In the case of two equal semi-axes we speak about an ellipsoid of revolution, and if all three semi-axes are equal, $a = b = c = r$, it becomes a sphere with the centre $S = [m, n, p]$ and radius r that is defined by the general equation

$$(x - m)^2 + (y - n)^2 + (z - p)^2 = r^2.$$

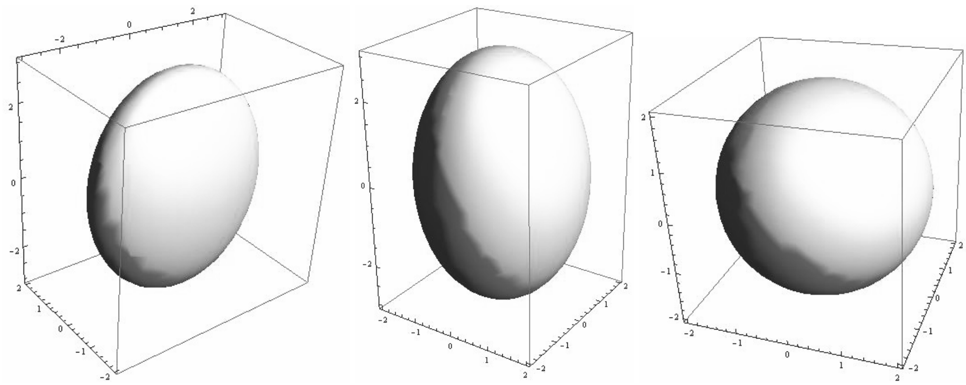


Fig. 1.24. Ellipsoids and sphere.

Hyperboloids

The equation of a hyperboloid of one sheet in the basic position (with its axis in the coordinate axis z and its centre of symmetry at origin) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

The intersections of this quadratic surface by coordinate planes *xz* and *yz* are hyperbolas with their centres at origin, their imaginary axes in the coordinate axis z and semiaxes (a, c) and (b, c), and the surface intersection by the coordinate plane *xy* is an ellipse with semi-axes (a, b) on coordinate axes x and y.

Analogously, the canonical equations of one-sheet hyperboloids in basic positions with axes in coordinate axes y and z can be derived as

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

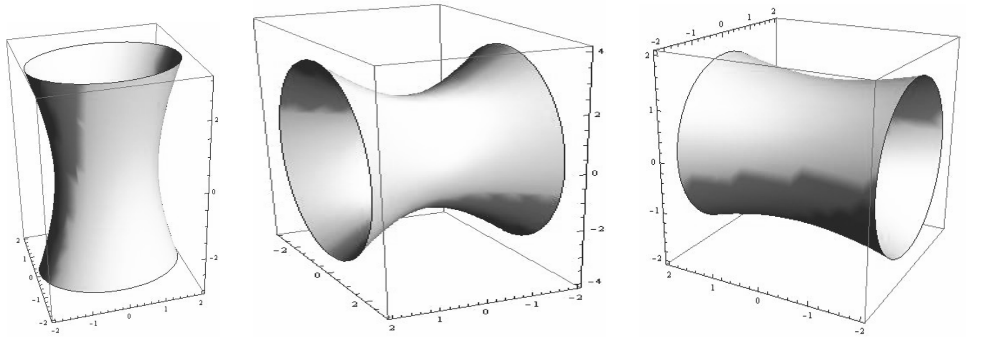


Fig. 1.25. Hyperboloids of one sheet.

A hyperboloid of two sheets in the basic position (with axis in the coordinate axis *z* and centre of symmetry at origin) with the equation

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,,$$

consists of two separated parts. The intersections of this quadratic surface by the coordinate planes *xz* and *yz* are hyperbolas with centres at origin and semiaxes (a, c) and (b, c). The coordinate plane *xy* does not intersect this two-sheet hyperboloid, but its intersections by planes $y = k$, $|k| > c$, are ellipses.

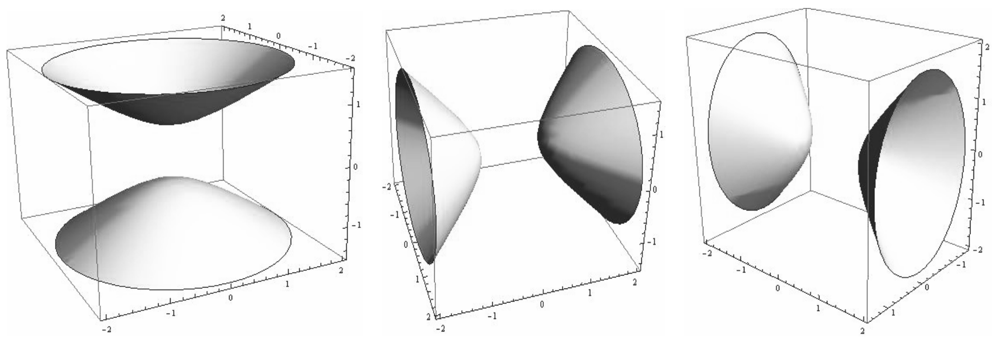


Fig. 1.26. Hyperboloids of two sheets.

Paraboloids

An elliptic paraboloid with its axis in the coordinate axis *z*, *y*, *x* and vertex at origin is determined by the canonical equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\pm cz=0,\quad \frac{x^2}{a^2}+\frac{z^2}{b^2}\pm cy=0,\quad \frac{y^2}{a^2}+\frac{z^2}{b^2}\pm cx=0,$$

while the + sign stands for the positive half-space where $z > 0$, $y > 0$, $x > 0$, and, obviously, the – sign for the negative half-space, where $z < 0$, $y < 0$, $x < 0$. A plane perpendicular to the paraboloid´s axis intersects the surface in ellipses, while for $a = b$ we speak about paraboloid of revolution with intersections that are circles. Planes passing through the paraboloid´s axis intersect it in parabolas.

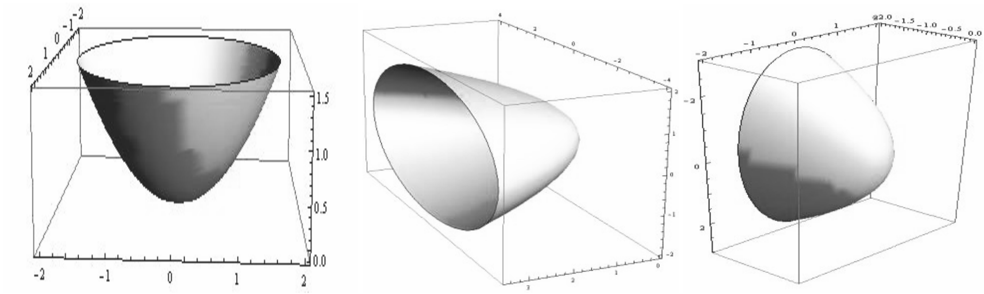


Fig. 1.27. Paraboloids.

A hyperbolic paraboloid with its axis in the coordinate axis *z*, *y*, *x* and its vertex – a saddle point at origin is determined by the canonical equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \pm cz = 0, \quad \frac{x^2}{a^2} - \frac{z^2}{b^2} \pm cy = 0, \quad \frac{y^2}{a^2} - \frac{z^2}{b^2} \pm cx = 0.$$

Its intersections by the coordinate planes are parabolas and/or hyperbolas, a plane passing through the vertex intersects this surface in a pair of intersecting lines (singular hyperbola). Contrary to other quadratic surfaces, there are no values of constants *a*, *b*, *c*, for which a hyperbolic paraboloid can be a surface of revolution.

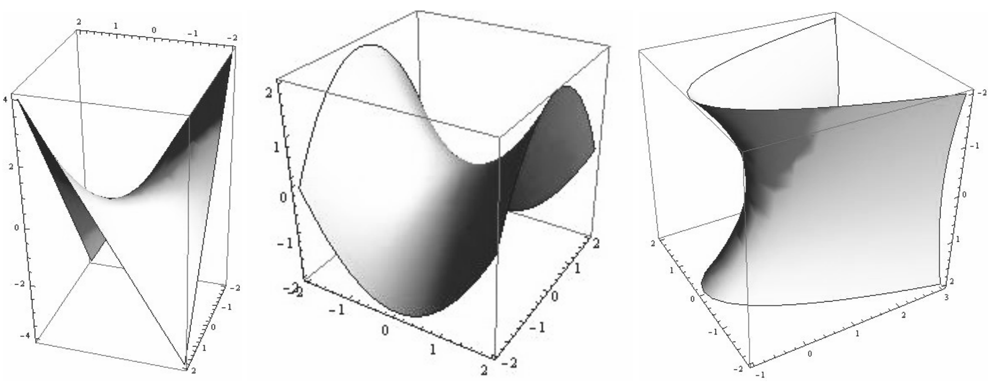


Fig. 1.28. Hyperbolic paraboloids.

Examples

1. An ellipsoid of revolution with axis in coordinate axis *x*, semi-axis $a = 2$, $b = c = 3$ is determined by the equation $9x^2 + 4y^2 + 4z^2 = 36$.
2. Equation $z = y^2 - x^2 = 0$ defines hyperbolic paraboloid with saddle point at origin.
3. Equation $z = \sqrt{4 + x^2 + y^2}$ defines the upper part of a two-sheet hyperboloid of revolution with its axis in coordinate axis *z* and semi-axes $a = b = c = 2$.
4. Canonical form of the equation of a parabolic cylindrical surface with its axis parallel to coordinate axis *y* and its basic parabola in coordinate plane *xz* with a vertex [0, 0, 2] and parameter $p = -2$ is $x^2 = -4(z - 2)$.

1.6 *n*-dimensional Euclidean space

A space consisting of all points determined by n-tuples of real numbers, $n \geq 1$, with the distance of two arbitrary points $X = [x_1, x_2, \dots, x_n]$, $Y = [y_1, y_2, \dots, y_n]$ defined by the Euclidean metric

$$d(X,Y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + ... + (y_n - x_n)^2}$$

is called an n-dimensional Euclidean space.
If $n = 1$, then, $d(X, Y) = |y_1 - x_1|$. For every natural number n and for any triplet of points $X, Y, Z \in \mathbf{E}^n$ the following properties can be proved:

1. $d(X,Y) \geq 0, \quad d(X,Y) = 0 \Leftrightarrow X = Y$
2. $d(X,Y) = d(Y,X)$
3. $d(X,Y) \leq d(X,Z) + d(Z,Y)$.

The function d: $\mathbf{E}^n \times \mathbf{E}^n \to \boldsymbol{R}$ is called a metric on $\mathbf{E}^n$, and the pair $(\mathbf{E}^n, d)$ is the metric space. Let X_0 be a point in the space $\mathbf{E}^n$, and $\varepsilon > 0$ be a real number. Then set

$$N_\varepsilon(X_0) = \{X \in \mathbf{E}^n : d(X, X_0) < \varepsilon\}$$

is the ε-neighbourhood of point X_0. $N_\varepsilon(X_0)$ is an open interval for $n = 1$, it is an open disc for $n = 2$, and a ball without its spherical boundary for $n = 3$.

Let M be a subset of $\mathbf{E}^n$. A point $X_0 \in M$ is called an interior point of the set M, if there exists $\varepsilon > 0$ such that $N_\varepsilon(X_0) \subset M$. The set of all interior points of the set M is called the interior of the set M. The set M is said to be open, if it consists of interior points only.
A point $X_0 \in \mathbf{E}^n$ is called a boundary point of the set $M \subset \mathbf{E}^n$, if each neighbourhood $N_\varepsilon(X_0)$ contains at least one point which belongs to the set M and at least one point that does not belong to the set M. The set of all boundary points of the set M is called the boundary of the set M. A set is called to be closed, if it contains all its boundary points.

Examples

1. The interior of set $M = \{[x, y] \in \mathbf{E}^2 : 0 \leq x \leq 1, 0 < y < 1\}$ is an open plane region with a square boundary with vertices at origin [0, 0], point [0, 1] on coordinate axis x, point [1, 0] on coordinate axis y and point with coordinates [1, 1].
2. The space region bounded by a unit cube with vertices on coordinate axes in $\mathbf{E}^3$ is the closed set $M = \{[x, y, z] \in \mathbf{E}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.
3. The set $M = \{[x, y, z] \in \mathbf{E}^3 : x^2 + y^2 + z^2 \leq 4\}$ is a closed ball in space $\mathbf{E}^3$ bounded by a sphere with centre at origin and radius of 2.

4. An ellipsoid of revolution with semi-axes $a = 2$, $b = 3$ and $c = 3$ on coordinate axes x, y, and z and centre at the origin of the coordinate system is the boundary of an open set determined as $M = \left\{ [x, y, z] \in \mathbf{E}^3 : \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{9} < 1 \right\}$.

Set $M \subset \mathbf{E}^n$ is called connected, if any pair of its points can be connected by a simple curve lying entirely in the set M. The set M is called simply connected, if it contains any bounded region with the boundary in a closed curve k in M, $k \subset M$. Simply connected set contains no holes.

Set $M \subset \mathbf{E}^n$ is call bounded, if a real number $r \in \boldsymbol{R}$ and a point $X_0 \in \mathbf{E}^n$ exist such that for each $X \in M$ holds $d(X, X_0) < r$.

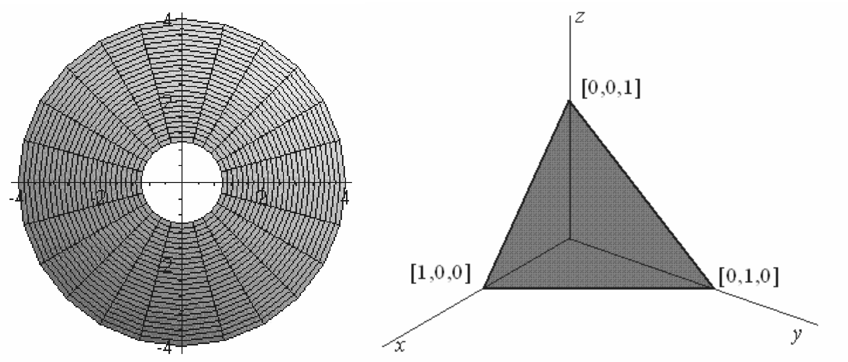


Fig. 1.29. Connected, simply connected and bounded sets.

Examples

1. The annulus $M = \left\{ [x, y] \in \mathbf{E}^2 : 1 \le x^2 + y^2 \le 16 \right\}$ determined by concentric circles with centres at origin and radii 1 and 4 is a connected set in the plane, but not simply connected. It is bounded, as for example for point $X_0 = [0, 0] \in \mathbf{E}^2$ and any $r > 4$, $d(X, X_0) < r$ for all $X \in M$, see in Fig. 1.29, left.
2. Tetrahedral region with vertices at origin and unit points on coordinate axes in $\mathbf{E}^3$ is set $M = \left\{ [x, y, z] \in \mathbf{E}^3 : x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1 \right\}$, which is closed, simply connected and bounded set, as for any $r > 1$ and point $X_0 = [0, 0, 0]$ it holds that $d(X, X_0) < r$ for all $X \in M$, see in Fig. 1.29, right.

A point is called an isolated point of a set $M \subset \mathbf{E}^n$, if such ε-neighbourhood of this point exists which contains no other point from the set M. A point is called a limit (cluster) point of a set $M \subset \mathbf{E}^n$, if any ε-neighbourhood of this point contains infinitely many points from the set M.

Any bounded infinite set $M \subset \mathbf{E}^n$ contains at least one limit point. A limit point is either an interior or a boundary point of this set. An interior point of the set is always an element of the set, an exterior point never belongs to the set. A boundary point can either belong to the set, or it can be not a point of the set.

Let $M \subset \mathbf{E}^n$ be a non-empty set and $X_0 \in \mathbf{E}^n$ be an arbitrary point. Then exactly one of the following holds:

1. Point X_0 belongs to the set M with at least one of its neighbourhoods $N_\varepsilon(X_0) \subset M$. Point X_0 is an interior point of the set M.
2. Point X_0 does not belong to the set M and neither does any point from at least one of its neighbourhoods, $N_\varepsilon(X_0) \cap M = \emptyset$. Point X_0 is an exterior point of the set M.
3. Any neighbourhood $N_\varepsilon(X_0)$ of the point X_0 contains at least one point of the set M and at least one point not in the set M. Point X_0 is a boundary point of the set M.
4. There exists such neighbourhood $N_\varepsilon(X_0)$ of the point X_0, for which $N_\varepsilon(X_0) \cap M = \{X_0\}$. Point X_0 is an isolated and a boundary point of the set M.

Any set of points in $\mathbf{E}^n$ which is opened and connected is called a region. A set of points in $\mathbf{E}^n$ which is the closure of a region is a closed region. Any closed region can be obtained from a suitable region by adding all its boundary points.
A set M is closed if and only if it contains all its limit points.

Examples

1. The set $M = \{[x, y, z] \in \mathbf{E}^3 : |z| > 1\}$ in Fig. 1.30, left, is a not connected and not bounded open region determined by all points in two half-spaces with boundary planes $z = -1$, and $z = 1$. Origin O is the exterior point of set M, all points in the boundary planes are limit points not contained in the set. Complement of set M is a closed region, a layer between the two planes $CM = \{[x, y, z] \in \mathbf{E}^3 : |z| \le 1\}$.
2. The region $M = \{[x, y] \in \mathbf{E}^2 : xy \ge 0, -2 \le x \le 2, -2 \le y \le 2\} \cup \{[-2,2],[2,-2]\}$ is a closed, simply connected, and bounded set, with boundaries in line segments on coordinate axes x and y joining points [–2, 0], [2, 0] and [0, –2], [0, 2] and both isolated points [–2, 2] and [2, –2], see in Fig. 1.30, right.

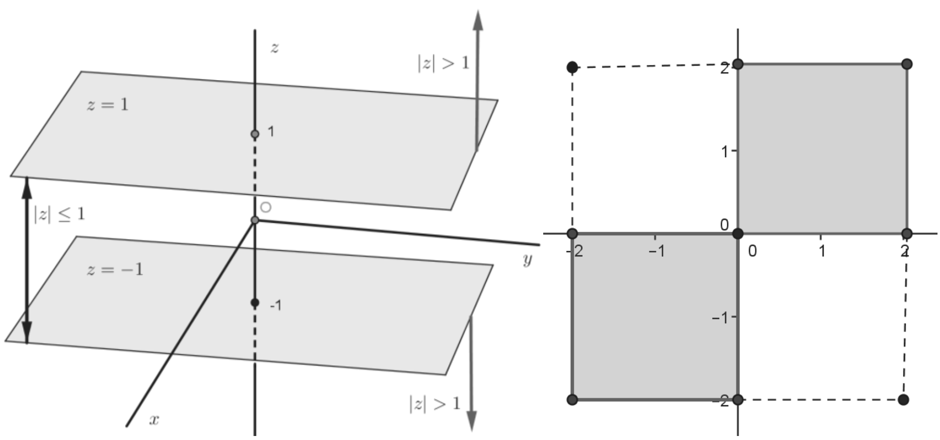


Fig. 1.30. Open region in space (on left), closed region with isolated points (right).

2 Differential calculus of multivariable functions

2.1 Definition of a function of more variables

Let M be a non-empty subset of the n-dimensional Euclidean space

$$M \subset \mathbf{E}^n,\ n \geq 1,\ M \neq \emptyset.$$

Any mapping f from the set M to $\boldsymbol{R}$, in which each point $X = [x_1, x_2, \dots, x_n] \in M$ is attached a unique real number, is called a real function of n real variables.

$$f: M \to \boldsymbol{R},\ X \to f(X).$$

Set M is called the domain of definition of function f, denoted as $D(f)$.
Image of the point $X = [x_1, x_2, \dots, x_n] \in M$ in the mapping f, the real number attached to the point X, denoted as $y = f(X) = f(x_1, x_2, \dots, x_n)$, is the value of the function f at the point X. The range of values of function f is the set

$$H(f) = \{y \in \boldsymbol{R} : \exists X \in D(f), y = f(X)\}.$$

For $n = 2$ we use the notation $f(x, y)$ instead of $f(x_1, x_2)$, and for $n = 3$ the notation $f(x, y, z)$ is used instead of $f(x_1, x_2, x_3)$.
A function of n variables can be uniquely determined by the domain of definition $D(f) \subset \mathbf{E}^n$ and by a formula (or function rule), due to which exactly one real number y can be attached to any point $X = [x_1, x_2, \dots, x_n] \in D(f)$ as a function value $y = f(X)$.
In the same way as for the functions of one real variable, a function rule can be determined in various forms:

- in words
- by a table of values
- by a graph
- analytically – by means of a mathematical expression or an equation.

Some of the significant concepts, for example boundedness (boundedness from below or boundedness from above), maximum or minimum of the function and operations on functions are defined analogously as in the real case for $n = 1$.

Let f be a function of n variables defined on the set M of points in the space $\mathbf{E}^n$, for $n \geq 1$. The graph of function f is a set $G(f) \subset \mathbf{E}^{n+1}$ of all ordered $(n + 1)$-tuples, points $[x_1, x_2, \dots, x_n, x_{n+1}] \in \mathbf{E}^{n+1}$, for which the following properties hold:

1. $[x_1, x_2, \dots, x_n] \in M$
2. $x_{n+1} = f(x_1, x_2, \dots, x_n)$.

Therefore

$$G(f) = \{[x_1, x_2, \dots, x_n, x_{n+1}] \in \mathbf{E}^{n+1} : X = [x_1, x_2, \dots, x_n] \in M, x_{n+1} = f(X)\}.$$

Geometrically we can directly visualise only graphs of functions of one or two variables. The graph of a continuous function of one real variable $y = f(x)$, $x \in I \subset R$ is a plane curve segment.

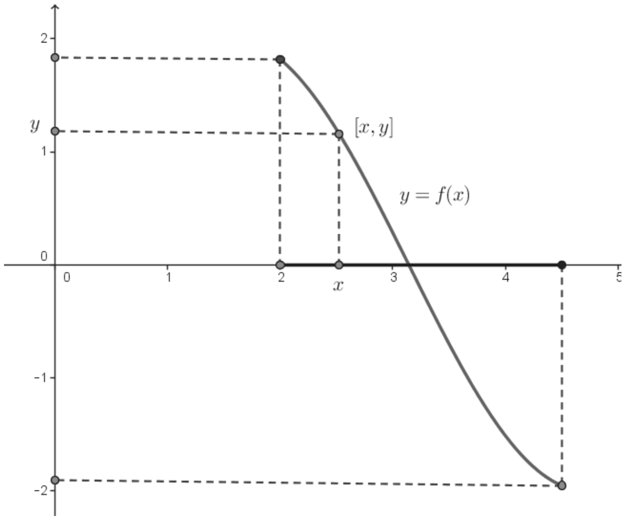


Fig. 2.1. Graph of function of one variable.

The graph of a function of two variables $f(x, y)$ is defined on the set $D(f)$ of points in the space $\mathbf{E}^2$ and it is the set of all such points $[x, y, z]$ in the space $\mathbf{E}^3$, for which:

1. $[x, y] \in D(f)$
2. $z = f(x, y)$

therefore

$$G(f) = \{[x, y, z] \in \mathbf{E}^3: [x, y] \in D(f), z = f(x, y)\}.$$

$G(f)$ is a set of those points $[x, y, z]$ in the space, whose coordinates satisfy the equation $z = f(x, y)$, and usually it can be geometrically visualised as a surface patch in $\mathbf{E}^3$.

Properties:

1. The orthogonal projection of the graph of function $f(x, y)$ to the plane xy is the function domain of definition $D(f)$.
2. Any line parallel to the coordinate axis z intersects the graph of function $f(x, y)$ in at most one point.

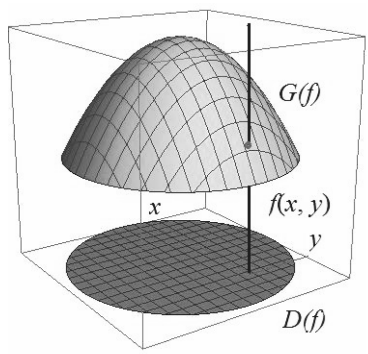


Fig. 2.2. Graph of function of two variables.

A surface that is a graph of function $f(x, y)$ of two variables can be projected into different views using the projection methods of Descriptive geometry, by means of one of the basic projection methods – Monge method (top view and front view or side views), or orthogonal axonometry (axonometric view).

Examples

1. Function $f(x,y)=\sqrt{1-xy}$ is defined on $D(f)=\{[x,y]\in\mathbf{E}^2 : xy\le 1\}$. Part of the domain of definition and the corresponding part of the graph of this function, and the contour plot of the graph are in the Fig. 2.3.

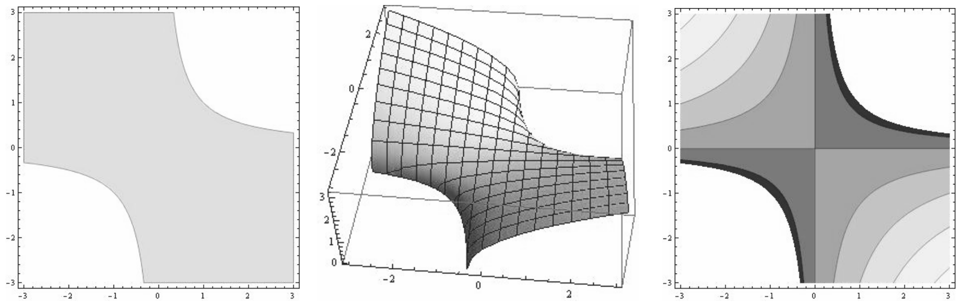


Fig. 2.3. Domain of definition and graphs of function with two variables.

2. Graph of function $f(x,y)=1+\arcsin(x+y)$, the contour plot of the function graph and its domain of definition $M=\{[x,y]\in\mathbf{E}^2 : -1\le x+y\le 1\}$ are in the Fig. 2.4.

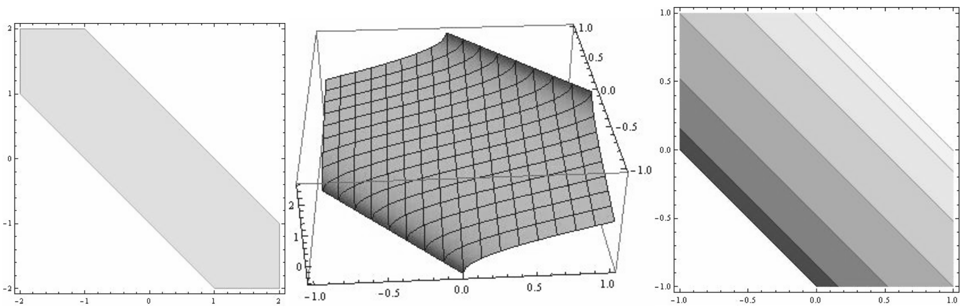


Fig. 2.4. Domain of definition and graphs of function with two variables.

2.2 Limit and continuity of functions of more variables

Let function $f(X)$ be defined on some neighbourhood of the point $A=[a_1, a_2, ..., a_n]$, which is the limit point of the function domain of definition $D(f)$.
Number b is said to be the limit of function $f(X)$ at the point A, if for all $\varepsilon>0$ there exists such $\delta>0$, that for all points $X\in N_\delta(A)$, $X\ne A$, it holds $f(X)\in N_\varepsilon(b)$:

$$\lim_{X\to A} f(X)=b \Leftrightarrow \forall\varepsilon>0\,\exists\delta>0: d(X,A)<\delta \Rightarrow \left|f(X)-b\right|<\varepsilon .$$

The limit of function $f(X)$ of two variables at the point $A=[x_0, y_0]$ can be also written

$$\lim_{X\to[x_0,y_0]} f(X)=b .$$

Let functions f and g have proper limits at the point A

$$\lim_{X \to A} f(X) = b_1,\ \lim_{X \to A} g(X) = b_2 .$$

Then there exists (at the point A) also the limit of functions:

1. $c_1 f + c_2 g$, where c_1, c_2 are arbitrary constants and

$$\lim_{X \to A}\left(c_1 f(X) + c_2 g(X)\right) = c_1 b_1 + c_2 b_2$$

2. $f.g$ and $\lim_{X \to A}\left(f(X).g(X)\right) = b_1 b_2$

3. $\dfrac{f}{g}$ and $\lim_{X \to A} \dfrac{f(X)}{g(X)} = \dfrac{b_1}{b_2},\ b_2 \neq 0 .$

An improper limit of a function of several variables at a proper point, which is the limit point of its domain of definition, is defined similarly to an improper limit of a function of one variable:

$$\lim_{X \to A} f(X) = \infty \Leftrightarrow \forall K > 0\ \exists \delta > 0 : d(X, A) < \delta \Rightarrow f(X) > K ,$$
$$\lim_{X \to A} f(X) = -\infty \Leftrightarrow \forall K > 0\ \exists \delta > 0 : d(X, A) < \delta \Rightarrow f(X) < K .$$

Examples

1. Function $f(x, y) = \dfrac{1}{x^2 + y^2}$ is not defined at the point [0, 0] that is the limit point of its domain of definition $D(f) = \mathbf{E}^2 - \{[0, 0]\}$, and it holds that $\lim_{X \to [0,0]} (x^2 + y^2) = 0$, anyhow there exists an improper limit of function $f(x, y)$ at the point [0, 0], $\lim_{X \to [0,0]} f(x, y) = \lim_{X \to [0,0]} \dfrac{1}{x^2 + y^2} = \infty$, see Fig. 2.5, left.

2. Function $f(x, y) = \ln(x + y^2)$, with domain $D(f) = \{[x, y] \in \mathbf{E}^2 : x + y^2 > 0\}$ is not defined at the points of parabola $y^2 = -x$, which are the limit points of its domain of definition, the function limit at these points is improper and it equals $-\infty$.

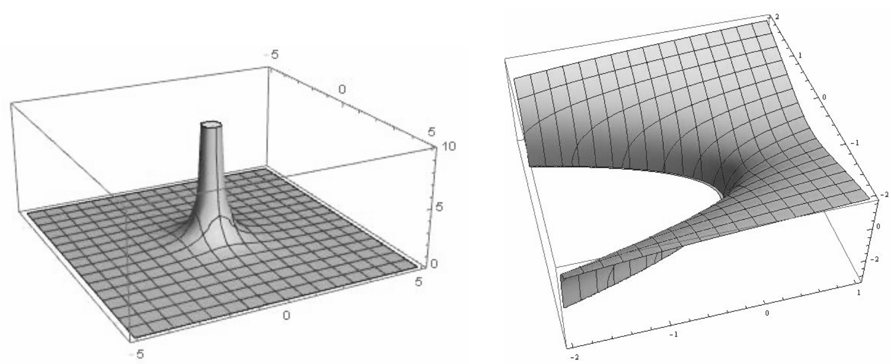


Fig. 2.5. Improper limits of functions at the limit points of their domains.

Let function $f(X)$ be defined on some neighbourhood of the limit point of its domain of definition $D(f)$, point $A = [a_1, a_2, ..., a_n]$. Function $f(X)$ is continuous at the point A, if a proper limit of function f exists at this point, and this limit equals to the value of function f at the point A

$$\lim_{X \to A} f(X) = f(A).$$

Function f continuous at all points of the set $M \subset D(f)$ is said to be continuous on the set M. If $M = D(f)$, we speak about a continuous function f.

Let functions f and g be defined on some neighbourhood of the point A and let them both be continuous at the point A. Then the functions

$$c_1 f + c_2 g,\ c_1, c_2 \in \boldsymbol{R}$$

$$f \cdot g$$

are also continuous at the point A.

If $g(A) \neq 0$, then the function $\dfrac{f}{g}$ is also continuous at the point A.

Remark. If A is a boundary point of the domain of definition of function f, then f cannot be continuous at A in the standard sense. This is the reason why we define a new concept of continuity of a function at a point with respect to a set, which is a kind of analogy of one-sided limits of real functions of one real variable.

Let function $f(X)$ be defined on a set $M \subset \boldsymbol{E}^n$ and let point $A \in M$. It is said that the function f is continuous at the point A with respect to the set M, if for each $\varepsilon > 0$ there exists $\delta > 0$ such, that if $X \in N_\delta(A) \cap M$ then $f(X) - f(A) < \varepsilon$. It is said that function f is continuous on the set M, if it is continuous at each point $X \in M$ with respect to the set M.

A function of more variables continuous on a closed region has similar properties as a function of one variable continuous on a closed interval.

Let function f be continuous on a bounded, connected and closed region $\Omega \subset \mathbf{E}^n$. Then the following assertions hold:

1. Function f is bounded on Ω, therefore there exists positive number $K > 0$ such, that $|f(X)| < K$ for any point $X \in \Omega$.
2. Function f has its maximum and minimum on the set Ω, therefore there exists at least one point $P_1 \in \Omega$ and at least one point $P_2 \in \Omega$ such, that

   $f(P_1) \leq f(X) \leq f(P_2)$ for all points $X \in \Omega$.
3. Let A and B be different points from the region Ω such, that $f(A) \neq f(B)$. Then function f reaches any value between $f(A)$ and $f(B)$ at the points from Ω, therefore there exists at least one point $C \in \Omega$ such, that $f(A) < f(C) < f(B)$.

The range of values of function f of two variables that is continuous on a bounded closed region Ω is a closed interval in $\boldsymbol{R}$ (or a one point set) that is the image of the region Ω in the mapping determined by the function f.

Examples

1. Range of values of function $f(x, y) = \sin(x^2 - y^2)$, with domain $D(f) = \mathbf{E}^2$, is the interval $H(f) = \langle -1, 1\rangle \subset \boldsymbol{R}$, the function graph on the set $M = \left\langle -\frac{2\pi}{3}, \frac{2\pi}{3} \right\rangle^2$ is illustrated in Fig. 2.6, left.
2. Interval $\langle -1, -0.5\rangle \subset \boldsymbol{R}$ is the range of function $f(x, y) = \dfrac{1}{x^2 + y^2 - 2}$ defined on the set $M = \{[x, y] \in \mathbf{E}^2 : x^2 + y^2 \le 1\}$. The graph of the function f is sketched in Fig. 2.6, on the right.

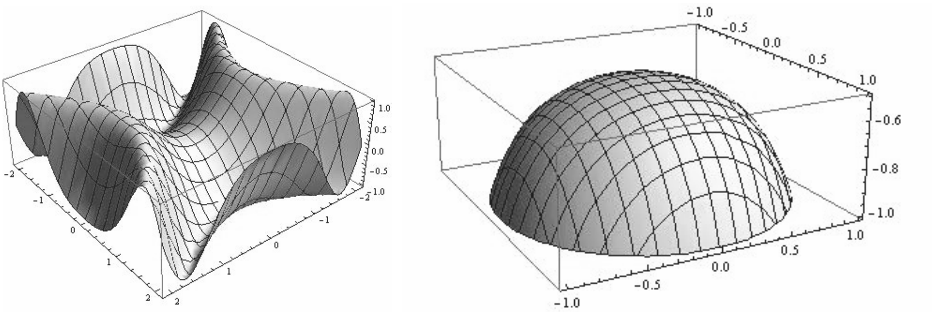


Fig. 2.6. Graphs of functions continuous on bounded closed regions.

2.3 Partial derivatives of functions of more variables

Let $z = f(x, y)$ be a function defined on certain neighbourhood of point $A = [x_0, y_0]$, which is the limit point of its domain of definition $D(f)$. Let us determine the set

$$M_x = \{x \in R : [x, y_0] \in D(f)\} \subset \mathbf{E}^1$$

and define the function

$$g : M_x \to R: \quad \forall x \in M_x \quad g(x) = f(x, y_0).$$

The derivative $g'(x_0)$ of the function $g(x)$ at the point x_0, if it exists, is said to be the partial derivative of function $f(x, y)$ at the point $A = [x_0, y_0]$ with respect to the variable x, denoted

$$f'_x(x_0, y_0) = f'_x(A) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial x}(A),$$

$$f'_x(x_0, y_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.$$

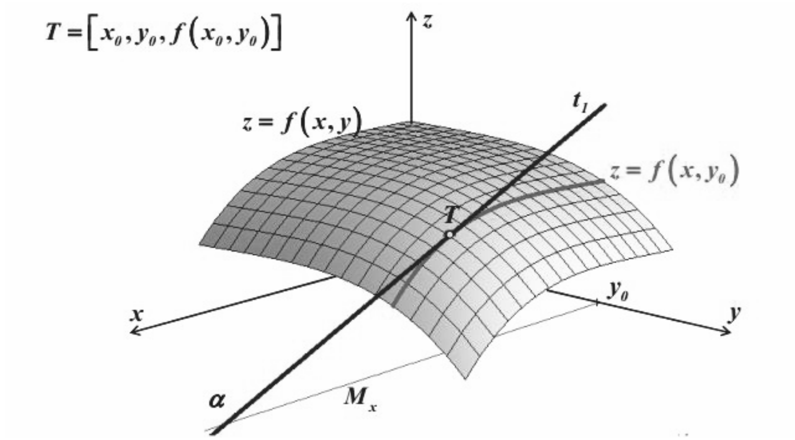


Fig. 2.7. Partial derivative of function $f(x, y)$ with respect to x at point $T = [x_0, y_0, f(x_0, y_0)]$.

Plane $y = y_0$ parallel to coordinate axis z and intersecting coordinate plane xy in the set M_x intersects the graph of function $f(x, y)$ in the curve $z = f(x, y_0)$. The partial derivative $f'_x(x_0, y_0)$ can be geometrically interpreted as the slope of the tangent line t_1 to this curve at the point $T = [x_0, y_0, f(x_0, y_0)]$, $f'_x(x_0, y_0) = \tan\alpha$, where α is the angle of line t_1 and the coordinate plane xy, while its direction vector is $\mathbf{s}_1 = (1{,}0, f'_x(x_0, y_0))$.

Analogously we can determine the set

$$M_y = \{y \in R : [x_0, y] \in D(f)\} \subset \mathbf{E}^1$$

and define the function

$$h : M_y \to R : \quad \forall y \in M_y \quad h(y) = f(x_0, y).$$

The derivative $h'(y_0)$ at the point y_0 of function $h(y)$, if it exists, is the partial derivative of function $f(x, y)$ at the point $A = [x_0, y_0]$ with respect to the variable y, denoted

$$f'_y(x_0, y_0) = f'_y(A) = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial f}{\partial y}(A),$$

$$f'_y(x_0, y_0) = \lim_{y \to y_0} \frac{h(y) - h(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.$$

Plane $x = x_0$ parallel to coordinate axis z and intersecting coordinate plane xy in the set M_y intersects the graph of function $f(x, y)$ in the curve $z = f(x_0, y)$. The partial derivative $f'_y(x_0, y_0)$ can be geometrically interpreted as the slope of the tangent line t_2 to this curve at the point $T = [x_0, y_0, f(x_0, y_0)]$, $f'_y(x_0, y_0) = \tan\beta$, where β is the angle that line t_2 forms with respect to the coordinate plane xy, while this line direction vector is $\mathbf{s}_2 = (0{,}1, f'_y(x_0, y_0))$.

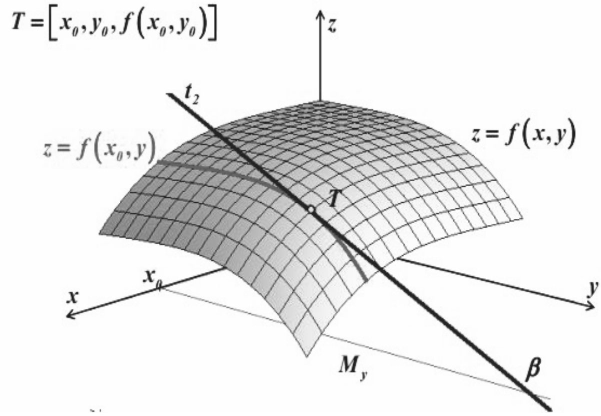


Fig. 2.8. Partial derivative of function $f(x, y)$ with respect to y at point $T = [x_0, y_0, f(x_0, y_0)]$.

Function $f(x, y)$ continuous at the point A may have no partial derivatives defined at the point A. Function $f(x, y)$, whose partial derivatives at the point A exist, need not be continuous at the point A.
The plane determined by both tangent lines, $\tau = t_1\, t_2$ is the tangent plane to the graph $G(f)$ of function $f(x, y)$ at the point $T = [x_0, y_0, f(x_0, y_0)] \in G(f)$.

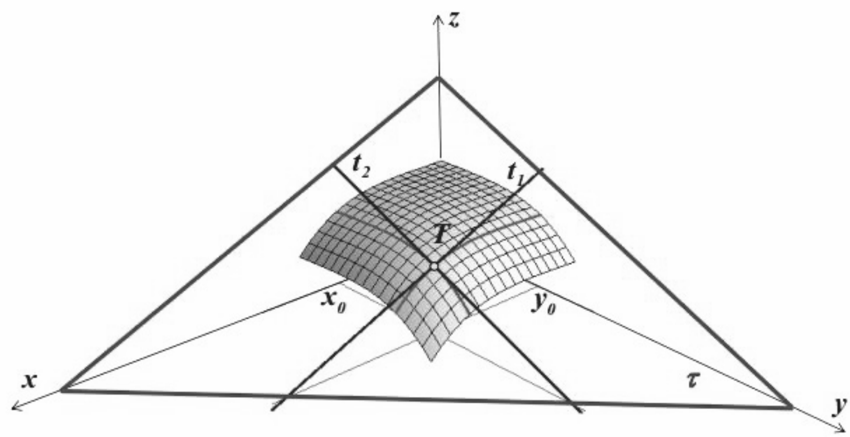


Fig. 2.9. Tangent plane to the graph of function $f(x, y)$ at the point $T = [x_0, y_0, f(x_0, y_0)]$.

Let $A = [x_0, y_0]$ be a limit point of the domain of definition $D(f\,)$ of function $f(x, y)$ and let there exist both partial derivatives of function f at A

$$f'_x(x_0, y_0) = f'_x(A),\ f'_y(x_0, y_0) = f'_y(A)\,.$$

Tangent plane τ to the graph of function f at the tangent point $T = [x_0, y_0, f(x_0, y_0)]$ is determined by the equation

$$z - f(A) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0)\,.$$

The normal vector to the tangent plane τ is represented as

$$\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2 = (-f_x'(x_0, y_0), -f_y'(x_0, y_0), 1)$$

and it is the direction vector of a straight line passing through the point T and perpendicular to the plane τ called the normal to graph $G(f)$ at T.

Examples

1. Function $f(x, y) = x^2 + y^2$, with domain $D(f) = \mathbf{E}^2$, has both partial derivatives at the point $A = [2, 3]$, while $f_x'(2,3) = 4, f_y'(2,3) = 6$, and $f(2, 2) = 13$, and the equation of the tangent plane at the point $T = [2, 3, 13]$ is $4x + 6y - z - 13 = 0$, while its normal vector is $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2 = (4, 6, -1)$, and parametric equations of the normal to the function graph are

   $x = 2 + 4t, y = 3 + 6t, z = 13 - t, t \in \boldsymbol{R}$.

   The graph of function $f(x, y)$, tangent plane and normal to the graph at the tangent point T are illustrated in Fig. 2.10, left.

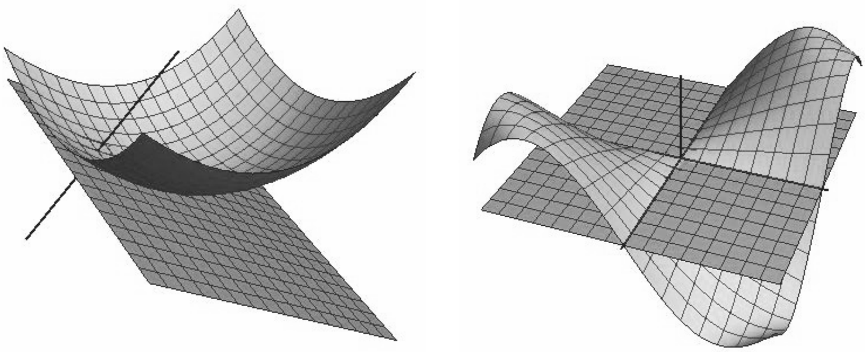


Fig. 2.10. Tangent planes and normals to graphs of functions $f(x, y)$.

2. Let function $f(x, y) = \sin xy$, be defined on the closed region $\langle -\pi/2, \pi/2\rangle \subset \mathbf{E}^2$. The tangent plane to the function graph at the point $O = [0, 0, 0]$ is the coordinate plane xy intersecting the graph of function f in the perpendicular line segments located on coordinate axes x and y and meeting at the origin of the coordinate system, see in Fig. 2.10, right. The tangent plane equation $z = 0$ can be determined from the partial derivatives

   $f_x'(x, y) = y\cos xy, \quad f_y'(x, y) = x\cos xy$,

   whose values at the point [0, 0] are equal to zero. The direction vectors of the tangent plane are the unit vectors of the coordinate axes x and y, $\mathbf{s}_1 = (1,0,0), \mathbf{s}_2 = (0,1,0)$, and the normal vector of the plane is the unit vector $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2 = (0,0,1)$, as the normal to the function graph is coordinate axis z with the parametric equations $x = 0, y = 0, z = t, t \in \boldsymbol{R}$.

3. Function $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$ defined on $\mathbf{E}^2$ has both partial derivatives

$$f_x'(x, y) = -\frac{2}{3} x (x^2 + y^2)^{-\frac{2}{3}}, \quad f_x'(x, y) = -\frac{2}{3} y (x^2 + y^2)^{-\frac{2}{3}}$$

defined on $\boldsymbol{E}^2$ except at the origin O = [0, 0] of the coordinate system. Their values at the point A = [2, 2] are equal, $f_x'(2, 2) = f_y'(2, 2) = -1/3$, and the equation of the tangent plane to the function graph at the point T = [2, 2, –1] can be derived in the form $x + y + 3z - 1 = 0$. The tangent plane intersects the graph of function in the curve with the double point at the tangent point T, see in Fig. 2.11, left, with the implicit equation

$$3\sqrt[3]{x^2 + y^2} - x - y - 2 = 0\,.$$

Vectors $\mathbf{s}_1 = (3, 0, -1), \mathbf{s}_2 = (0, 3, -1)$ can be chosen as the tangent plane direction vectors, while its normal vector is then $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2 = (3, 3, 9)$, or collinear vector $\mathbf{n}_1 = (1, 1, 3)$. The normal to the function graph can be determined parametrically, as line $x = 2 + t, y = 2 + t, z = -1 + 3t, t \in \boldsymbol{R}$.

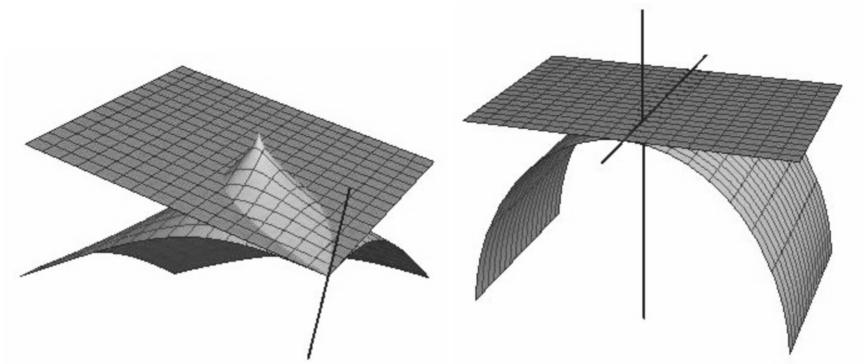


Fig. 2.11. Tangent planes and normals to graphs of functions $f(x, y)$.

4. Equation of the tangent plane to the graph of function $f(x, y) = \sqrt{4 - y^2}$ at the point T = [1, 0, f(1, 0) = 2] is $z = 2$, because both partial derivatives at the point [1, 0] have zero values

$$f_x'(X) = 0, \quad f_y'(X) = -y(4 - y^2)^{-\frac{1}{2}}, \quad f_y'(1, 0) = 0\,.$$

The tangent plane is parallel to the coordinate plane xy, direction vectors are $\mathbf{s}_1 = (1, 0, 0), \mathbf{s}_2 = (0, 1, 0)$, and it is tangent to the function graph in the line parallel to the coordinate axis z given by equations $y = 0$, $z = 2$, as illustrated in Fig. 2.11, right. Normal to the graph of function can be determined parametrically as line $x = 1, y = 0, z = t, t \in \boldsymbol{R}$.

2.4 Total differential of functions of more variables

Let $M \subset \boldsymbol{E}^2$ be set of such points from $D(f)$, at which both partial derivatives of function $f(x, y)$ with respect to x or y exist.
Function determined on the set M, in which any point $A \in M$ is attached the partial derivative of function $f(x, y)$ at the point A with respect to x or y, is said to be the partial derivative of function $f(x, y)$ with respect to x or y

$$f'_x = \frac{\partial f}{\partial x} \qquad\qquad f'_y = \frac{\partial f}{\partial y}$$

$$f'_x : M \to \boldsymbol{R} \qquad\qquad f'_y : M \to \boldsymbol{R}$$

$$f'_x : A \to f'_x(A) \qquad\qquad f'_y : A \to f'_y(A)$$

Suppose that $A = [x_0, y_0]$ is the limit point of the domain $D(f)$ of definition of function $f(x, y)$ and let both partial derivatives $f'_x(A), f'_y(A)$ of function f exist and be continuous at the point A. Then function f is said to be differentiable at the point A, and function increment $\Delta f = f(X) - f(A)$ is expressible in the form

$$f(X) - f(A) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0) + \omega(X)\, d(X, A),$$

where $d(X, A)$ is the distance of points X and A and $\omega(X)$ is the function continuous at A and such that $\omega(A) = 0$.

Remark. It can be simply proved that the differentiability of f at A implies its continuity at A. On the other hand, differentiability of f at A does not follow from continuity. A sufficient condition of the function differentiability at a point A is the existence and the continuity of its function partial derivatives at A.

The total differential of function f at the point A is the expression

$$df_A(x, y) = f'_x(A)(x - x_0) + f'_y(A)(y - y_0).$$

An equation of the tangent plane to the graph of function f can be then rewritten in the form

$$z - f(A) = df_A(X),$$

from which the geometric interpretation of function f total differential at the point A can be derived, as illustrated in Fig. 2.12.

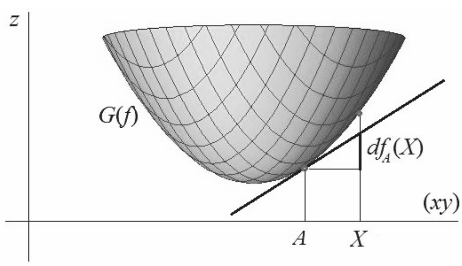


Fig. 2.12. Total differential of function $f(x, y)$.

If: 1. $\lim\limits_{X \to A} d(X, A) = 0$,

2. function f is continuous at the point A,
3. partial derivatives $f_x'(A), f_y'(A)$ exist,

then the condition of the differentiability of function f can be written in the form

$$\Delta f(X) = f(X) - f(A) \doteq df_A(x, y) + \omega(X)\, d(X, A).$$

Omitting the last member of this equality we can obtain an approximation formula often used in numerical mathematics for the estimation of values of function f in the neighbourhood of point A

$$f(X) - f(A) \doteq df_A(X) \Rightarrow f(X) \doteq f(A) + df_A(X).$$

Examples

1. The total differential of function $f(x, y) = xy(x^2 + y^2)^{-\frac{1}{2}}$ at the point $A = [1, -1]$ is determined by the values of partial derivatives $f_x'(A), f_y'(A)$, while

$$f_x'(x, y) = y^3(x^2 + y^2)^{-\frac{3}{2}}, \quad f_y'(x, y) = x^3(x^2 + y^2)^{-\frac{3}{2}},$$
$$df_A(x, y) = -\frac{1}{2\sqrt{2}}(x - 1) + \frac{1}{2\sqrt{2}}(y + 1).$$

The equation of the tangent plane to the function graph at point $T = [1, -1, f(A)]$ illustrated in Fig. 2.13 can be determined as $z - f(A) = df_A(x, y)$ in the form

$$z + \frac{1}{\sqrt{2}} = -\frac{1}{2\sqrt{2}}(x - 1) + \frac{1}{2\sqrt{2}}(y + 1) \Leftrightarrow x - y + 2\sqrt{2}z = 0.$$

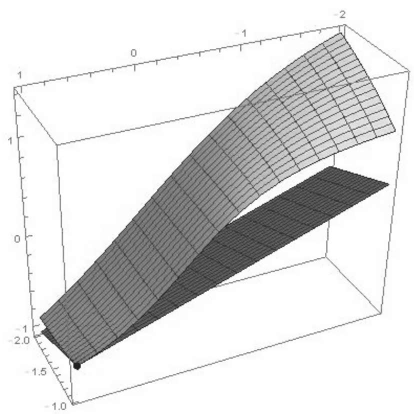


Fig. 2.13. Tangent plane to the graph of function $f(x, y)$.

2. An approximate value of the number $\sqrt{2.1 \cdot 8.05}$ can be estimated, for instance, using the total differential of function $f(x,y)=\sqrt{xy}$ at the point A = [2, 8], which is determined by the partial derivatives of this function

$$f'_x(x,y)=\frac{y}{2\sqrt{xy}}, \quad f'_x(2{,}8)=1, \quad f'_y(x,y)=\frac{x}{2\sqrt{xy}}, \quad f'_y(2{,}8)=\frac{1}{4}$$

and the value of function $f(A) = f(2, 8) = 4$. Then

$$f(X) \doteq f(2{,}8)+f'_x(2{,}8)(x-2)+f'_y(2{,}8)(y-8)$$

$$f(X) \doteq 4+x-2+0.25y-2 = x+0.25y$$

$$f(2.1, 8.5) \doteq 2.1+0.25 \cdot 8.05 = 4.1125$$

while the exact value is 4.11157 .

2.5 Partial derivatives of higher orders

Suppose that a function of two variables $f(x, y)$ defined on a set M has both partial derivatives $f'_x(X)$ and $f'_y(X)$. These functions of two variables can again possess partial derivatives with respect to each of the variables. If such partial derivatives exist we denote them the second partial derivatives or partial derivatives of the second order of the function $f(x, y)$. According to the order of differentiation we obtain four second-order partial derivatives denoted in one from the following ways:

$$f''_{xx}=[f'_x]'_x=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}, \qquad f''_{xy}=[f'_x]'_y=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x \partial y},$$

$$f''_{yx}=[f'_y]'_x=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y \partial x}, \qquad f''_{yy}=[f'_y]'_y=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}.$$

Derivatives $f''_{xy}(X)$, and $f''_{yx}(X)$ are called the mixed second partial derivatives. For most of the functions they are equal. Generally, if they are both continuous, then they are identical.

Function f of n variables, possessing n partial derivatives with respect to all n variables has up to n^2 partial derivatives of the second order. In case $n = 2$ there exist $2^2 = 4$ partial derivatives of the second order, for $n = 3$ there are 9 second-order partial derivatives, etc.

Examples

1. The second partial derivatives of function $f(x,y)=x^2y+x^4y^3$ are the following:

$$f''_{xx}(x,y)=2y+12x^2y^3,$$

$$f''_{xy}(x,y)=f''_{yx}(x,y)=2x+12x^3y^2,$$

$$f''_{yy}(x,y)=6x^4y.$$

2. Function $f(x,y) = x^y$ has partial derivatives $f'_x(x,y) = yx^{y-1}$, $f'_y(x,y) = x^y \ln x$ and its second partial derivatives are

$$f''_{xx}(x,y) = y(y-1)x^{y-2}, \quad f''_{yy}(x,y) = x^y \ln^2 x,$$
$$f''_{xy}(x,y) = f''_{yx}(x,y) = x^{y-1}(1 + y\ln x).$$

3. Function $f(x,y,z) = ze^{xy}$ of three variables has the first partial derivatives

$$f'_x(x,y,z) = yze^{xy}, \quad f'_y(x,y,z) = xze^{xy}, \quad f'_z(x,y,z) = e^{xy},$$

and the following partial derivatives of the second order

$$f''_{xx}(x,y,z) = y^2ze^{xy}, \quad f''_{xy}(x,y,z) = z(1+xy)e^{xy}, \quad f''_{xz}(x,y,z) = ye^{xy},$$
$$f''_{yx}(x,y,z) = z(1+xy)e^{xy}, \quad f''_{yy}(x,y,z) = x^2ze^{xy}, \quad f''_{yz}(x,y,z) = xe^{xy},$$
$$f''_{zx}(x,y,z) = ye^{xy}, \quad f''_{zy}(x,y,z) = xe^{xy}, \quad f''_{zz}(x,y,z) = 0.$$

4. Partial derivatives of the first and second order of function $f(x,y) = \ln xy^{-1}$, defined on the set $D(f) = \{[x,y] \in \boldsymbol{E^2} : xy^{-1} > 0\}$, are functions

$$f'_x(x,y) = x^{-1}, \quad f'_y(x,y) = -y^{-1},$$
$$f''_{xx}(x,y) = -x^{-2}, \quad f''_{yy}(x,y) = x^{-1}y^{-2},$$
$$f''_{xy}(x,y) = f''_{yx}(x,y) = 0,$$

defined on the same domain.

The partial derivatives of the third and higher orders are defined similarly.
If the second partial derivatives of the function $f(x, y)$ have partial derivatives, then these are called the third partial derivatives or partial derivatives of the third order of the function f. The number of the third-order partial derivatives of the function of two variables is $2^2 \times 2 = 2^3 = 8$. If we differentiate function $f(x, y)$ three times with respect to both variables, we receive

$$f'''_{xxx} = [f''_{xx}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}, \qquad f'''_{xxy} = [f''_{xx}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^2 \partial y},$$

$$f'''_{xyx} = [f''_{xy}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}, \qquad f'''_{xyy} = [f''_{xy}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x \partial y^2},$$

$$f'''_{yxx} = [f''_{yx}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y \partial x^2}, \qquad f'''_{yxy} = [f''_{yx}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y \partial x \partial y},$$

$$f'''_{yyx} = [f''_{yy}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}, \qquad f'''_{yyy} = [f''_{y}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}.$$

The partial derivatives of partial derivatives of order $n-1$ are called the n-th partial derivatives or partial derivatives of order n. The number of partial derivatives of function of two variables of order n is $2^{n-1} \times 2 = 2^n$, and they can be represented as follows

$$f^{(n)}_{\underbrace{xx\ldots x}_{n}} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-1}}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial x^n}, \qquad f^{(n)}_{\underbrace{xx\ldots x}_{n-1}y} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-1}}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-1}}\right) = \frac{\partial^n f}{\partial x^{n-1}\partial y},$$

$$f^{(n)}_{\underbrace{xx\ldots x}_{n-2}yx} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-2}y}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y}\right) = \frac{\partial^n f}{\partial x^{n-2}\partial y\partial x},$$

$$f^{(n)}_{\underbrace{xx\ldots x}_{n-2}yy} = [f^{(n-1)}_{\underbrace{xx\ldots x}_{n-2}y}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y}\right) = \frac{\partial^n f}{\partial x^{n-2}\partial y^2},$$

...

$$f^{(n)}_{\underbrace{yy\ldots y}_{n-1}x} = [f^{(n-1)}_{\underbrace{yy\ldots y}_{n-1}}]'_x = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f}{\partial y^{n-1}}\right) = \frac{\partial^n f}{\partial y^{n-1}\partial x}, \quad f^{(n)}_{\underbrace{yy\ldots y}_{n}} = [f^{(n-1)}_{\underbrace{yy\ldots y}_{n-1}}]'_y = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f}{\partial y^{n-1}}\right) = \frac{\partial^n f}{\partial y^n}.$$

Examples

1. Function $f(x,y) = 3x^2y - xy^2$ has non-zero partial derivatives up to order 3, where mixed third-order derivatives are non-zero constants.

   $f'_x(x,y) = 6xy - y^2, \quad f'_y(x,y) = 3x^2 - 2xy,$

   $f''_{xx}(x,y) = 6y, \quad f''_{xy}(x,y) = f''_{yx}(x,y) = 6x - 2y, \quad f''_{yy}(x,y) = -2x,$

   $f'''_{xxx}(x,y) = f'''_{yyy}(x,y) = 0,$

   $f'''_{xxy}(x,y) = f'''_{xyx}(x,y) = f'''_{yxx}(x,y) = 6, \quad f'''_{xyy}(x,y) = f'''_{yyx}(x,y) = f'''_{yxy}(x,y) = -2.$

   All partial derivatives of function f of order 4 and higher orders are equal to zero.

2. Partial derivatives of function $f(x,y) = \ln(x-y)$ are functions

   $f'_x(x,y) = (x-y)^{-1}, \quad f'_y(x,y) = -(x-y)^{-1},$

   $f''_{xx}(x,y) = -(x-y)^{-2}, \quad f''_{xy}(x,y) = f''_{yx}(x,y) = (x-y)^{-2}, \quad f''_{yy}(x,y) = (x-y)^{-2},$

   $f'''_{xxx}(x,y) = f'''_{xyy}(x,y) = f'''_{yxy}(x,y) = f'''_{yyx}(x,y) = 2(x-y)^{-3},$

   $f'''_{xxy}(x,y) = f'''_{xyx}(x,y) = f'''_{yxx}(x,y) = f'''_{yyy}(x,y) = -2(x-y)^{-3},$

   therefore $f^{(n)}_{\underbrace{x..y..x}_{n}} = \pm(n-1)!(x-y)^{-n}$,

   where the sign of a particular derivative depends on the parity of the derivative order and on the number of differentiations with respect to variable y. In the case of an odd order of the derivative its sign is positive if the number of differentiations with respect to variable y is an even number or zero, and it is negative for odd number of differentiations with respect to variable y. For an even order of the derivative, the sign is negative for the even number of differentiations with respect to variable y, and it is positive for their odd number.

Analogously to the properties studied in calculus of a real function of one real variable we can investigate various properties of functions of more variables by means of their partial derivatives of higher orders. This means that we can

- determine all stationary and critical points in the function domain $D(f)$
- identify the existence of points of the local extremes of function, i.e. points at which function reaches its locally minimal or maximal values
- find points at which function reaches global extremes on a closed region
- investigate other special points on the graph of function, e.g. saddle points.

2.6 Local extremes of functions of more variables

Let f be a function of n variables defined on $D(f) \subset \mathbf{E}^n$ and let A be an arbitrary point from its domain of definition. It is said that the value $f(A)$ is the local maximum, or local minimum of function f, if there exists such neighbourhood $N_\varepsilon(A)$ of point A that for each point $X \in N_\varepsilon(A) \cap D(f)$ the following inequality holds:

$$f(X) \geqq f(A), \text{ or } f(X) \leqq f(A).$$

If for all points $X \in N_\varepsilon(A) \cap D(f)$, $X \neq A$ is

$$f(X) > f(A), \text{ or } f(X) < f(A),$$

the function value $f(A)$ is said to be the strict local maximum, or strict local minimum. The point A is called the point of (the strict) local maximum, or (the strict) local minimum.
Function f is said to have local minimum, or local maximum $f(A)$ at the point A, while in case of sharp inequalities we speak about function strict local minimum, or strict local maximum $f(A)$ at the point A. Together, the local minima and the local maxima of a function are called local extremes of a function.

Function f can reach local extremes in the following points only:

1. stationary points, at which all partial derivatives, if they exist, are equal to 0
2. points, at which partial derivatives do not exist.

Let f be a function of two variables. Point $A \in D(f)$ is said to be a critical point of the function, if $f'_x(A)$ and $f'_y(A)$ vanish or they do not exist. It can be proved that function f can possess local extremes only at its critical points.
If $f'_x(A) = 0,$ and $f'_y(A) = 0$ but $f(A)$ is not any local extreme of function f, then the point A is called the saddle point of function f.

The geometric interpretation for the function of two variables is straightforward.
Let $[x_0, y_0] \in D(f)$ be a stationary point of function $f(x, y)$. The total differential of function f equals to zero at the function stationary point. Then the tangent plane to the graph of function f at the point $T = [x_0, y_0, f(x_0, y_0)]$ on the surface $G(f) \subset \mathbf{E}^3$ has the equation $z = f(x_0, y_0)$ and it is parallel to the coordinate plane xy.

An illustration can be seen in Fig. 2.11., right, where function $f(x,y)=\sqrt{4-y^2}$ reaches its local maximum at the point A = [1, 0], at which both partial derivatives vanish. The tangent plane with equation $z = f(A) = 2$ is parallel to the plane xy. On the contrary, function $f(x, y) = \sin xy$ reaches no extreme value at the point A = [0,0], as can be recognised from Fig. 2.10, right, while both partial derivatives vanish at this point, A is the saddle point of the function and tangent plane intersects its graph.

The stationarity of point A is the necessary but not the satisfactory condition for the existence of a local extreme of the function at this point. The function can reach local extremes also in such points, at which it is not differentiable.

Examples

1. Function $f(x,y)=\sqrt{4+x^2-y^2}$ defined on $D(f)=\{[x,y]\in \mathbf{E}^2 : 4+x^2-y^2\geq 0\}$ is not differentiable at the points of a hyperbola with the equation $4 + x^2 - y^2 = 0$, as both partial derivatives, $f'_x(x,y)=\dfrac{x}{\sqrt{4+x^2-y^2}}$, $f'_y(x,y)=\dfrac{-y}{\sqrt{4+x^2-y^2}}$, are not defined at these points. The function reaches its minimal value 0 at these points, and its range is interval $\langle 0, \infty\rangle$. The stationary point A = [0, 0] of this function, at which partial derivatives are vanishing, is the saddle point of function f, tangent plane to the function graph at this point has equation $z = 2$ and it intersects the graph in lines with equations $x = y$, $z = 2$, and $x = -y$, $z = 2$, see in Fig. 2.14, left.

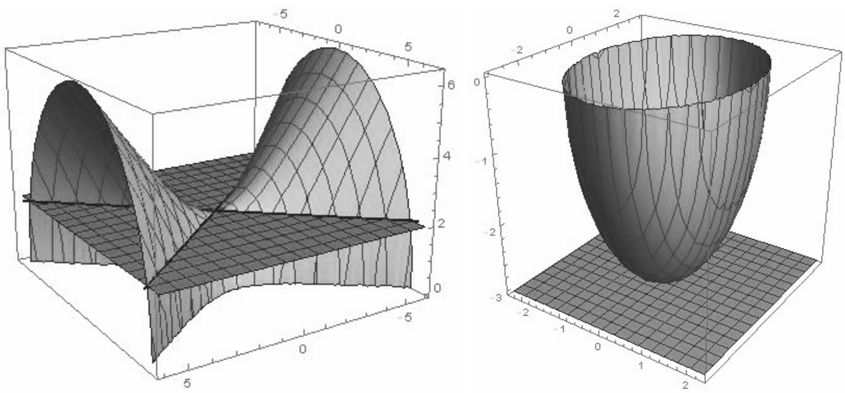


Fig. 2.14. Saddle point and points of local extremes of functions $f(x, y)$.

2. Function $f(x,y)=-\sqrt{9-3x^2-y^2}$ has a minimal value –3 at the stationary point A = [0, 0], and its partial derivatives vanish at this point. Partial derivatives $f'_x(x,y)=\dfrac{3x}{\sqrt{9-3x^2-y^2}}$, $f'_y(x,y)=\dfrac{y}{\sqrt{9-3x^2-y^2}}$ are not defined at the points of an ellipse in the plane xy with the equation $9 - 3x^2 - y^2 = 0$, anyhow, the function reaches its maximum value 0 at this points, and its range is interval $\langle -3, 0\rangle$, as is illustrated in Fig. 2.14, right.

From the above examples it is clear that the simple fact of vanishing partial derivatives $f'_x(A) = f'_y(A) = 0$ does not itself guarantee that the function value $f(A)$ is a local extreme of function. However, if f and its first and second partial derivatives are continuous on some neighbourhood $N_\varepsilon(A)$ of point A, the second derivative test exists, and it may verify the behaviour of the function f at the point A.

Let $A = [x_0, y_0]$ be a stationary point of function $f(x, y)$ of two variables, and let there exist continuous first and second partial derivatives of function f on some neighbourhood $N_\varepsilon(A)$ of point A, while $f''_{xy}(A) = f''_{yx}(A)$. Let

$$D(x_0, y_0) = \begin{vmatrix} f''_{xx}(x_0, y_0) & f''_{xy}(x_0, y_0) \\ f''_{yx}(x_0, y_0) & f''_{yy}(x_0, y_0) \end{vmatrix} = f''_{xx}(x_0, y_0)\, f''_{yy}(x_0, y_0) - \left(f''_{xy}(x_0, y_0)\right)^2 .$$

Then

a) function f has a strict local extreme $f(x_0, y_0)$ at the point A, if $D(x_0, y_0) > 0$, which is

a strict local minimum, if $f''_{xx}(x_0, y_0) > 0$, or $f''_{yy}(x_0, y_0) > 0$

a strict local maximum, if $f''_{xx}(x_0, y_0) < 0$, or $f''_{yy}(x_0, y_0) < 0$,

b) function f does not have a sharp local extreme at the point A, if $D(x_0, y_0) < 0$, but A is a saddle point of the function graph,

c) the test fails if $D(x_0, y_0) = 0$.

Determinant D is called the Hesse determinant (Hessian) of function $f(x, y)$ of two variables at the point A from its domain of definition.

The investigation of local extremes of the function of two variables can be therefore performed in the following steps:

1. Find all stationary points of the function and points, at which the function does not have partial derivatives.
2. Assess all critical points from step 1 and analyse possible existence of function extremes at these points.
   - At stationary points, at which second partial derivatives are continuous, Hesse determinant of the function at this point can be used to decide about the existence of local extremes ($D(A) > 0$) or saddle points ($D(A) < 0$).
   - In the case of a vanishing Hesse determinant ($D(A) = 0$) at the stationary point, the behaviour of the function on the neighbourhood of this point must be investigated by means of definition of local extremes.
   - The existence of local extremes at such points, at which partial derivatives do not exist, can be proved from the definition.

Examples

1. To find all local extremes of function $f(x, y) = x^3 + xy^2 - 27x$ we must determine its both first derivatives and all four second partial derivatives, and find all critical points. This function is defined on $\mathbf{E}^2$, and there are also defined its first partial derivatives $f'_x(x,y) = 3x^2 + y^2 - 27,\ f'_y(x,y) = 2xy,$ which are vanishing at the stationary points whose coordinates satisfy two equations

   $$3x^2 + y^2 - 27 = 0,\ 2xy = 0\,.$$

   From the second equation it follows that either $x = 0$ or $y = 0$, and substituting this condition to the second equation we receive the values of corresponding coordinates, $y = \pm 3\sqrt{3},$ or $x = \pm 3$. Four stationary points exist to be investigated, $A_1 = [0,-3\sqrt{3}], A_2 = [0,+3\sqrt{3}], A_3 = [-3,0], A_4 = [3,0]$. The function second partial derivatives are $f''_{xx}(x,y) = 6x,\ f''_{xy}(x,y) = f''_{yx}(x,y) = 2y,\ f''_{yy}(x,y) = 2x,$ while the Hesse determinant can be determined as function of two variables in the form

   $$D(x,y) = \begin{vmatrix} 6x & 2y \\ 2y & 2x \end{vmatrix} = 12x^2 - 4y^2\,.$$

   Applying the second derivative test we receive:
   $D(0,-3\sqrt{3}) = -108 < 0,\ D(0,3\sqrt{3}) = -108 < 0,$ Hessian value is negative at the points A_1 and A_2, therefore these are saddle points of function, tangent plane with the equation $z = 0$, as $f(0,-3\sqrt{3}) = f(0,3\sqrt{3}) = 0$, intersects graph of function in the line $x = 0$ and the circle $x^2 + y^2 = 27$, which are presented in Fig. 2.15, right.
   $D(-3,0) = 108 > 0,\ f''_{xx}(-3,0) = -18 < 0,$ Hessian value is positive at the point A_3, and there is the local maximum at this point with the value $f(-3, 0) = 54$, Hessian value $D(3,0) = 108 > 0,\ f''_{xx}(3,0) = 18 < 0$ is positive at the point A_4, and there is the local minimum at this point with the value $f(3, 0) = -54$, Fig. 2.15, left.

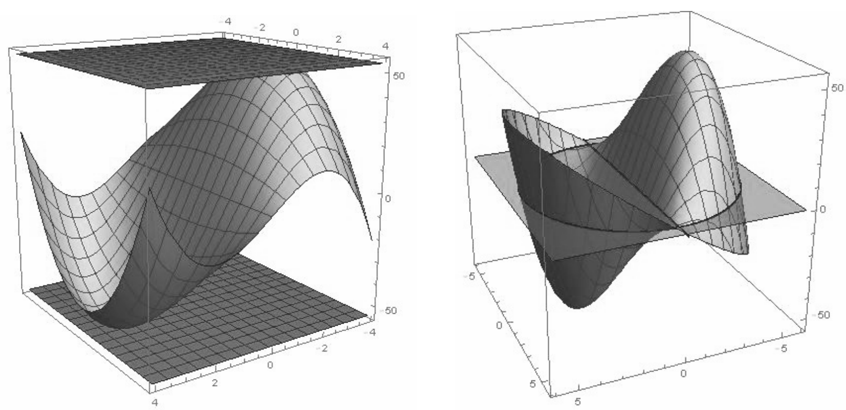


Fig. 2.15. Saddle points and points of local extremes of functions $f(x, y)$.

2.7 Constrained and global extremes of functions of more variables

In problems involving the determination of extremes of functions of two variables we often encounter the so called constrained (conditional) extremes. Let there be given a function $f(x, y)$ and a set (for example a curve) $M \subset D(f)$. The problem is to find a point $A \in M$ such that the value $f(A)$ is the greatest or the least, compared to the values of f at the points of the set M, lying close to the point A. Point A of this kind is called a point of the constrained extreme.

Let f be a function of two variables defined on $D(f) \subset \mathbf{E}^2$ and let a set $M \subset D(f)$. Then a point $A \in M$ is called the point of a constrained local maximum (minimum) if such neighbourhood $N_\varepsilon(A)$ exists that for each $X \in M \cap N_\varepsilon(A)$ it is valid

$$f(X) \le f(A) \qquad (f(X) \ge f(A)).$$

The set M is mostly given as a set of points from $D(f)$ satisfying a condition given by the equation $g(x, y) = 0$,

$$M = \{[x, y] \in D(f): g(x, y) = 0\} \subset D(f).$$

The condition determined by the equation, which is satisfied by coordinates of all points from the function f domain of definition $D(f)$ that are in the set M, is called a constraint. Extremes of function f, attained on the set $M \subset D(f)$ determined by the constraint are called the constrained local extremes of function f .

Point $A = [x_0, y_0]$ is the point of constrained local maximum (minimum) of function f for the constraint $g(x, y) = 0$, if such neighbourhood $N_\varepsilon(A)$ of point A exists, that for all $X \in N_\varepsilon(A)$, whose coordinates satisfy the given constraint it holds that

$$f(X) \le f(A) \qquad (f(X) \ge f(A)).$$

In case of strict inequalities we speak about a strict constrained local maximum or minimum. Constrained local minima and maxima of a function are called constrained local extremes of function.

How to determine all constrained local extremes of function $f(x, y)$?

It is straightforward and easier to solve this problem in the case, when it is possible to express one from the variables x or y as a function of the other variable from the constraint equation $g(x, y) = 0$. If, for example, we can obtain $y = h(x)$ from the constraint, then, substituting this expression for y to the original function f, we obtain a function of one variable $F(x) = f(x, h(x))$. In this way, instead of determining constrained local extremes of the function of two variables $f(x, y)$, we look for local extremes of the function of one variable $F(x)$.

Similarly, if $x = h(y)$, by substitution we receive function $G(y) = f(h(y), y)$ of one variable y and we look for its local extremes.

Examples

1. Constrained local extremes of function $f(x,y) = x^2 + y^2 + 1$, if the constraint is given by the equation $x + y - 1 = 0$, can be determined as local extremes of function of one variable, because the function $y = 1 - x$ can be substituted to the

function $f(x, y)$, thus obtaining function $F(x) = x^2 + (1 - x)^2 + 1 = 2x^2 - 2x + 2$. This function has the derivative $F'(x,y) = 4x - 2$ vanishing at the point $x = 0.5$, while value of the second derivative $F''(x,y) = 4 > 0$ is always positive, and function has local minimum at the stationary point $x = 0.5$, which is $F(0.5) = 1.5$. This is also the value of the constrained local minimum of function $f(x, y)$ at the point [0.5, 0.5], as $f(0.5, 0.5) = 1.5$. Geometric meaning of this problem is to find a point with the minimal z coordinate on the curve, which is the intersection of the function graph $G(f)$ - paraboloid of revolution with axis in the coordinate axis z, and plane passing in the direction of axis z through the line $x + y = 1$ in the coordinate plane xy. It is the vertex of the intersection parabola, in Fig. 2.16, left.

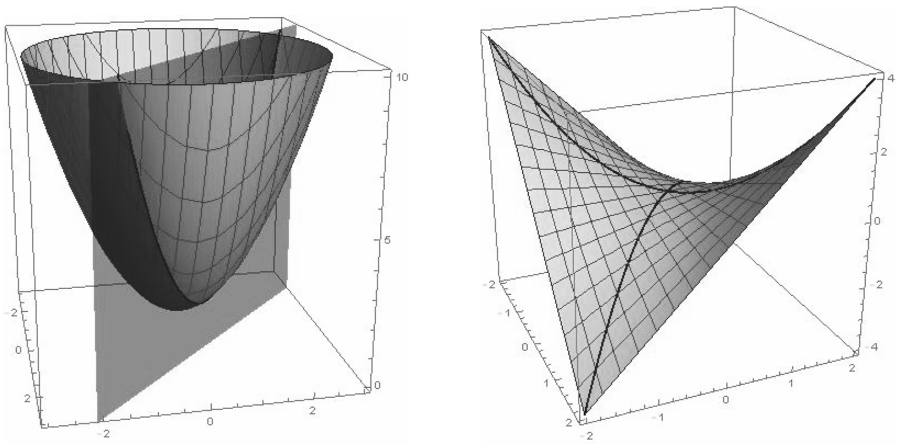


Fig. 2.16. Geometric interpretation of constrained local extremes of functions $f(x, y)$.

2. Function $f(x, y) = xy$ defined on $\mathbf{E}^2$, whose graph $G(f)$ is the hyperbolic paraboloid, has a saddle point $A = [0, 0]$, as it is the stationary point of its partial derivatives $f'_x(x,y) = y, f'_y(x,y) = x$, and from the values of the function second partial derivatives $f''_{xx}(x,y) = f''_{yy}(x,y) = 0, f''_{xy}(x,y) = f''_{yx}(x,y) = 1$ follows the constant value 1 of the Hesse determinant. Constrained extremes of function $f(x, y)$ for constraints $x - y = 0$ and $x + y = 0$ can be found expressing one variable as function of the other one. Function $F(x) = x^2$, received from the first constraint, with derivatives $F'(x) = 2x, F''(x) = 2$, has its local minimum at the point $x = 0$ with value $F(0) = 0$, which is the local constrained minimum of function $f(x, y) = xy$ at [0, 0]. Function $G(y) = -y^2$, derived from the second constraint with derivatives $G'(y) = -2y, G''(y) = -2$, has its local maximum at the point $y = 0$ of value $G(0) = 0$ that is the local constrained maximum of function $f(x, y) = xy$ at the point [0, 0]. Intersection parabolas of graph $G(f)$ and planes passing in direction of coordinate axis z through the lines with equations defining constraints in the coordinate plane xy meet at the point [0, 0, 0], whose z coordinate defines values of both local constrained extremes, and which is the

highest point at one and the lowest point at the other parabola, as can be observed in Fig. 2.16, right.

However, if the constraint $g(x, y) = 0$ is too complicated to express one of the variables in terms of the other, then another method can be used, namely the method of the Lagrange multipliers. To determine the points at which the function f attains constrained local extremes we form an auxiliary function

$$F(x, y) = f(x, y) + \lambda g(x, y),$$

where λ is a suitable constant called the Lagrange multiplier. It is clear that the function $F(x, y)$ is defined on the set M and moreover

$$F(x, y) = f(x, y) \text{ for each } [x, y] \in M.$$

It can be easily proved, that if a point $A = [x_0, y_0] \in M$ is a point of a local extreme of the function $F(x, y)$, then A is a point of a constrained local extreme of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$. The converse of the latter proposition is not valid. It might not be possible to find all constrained local extremes with the aid of this method.

Applying the method of the Lagrange multipliers, we create first the following system of three equations in three unknowns x, y and λ

$$F'_x(x,y) = f'_x(x,y) + \lambda g'_x(x,y) = 0$$
$$F'_y(x,y) = f'_y(x,y) + \lambda g'_y(x,y) = 0$$
$$g(x,y) = 0$$

Solving this system we obtain the critical points of the function $F(x, y)$. Then we must verify (for example by means of the second derivative test) whether these points are points of extremes.

Examples

1. Optimisation problem "At which point of the circle $x^2 + y^2 = 1$ does the sum $x + y$ have the extreme value?" can be solved by determining the constrained local extremes of the function of two variables $f(x, y) = x + y$ with the constraint defined by equation $x^2 + y^2 - 1 = 0$. None of the variables can be represented as function of the other one, so we form function $F(x, y) = x + y + \lambda(x^2 + y^2 - 1)$ with the Lagrange multiplier λ and find the solution of the system of equations

$$F'_x(x,y) = 1 + 2x\lambda = 0$$
$$F'_y(x,y) = 1 + 2y\lambda = 0$$
$$x^2 + y^2 - 1 = 0$$

From the first two equations it follows that $x = y = -\dfrac{1}{2\lambda}$, which after substituting to the third equation gives the values $\lambda_{1,2} = \pm\sqrt{2}/2$. Thus we receive two stationary points of function $F(x, y)$ that have to be investigated, namely

$A_1 = [-\sqrt{2}/2, -\sqrt{2}/2], A_2 = [\sqrt{2}/2, \sqrt{2}/2]$.

The second partial derivatives of function $F(x, y)$ are

$$F''_{xx}(x,y) = F''_{yy}(x,y) = 2\lambda, \quad F''_{xy}(x,y) = F''_{yx}(x,y) = 0,$$

and the value of the Hesse determinant is $D(X) = 4\lambda^2 > 0$, so function F attains local constrained extremes at both points. It is a local constrained minimum at point A_1 determined by the positive value of λ_1 with value $F(A_1) = f(A_1) = -\sqrt{2}$. At the point A_2, determined by the negative value of λ_2, function F attains a local constrained maximum with the value $F(A_2) = f(A_2) = \sqrt{2}$.

This situation can be interpreted geometrically as looking for such points on an ellipse that are extremely located with respect to their distance to the coordinate plane xy. The ellipse is the intersection curve of a plane determined by equation $x + y - z = 0$ and a cylindrical surface of revolution $x^2 + y^2 = 1$. The axis of this cylindrical surface is in the coordinate axis z and its radius equals 1. The values of the constrained extremes are equal to z coordinates of the two extremely located points, as illustrated in Fig. 2.17, left. Then the answer to the optimization problem is the following: "The sum $x + y$ has its maximum value $\sqrt{2}$ at the point A_1 on the circle $x^2 + y^2 = 1$ and it has the minimum value $-\sqrt{2}$ at the point A_2 of the circle."

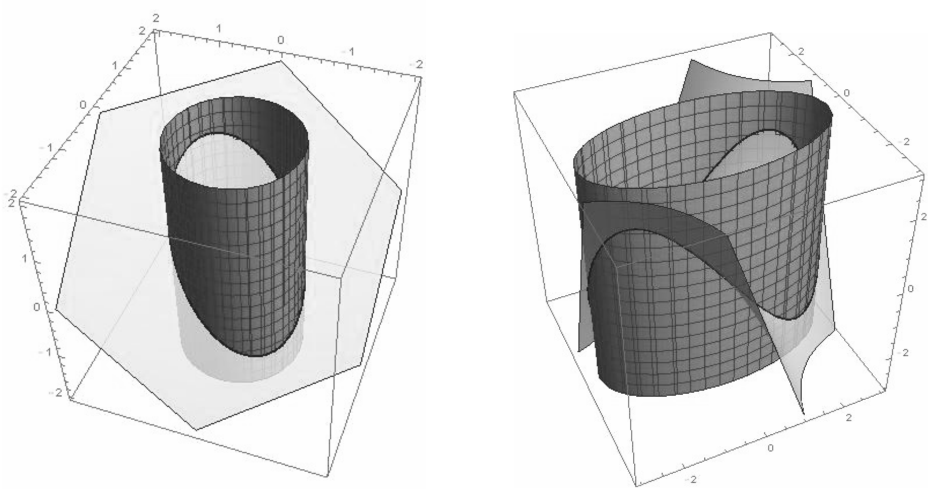


Fig. 2.17. Geometric interpretation of constrained local extremes of functions $f(x, y)$.

2. Finding the greatest and the least values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ we are looking for constrained local extremes that could be investigated by the method of the Lagrange multipliers, as the constraint equation does not allow elimination of any of the variables. Forming the new function

$F(x, y) = xy + \lambda(x^2 + 4y^2 - 8)$ we can define the system of three equations for determination of its stationary points

$$F'_x(x, y) = y + 2x\lambda = 0,$$

$$F'_y(x, y) = x + 8y\lambda = 0,$$

$$x^2 + 4y^2 - 8 = 0.$$

From the first and the second equations follows $\lambda = -\frac{y}{2x} = -\frac{x}{8y}$, from which after some manipulations we receive $y = \pm x$. Substitution into the third equation gives values $y_{1,2} = \pm 1$ and therefore $x_{1,2} = \pm 2$. Thus we receive four stationary points of function $F(x, y)$ that have to be investigated, namely

$$A_1 = [-2,-1], A_2 = [-2,1], A_3 = [2,-1], A_4 = [2,1],$$

while the corresponding values of the respective Lagrange multipliers are

$$\lambda_1 = -1/4, \lambda_2 = 1/4, \lambda_3 = 1/4, \lambda_4 = -1/4.$$

The second partial derivatives of function $F(x, y)$ are

$$F''_{xx}(x, y) = 2\lambda, F''_{yy}(x, y) = 8\lambda, F''_{xy}(x, y) = F''_{yx}(x, y) = 1$$

and the value of the Hesse determinant is $D(X) = 16\lambda^2 - 1$, which gives $D(A_i) = 0$ for all $i = 1, 2, 3, 4$. Thus the method fails and function F must be investigated in the neighbourhoods of all stationary points. It attains equal values at pairs of points, as $F(A_1) = F(A_4) = 2$ and A_1, A_4 are points of local constrained maxima, while $F(A_2) = F(A_3) = -2$, and points A_2, A_3 are points of local constrained minima, see in Fig. 2.17, right. Then, the greatest value that function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ equal to 2 at the points $A_1 = [-2,-1]$, $A_4 = [2, 1]$, while the least value equal to -2 is reached at the points $A_2 = [-2, 1]$, $A_3 = [2, -1]$.

Investigation of the constrained local extremes of function $f(x, y)$ of two variables can be therefore summarized in the following steps:

1. Variable y can be extracted from the constraint $g(x, y) = 0$ and determined as a function of variable x, $y = h(x)$. This function can be substituted into the function $f(x, y)$, while a composite function of one variable x defined on the set M can be obtained, $f(x, h(x)) = F(x)$. All local extremes of function $F(x)$ on the set M are constrained local extremes of function $f(x, y)$ of two variables on the set M.
2. Variable x can be extracted from the constraint $g(x, y) = 0$ and determined as a function of variable y, $x = h(y)$. This function can be substituted into the function $f(x, y)$, while a composite function of one variable y defined on the set M can be obtained, $f(h(y), y) = F(y)$. All local extreme of function $F(y)$ on the set M are constrained local extremes of function $f(x, y)$ of two variables on the set M.
3. In the case that none from the variables x or y can be extracted from the constraint $g(x, y) = 0$ and expressed in terms of the other, the method of the Lagrange multipliers can be used. We define an auxiliary function called Lagrange function

$F(x, y) = f(x, y) + \lambda\, g(x, y),$

where λ is an arbitrary constant called the Lagrange multiplier. Function $F(x, y)$ is defined on set $D(f)$, and moreover, as $g(x, y) = 0$ in the points of set M, it holds that $F(x, y) = f(x, y)$ on M. If any point $A = [x_0, y_0] \in M$ is the point of a local extreme of function $F = f + \lambda g$, then point A is the point of a constrained local extreme of function f for the constraint $g(x, y) = 0$.

Geometric interpretation of constrained local extremes of function f can be derived as z-coordinates of extremely located points on a curve that is the intersection of graph $G(f)$ of function f with the cylindrical surface determined by a curve defined in the plane xy by the respective constraint, while lines on this surface are in the direction of coordinate axis z.

In many, especially in optimization problems, we are interested in the greatest or the least value of a function on a subset of its domain of definition, in other words, in the global extremes of a function on a set. Global extremes of a function of more variables are defined in the following.

Let f be a function of n variables, $n \geq 1$, defined on the set $M \subset D(f)$. Maximum (minimum) of the set $H(f)$, which is the range of function f for all $X \in M$ is called the global, or absolute maximum (minimum) of function f on the set M. Global maxima and minima are referred to as the global extremes of function f on the set M. If M is an open region, function f may not attain any global extremes on this set. If M is a closed bounded set and f is a function continuous on M, then the global extremes on M are attained, and they can be found in the following steps:

1. We find all local extremes of function f inside set M, while it is sufficient to find values at all critical interior points of M.
2. We find all local extremes of function f on the boundary of set M, which are constrained local extremes of function f on the boundary of set M.
3. Global maximum (minimum) of function f on the set M is then the greatest (least) from all found values, local extremes of f inside the set M and constrained local extremes of f on the boundary of the set M.

Examples

1. Let us find all global extremes of function $f(x, y) = 2x^2 + y^2$ on the closed disc in the plane xy, set $M = \{[x, y] \in \mathbf{E}^2\colon x^2 + y^2 \leq 4\}$. Local extremes of function f inside set M can be found at stationary points obtained from vanishing conditions of the first partial derivatives $f'_x(x,y) = 4x,\ f'_y(x,y) = 2y$, while one point $A = [0, 0]$ is determined, which is the point in the set M. The Hesse determinant value can be calculated as $D(x, y) = 8 > 0$, from the constant values of the second partial derivatives $f''_{xx}(x,y) = 4,\ f''_{yy}(x,y) = 2,\ f''_{xy}(x,y) = f''_{yx}(x,y) = 0$, therefore point A is the point of a local minimum, the value is $f(0, 0) = 0$. Constrained local extremes on the boundary of set M, which is the circle $x^2 + y^2 = 4$, can be

investigated introducing the function $F(x, y) = 2x^2 + y^2 + \lambda(x^2 + y^2 - 4)$. From the partial derivatives of this function the system of equations is formed

$$F'_x(x,y) = 4x + 2x\lambda = 0,$$

$$F'_y(x,y) = 2y + 2y\lambda = 0,$$

$$x^2 + y^2 - 4 = 0,$$

and stationary points can be found, as follows.
For $x = 0$, $\lambda = -1$, $y = \pm 2$, and $A_1 = [0, -2]$, $A_2 = [0, 2]$,
for $y = 0$, $\lambda = -2$, $x = \pm 2$, and $A_3 = [-2, 0]$, $A_4 = [2, 0]$.
The second partial derivatives of function $F(x, y)$ are

$$F''_{xx}(x,y) = 4 + 2\lambda, \quad F''_{yy}(x,y) = 2 + 2\lambda, \quad F''_{xy}(x,y) = F''_{yx}(x,y) = 0$$

and the Hesse determinant equals $D(X) = 4\lambda^2 + 12\lambda + 8$, which gives for both values of λ equal values $D(A_i) = 0$ for $i = 1, 2, 3, 4$. Thus the method fails and function F must be investigated in the neighbourhoods of all stationary points. It attains equal values at pairs of points, as $F(A_1) = F(A_2) = 4$, and A_1, A_2 are points of local constrained minima of function $f(x, y)$, while $F(A_3) = F(A_4) = 8$, and points A_3, A_4 are points of local constrained maxima of function $f(x, y)$, see geometrically represented in Fig. 2.18. Here the graph of function $f(x, y)$ is an elliptic paraboloid with axis in the coordinate axis z and vertex at origin, while boundary of the set M, circle $x^2 + y^2 = 4$ determining constraint is represented by a cylindrical surface of revolution with the axis in coordinate axis z and radius 2. The constrained extremes are z coordinates of the extremely located points on the intersection curve of the two surfaces. Comparing values of local minimum and constrained minimum we obtain the final solution of the initially posed problem. Global extremes of function $f(x, y) = 2x^2 + y^2$ on the closed disc M are attained at the following points - global minimum at the point $A = [0, 0]$, value is $f(0, 0) = 0$, global maximum at points $A_3 = [-2, 0]$, $A_4 = [2, 0]$, value is $f(A_3) = f(A_4) = 8$.

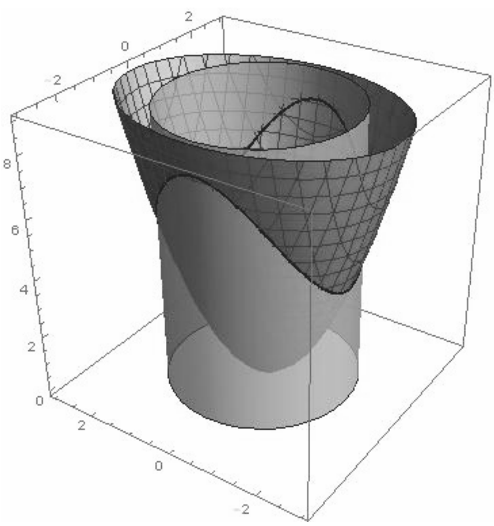


Fig. 2.18. Global extremes of functions $f(x, y)$ on set M.

3 Integral calculus of multivariable functions

3.1 Basic concepts of multiple integration

Integration of functions in several variables is done applying the principles of integration of functions of one real variable. There, for example, we calculated the area under a graph of a continuous non-negative function $f(x)$ defined on an interval in real numbers, $I = \langle a, b\rangle \subset \boldsymbol{R}$, by accumulating the area. First we divided interval I into n not overlapping sub-intervals $\langle x_{i-1}, x_i\rangle$, $i = 1, \ldots, n$, then we chose an arbitrary point $\xi_i \in (x_{i-1}, x_i)$ in each of them, and formed the sum of areas of small rectangles with sides equal to the lengths of sub-intervals $\Delta x_i = x_i - x_{i-1}$ and the value of function $f(\xi_i)$ at the chosen point from the respective interval. Thus we obtained an approximate formula

$$A = \sum_{i=1}^{n} f(\xi_i).\Delta x_i$$

for the calculation of the area of the defined curvilinear trapezoid, see in Fig. 3.1. The denser the division, the more precise results can be obtained. In a limit process for number of sub-intervals n tending to infinity we arrive to the concept of definite integral of function $f(x)$ over an integration domain $I = \langle a, b\rangle$

$$A = \lim_{i\to n} \sum_{i=1}^{n} f(\xi_i).\Delta x_i = \int_a^b f(x)dx\,.$$

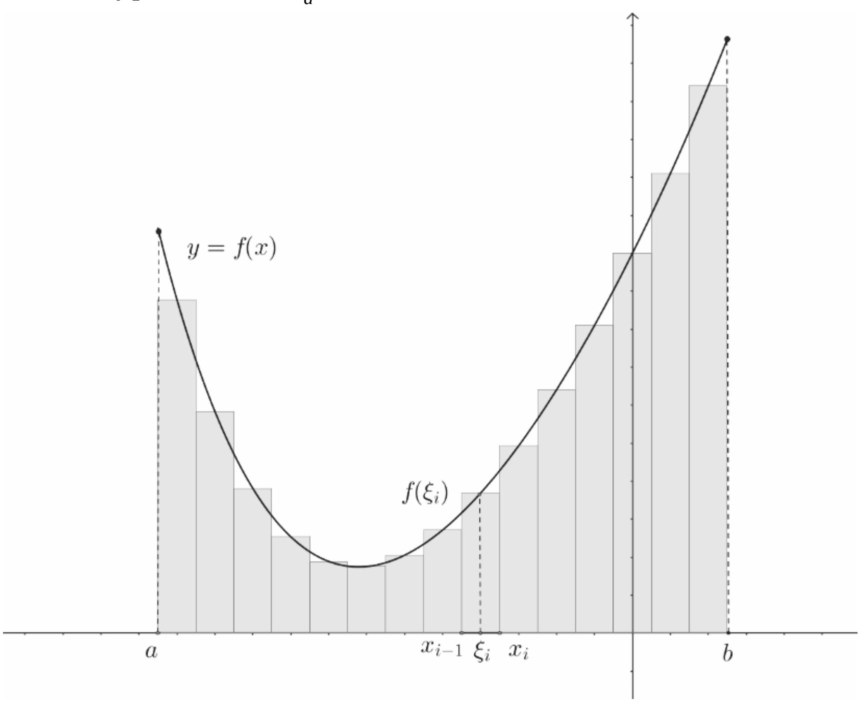


Fig. 3.1. Area of a curvilinear trapezoid.

Consider now the following problem.

Let $D \subset \mathbf{E}^2$ be a bounded region and $f(x, y)$ be a non-negative continuous function of two variables defined on D. We want to find the volume of a curvilinear cylinder determined by D and $f(x, y)$, it means the volume of the solid bounded from below by D lying in the plane xy, by the surface $G(f)$ that is the graph of function $f(x, y)$ from above, and by the corresponding cylindrical surface generated by lines passing through the boundary points of D and parallel to the coordinate axis z. The procedure is analogous to that for computing the area of a curvilinear trapezoid.
First we divide D into n sub-regions $D_1, D_2, \dots, D_n$ not overlapping and such that areas $A(D_1), A(D_2), \dots, A(D_n)$ can be computed. Then we choose an arbitrary point from each sub-region $[\xi_i, \eta_i] \in D_i$, $i = 1, 2, \dots, n$. Finally we form the sum

$$V = \sum_{i=1}^{n} f(\xi_i, \eta_i).A(D_i).$$

This number is equal to the volume of a solid bounded by parts of planes $z = f(\xi_i, \eta_i)$ from above, therefore depending on the division of the region D and the choice of points $[\xi_i, \eta_i] \in D_i$, $i = 1, 2, \dots, n$. It is natural to consider the acquired sum as an approximation of the desired volume of a given curvilinear cylinder, in Fig. 3.2.

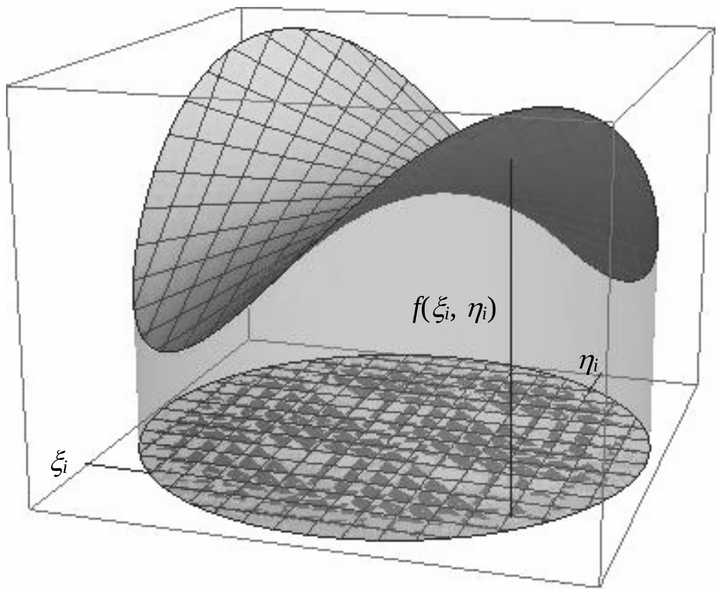


Fig. 3.2. Volume of a curvilinear cylinder.

This idea leads to the concept of double integrals for functions of two variables, over plane regions. We will first discuss a simpler case, with the region D considered as a two-dimensional interval, i. e. a rectangle, and function $f(x, y)$ is not necessarily non-negative.

3.2 Double integrals

Let $I \subset \mathbf{E}^2$ be a two-dimensional interval, which is the Cartesian product of two closed intervals $\langle a, b\rangle$ and $\langle c, d\rangle$, i.e. a rectangular region

$$I = \{[x, y] \in \mathbf{E}^2: a \le x \le b, c \le y \le d\} = \langle a, b\rangle \times \langle c, d\rangle.$$

Let us take an arbitrary division of the interval $\langle a, b\rangle$

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

and an arbitrary division of the interval $\langle c, d\rangle$

$$c = y_0 < y_1 < \ldots < y_{m-1} < y_m = d$$

where n and m are any natural numbers. By means of these two divisions, a division is given of the two-dimensional interval (rectangle) I consisting of $n.m$ two-dimensional sub-intervals (rectangles)

$$I_{ij} = \langle x_{i-1}, x_i\rangle \times \langle y_{j-1}, y_j\rangle,\ i = 1, \ldots , n, j = 1, \ldots , m$$

such that

$$I = \bigcup_{i,j=1}^{n,m} I_{ij} \quad \text{and} \quad A(I) = \sum_{i,j=1}^{n,m} A(I_{ij}).$$

Now let $f(x, y)$ be any function of two variables, defined and bounded on I. In a similar way to above we can compute the sum $\sum_{i,j=1}^{n,m} f(\xi_i, \eta_j).A(I_{ij})$ for any division of I and any choice of points $[\xi_i, \eta_j] \in I_{ij}$, $i = 1, 2, \ldots , n$, $j = 1, \ldots , m$, as the area of the two-dimensional sub-intervals equals $A(I_{ij}) = \Delta x_i \Delta y_j$.
This number is called the integral sum of function $f(x, y)$ over the rectangular region, two-dimensional interval I.

If the limit of the integral sums exists as the area of the greatest two-dimensional sub-interval (rectangle) approaches zero, it is called the double integral of function $f(x, y)$ on (over) region I and denoted by $\iint_I f(x, y)dxdy$. Therefore

$$\iint_I f(x, y)dxdy = \lim_{\max A(I_{ij}) \to 0} \sum_{i,j=1}^{n,m} f(\xi_i, \eta_j).A(I_{ij}).$$

Function $f(x, y)$ is then called integrable on I.

Sufficient condition of integrability
If a bounded function of two variables possesses only a finite number of points of discontinuity on any two-dimensional interval $I \subset \mathbf{E}^2$, then it is integrable on this interval.

Corollary. Every function of two variables, continuous on a two-dimensional interval $I \subset \mathbf{E}^2$ is integrable on I.

Examples

1. Any function defined by the relations: $f(x, y) = 1$ if $x.y$ is rational, and $f(x, y) = 0$ if $x.y$ is irrational, is not integrable on any two-dimensional interval in $\mathbf{E}^2$.
2. Function $f(x, y) = c$, where c is an arbitrary constant, is integrable on any two-dimensional interval and

$$\iint_I f(x, y)\,dxdy = cA(I).$$

Geometric interpretation of double integral of function $f(x, y) \geq 0$ on region I is the volume of solid T in $\mathbf{E}^3$ that is bounded by planes $z = 0$, $x = a$, $x = b$, $y = c$, $y = d$ and by the graph of function f, surface patch $G(f)$ with equation $z = f(x, y)$

$$T = \{[x, y, z] \in \mathbf{E}^3 : [x, y] \in I,\ 0 \leq z \leq f(x, y)\}.$$

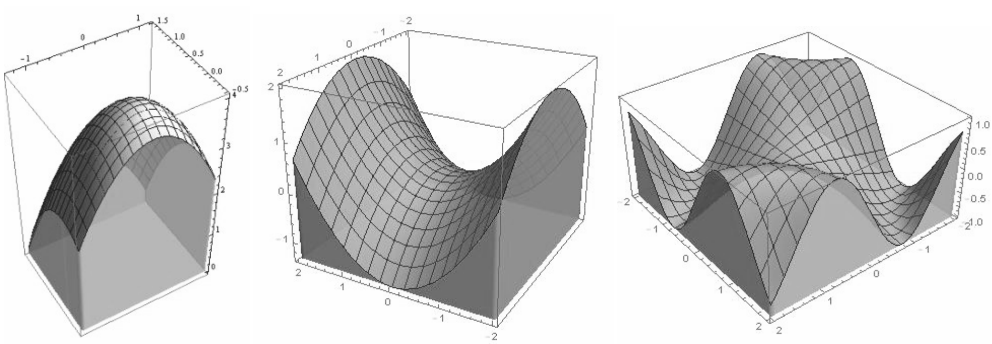


Fig. 3.3. Solids determined by various functions.

Fubini theorem (simple form)

Let function $f(x, y)$ be continuous on a rectangular region $I = \langle a, b\rangle \times \langle c, d\rangle$, then

$$\begin{aligned}
&\iint_I f(x, y)\,dx\,dy = \\
&= \int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left(\int_a^b f(x, y)\,dx \right) dy = \\
&= \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx.
\end{aligned}$$

Two-dimensional intervals are planar regions with measurable areas. Any region, whose area is measurable is called a measurable region. All of the above considerations can therefore be rewritten for double integrals defined on measurable regions. Some basic properties of double integrals on measurable regions are presented in the following.

Properties of double integrals:

1. Linearity: Let functions $f_1, f_2, ..., f_k$ be integrable on a measurable region $M \subset \mathbf{E}^2$ and let $c_1, c_2, ..., c_k$ be real numbers, then
$$\iint_M \big(c_1 f_1(x,y) + c_2 f_2(x,y) + ... + c_k f_k(x,y)\big) dxdy =$$
$$= c_1 \iint_M f_1(x,y)dxdy + c_2 \iint_M f_2(x,y)dxdy + ... + c_k \iint_M f_k(x,y)dxdy$$
2. Additivity: Let function f be integrable on a measurable region $M \subset \mathbf{E}^2$ that is the union of a finite number of measurable regions $M_i \subset \mathbf{E}^2$ with no common interior points, then
$$M = \bigcup_{i=1}^{k} M_i \qquad \iint_M f(x,y)\,dx\,dy = \sum_{i=1}^{k} \iint_{M_i} f(x,y)\,dx\,dy .$$
3. Monotonicity: Let functions f, g be integrable on a measurable region $M \subset \mathbf{E}^2$ and let for all points $X = [x, y] \in M$ hold that $f(x, y) \le g(x, y)$, then
$$\iint_M f(x,y)\,dx\,dy \le \iint_M g(x,y)\,dx\,dy .$$
4. Positivity: Let function f be integrable on a measurable region $M \subset \mathbf{E}^2$ and let $f(x, y) \ge 0$ for all $X = [x, y] \in M$, then
$$\iint_M f(x,y)\,dx\,dy \ge 0 .$$
5. Let function f be integrable on a measurable region $M \subset \mathbf{E}^2$, then function $|f|$ is also integrable on M and
$$\left| \iint_M f(x,y)\,dx\,dy \right| \le \iint_M |f(x,y)|\,dx\,dy .$$

Examples

1. Double integral of function $f(x, y) = x^3 - 2xy + y^2$ over a two-dimensional interval $\langle -1, 1\rangle \times \langle 0, 2\rangle$ can be evaluated in two ways:
$$\int_0^2 \int_{-1}^1 \left(x^3 - 2xy + y^2\right) dx\,dy = \int_0^2 \left(\int_{-1}^1 \left(x^3 - 2xy + y^2\right) dx \right) dy =$$
$$= \int_0^2 \left[\frac{x^4}{4} - x^2 y + xy^2 \right]_{-1}^1 dy = \int_0^2 2y^2 dy = \left[\frac{2y^3}{3} \right]_0^2 = \frac{16}{3},$$
$$\int_0^2 \int_{-1}^1 \left(x^3 - 2xy + y^2\right) dx\,dy = \int_{-1}^1 \left(\int_0^2 \left(x^3 - 2xy + y^2\right) dy \right) dx =$$
$$= \int_{-1}^1 \left[x^3 y - xy^2 + \frac{y^3}{3} \right]_0^2 dx = \int_{-1}^1 \left(2x^3 - 4x + \frac{8}{3} \right) dx = \left[\frac{x^4}{2} - 2x^2 + \frac{8x}{3} \right]_{-1}^1 = \frac{16}{3}.$$

2. Double integral of function $f(x, y) = \sin(x - y)$ over a two-dimensional interval $\langle -\pi, \pi \rangle \times \langle 0, \pi \rangle$ can be evaluated as follows:

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} \sin(x-y)dx\, dy = \int_{-\pi}^{\pi}\left(\int_{0}^{\pi} \sin(x-y)\, dx\right) dy = \int_{-\pi}^{\pi}\left[-\cos(x-y)\right]_0^{\pi} dy =$$

$$= \int_{-\pi}^{\pi}\left(\cos y - \cos(\pi - y)\right) dy = \left[\sin y + \sin(\pi - y)\right]_{-\pi}^{\pi} = 0, \text{ or}$$

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} \sin(x-y)dx\, dy = \int_{0}^{\pi}\left(\int_{-\pi}^{\pi} \sin(x-y)dy\right) dx = \int_{0}^{\pi}\left[\cos(x-y)\right]_{-\pi}^{\pi} dx =$$

$$= \int_{0}^{\pi}\left(\cos(x-\pi) - \cos(x+\pi)\right)dx = \left[\sin(x-\pi) - \sin(x+\pi)\right]_0^{\pi} = 0.$$

The plane region

$$M_{xy} = \{[x, y] \in \mathbf{E}^2 \colon a \le x \le b,\ g(x) \le y \le h(x)\},$$

where $a, b \in R$, $a < b$, g and h are continuous functions defined on interval $\langle a, b \rangle$, while for all $x \in \langle a, b \rangle$ it holds that $g(x) \le h(x)$, is called regular region of type *xy*. Various forms of regular regions of type *xy* are illustrated in Fig. 3.4.

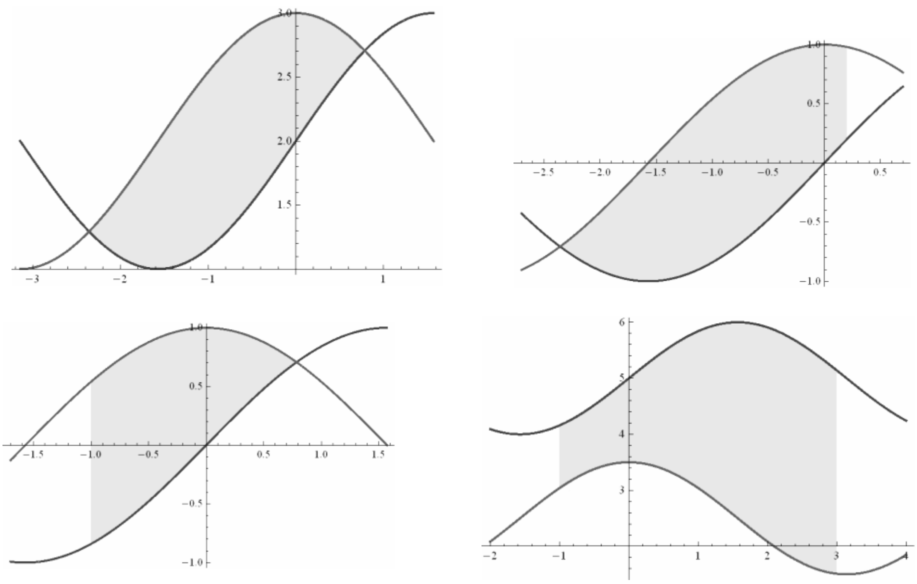


Fig. 3.4. Regular regions of type *xy*.

Similarly, various forms of regular regions M_{yx} of type *yx* can be determined, while the description of a regular region M_{yx} can be derived analogously to the description of the regular region M_{xy}, simply by exchanging variables. Therefore the independent variable x becomes a dependent variable and vice-versa, so that variable y is considered as an independently changing variable.

The plane region

$$M_{yx} = \{[x, y] \in \mathbf{E}^2: c \le y \le d,\, g(y) \le x \le h(y)\},$$

where $c, d \in R$, $c < d$, and functions g and h are continuous on interval $\langle c, d\rangle$, while for all $y \in \langle c, d\rangle$ it holds that $g(y) \le h(y)$, is called a regular region of type yx.

Examples

1. Set $M = \{[x, y] \in \mathbf{E}^2 : 1 \le x \le y \le 2\}$ can be described in both ways, as a regular region of type xy, $M_{xy} = \{[x, y] \in \mathbf{E}^2 : 1 \le x \le 2,\ x \le y \le 2\}$, or as a regular region of type yx, $M_{yx} = \{[x, y] \in \mathbf{E}^2 : 1 \le y \le 2,\ 1 \le x \le y\}$.
2. Region $M = \{[x, y] \in \mathbf{E}^2 : x^2 + y^2 \ge 1 \wedge x^2 + 4y^2 \le 4\}$ can be described as a union of regular regions sketched in Fig. 3.5, $M = {}^1M_{yx} \cup {}^2M_{yx}$, where
   ${}^1M_{yx} = \{[x, y] \in \mathbf{E}^2 : -1 \le y \le 1,\ -\sqrt{4-4y^2} \le x \le -\sqrt{1-y^2}\}$,
   ${}^2M_{yx} = \{[x, y] \in \mathbf{E}^2 : -1 \le y \le 1,\ \sqrt{1-y^2} \le x \le \sqrt{4-4y^2}\}$.

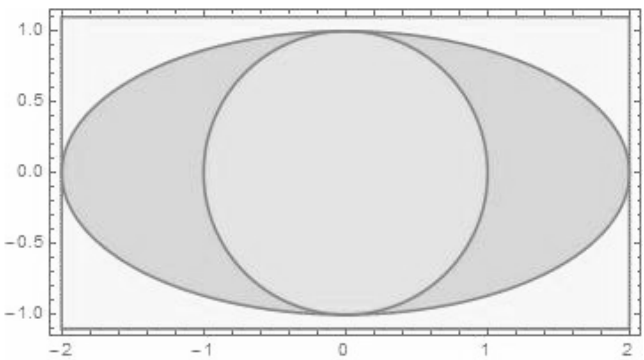


Fig. 3.5. Regular region M.

The double integral of function $f(x, y)$ of two variables on a regular region M_x or M_y can be defined similarly to a double integral on the double interval $I = \langle a, b\rangle \times \langle c, d\rangle$.

Let function f be integrable on a regular region M, then the double integral of f over M exists and it is denoted

$$\iint_M f(x, y)\,dxdy\,.$$

Analogous properties to those stated for integrals over measurable regions (double intervals) are valid also for double integrals on regular regions M_x or M_y, or on any union of a finite number of regular regions. Areas of these measurable regions can then be evaluated using these properties as double integrals of function $f(x, y) = 1$ over the respective regions.

These properties are:

0. Sufficient condition of integrability
1. Linearity
2. Additivity
3. Monotonicity
4. Positivity
5. $\left|\iint\limits_M f(x,y)\,dx\,dy\right| \le \iint\limits_M |f(x,y)|\,dx\,dy$

For the evaluation of double integrals on regular regions, a strong form of the Fubini theorem can be used.

Fubini theorem (strong form)
Let function $f(x, y)$ be continuous on a regular region of type *x*,

$$M_x = \{[x, y] \in \mathbf{E}^2: a \le x \le b, g(x) \le y \le h(x)\},$$

then

$$\iint\limits_M f(x,y)dxdy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx \;.$$

For function $f(x, y)$ continuous on regular region of type *y*

$$M_y = \{[x, y] \in \mathbf{E}^2: g(y) \le x \le h(y), c \le y \le d\}$$

it holds that

$$\iint\limits_M f(x,y)dxdy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy \,.$$

Examples

1. The double integral of function $f(x,y) = \dfrac{x}{y}$ over set $M = \{[x,y] \in \mathbf{E}^2 : 1 \le x \le y \le 2\}$

   can be evaluated as a double integral over a regular region of type *xy*,

$$\iint\limits_M \frac{x}{y}\,dx\,dy = \int_1^2 \left(\int_x^2 \frac{x}{y}\,dy \right) dx = \int_1^2 \left[x \ln|y| \right]_x^2 dx = \int_1^2 \left(x\ln 2 - x\ln|x| \right) dx =$$

$$= \left[\frac{x^2}{2} \ln 2 \right]_1^2 - \left(\left[\frac{x^2}{2} \ln|x| \right]_1^2 - \int_1^2 \frac{x}{2}\,dx \right) = 2\ln 2 - \frac{1}{2}\ln 2 - 2\ln 2 + \left[\frac{x^2}{4} \right]_1^2 = \frac{3}{4} - \ln\sqrt{2}.$$

2. The double integral of function $f(x,y) = e^{y^2}$ over $M = \{[x,y] \in \mathbf{E}^2 : 0 \le x \le y \le 1\}$ can be evaluated as a double integral over a regular region of type *yx* only, $M_{yx} = \{[x,y] \in \mathbf{E}^2 : 0 \le y \le 1, 0 \le x \le y\}$, it cannot be evaluated over region M_{xy}

$$\iint\limits_M e^{y^2}\,dx\,dy = \int_0^1 \left(\int_0^y e^{y^2}\,dx \right) dy = \int_0^1 \left[x e^{y^2} \right]_0^y dy = \int_0^1 y e^{y^2}\,dy = \frac{1}{2}\left[e^{y^2} \right]_0^1 = \frac{e-1}{2}.$$

3. The double integral of function $f(x,y) = x^2 + y$ over the set bounded by two parabolas with equations $y = x^2$ and $x = y^2$ can be evaluated as an integral over a regular region of type *xy*, see in Fig. 3.6.

$$M_{xy} = \{[x,y] \in \mathbf{E}^2 : 0 \le x \le 1, x^2 \le y \le \sqrt{x}\},$$

$$\iint\limits_{M_{xy}} (x^2 + y)\,dx\,dy = \int_0^1 \left(\int_{x^2}^{\sqrt{x}} (x^2 + y)\,dy \right) dx = \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx =$$

$$= \int_0^1 \left(\sqrt{x^5} + \frac{x}{2} - \frac{3x^4}{2} \right) dx = \left[\frac{2\sqrt{x^7}}{7} + \frac{x^2}{4} - \frac{3x^5}{10} \right]_0^1 = \frac{33}{140}.$$

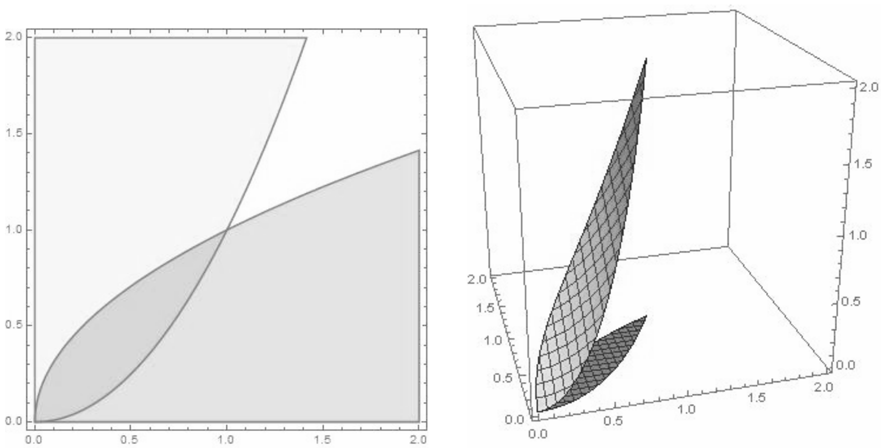


Fig. 3.6. Regular region and graph of function of two variables.

4. The double integral of function $f(x,y) = e^{x+y}$ over unit disc with centre in the origin, $M_{yx} = \left\{ [x,y] \in \mathbf{E}^2 : -1 \le y \le 1, -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \right\}$,

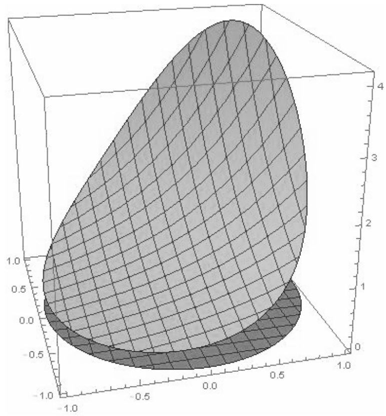
cannot be evaluated analytically. Analogously, the double integral of function f over region bounded by two parabolas with equations $y = x^2 - 1$ and $y = -x^2$,

$$M_{xy} = \{[x,y] \in \mathbf{E}^2 : -\frac{\sqrt{2}}{2} \le x \le \frac{\sqrt{2}}{2}, x^2 - 1 \le y \le -x^2\},$$

but these double integrals can be evaluated numerically, e.g. using the symbolic algebra system Mathematica we can receive the following approximated results

$$\iint\limits_{M_{yx}} e^{x+y} dxdy \cong 3{,}99524, \quad \iint\limits_{M_{xy}} e^{x+y} dxdy \cong 0{,}371421.$$

Both regular regions with graphs of function $f(x,y) = e^{x+y}$ over them are presented in Fig. 3.7.



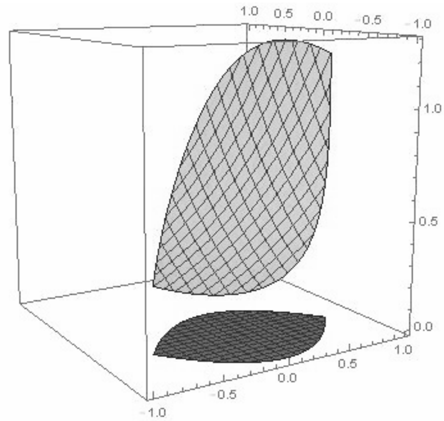
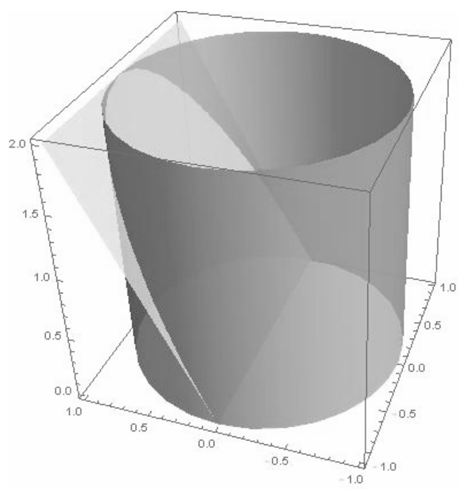


Fig. 3.7. Regular region and graphs of function of two variables.

5. Archimedes (287 BC - 212 BC) discovered relation between volumes of basic geometric solids (Fig. 3.8), which can be verified by the evaluation of volume by means of double integrals.
   A1. Volume of a parabolic section of a cylinder of revolution equals to one-sixths of the volume of a prism circumscribed to this cylinder.

$$\iint_{M_{xy}} 2y\,dx\,dy = \int_{-1}^{1}\left(\int_{0}^{\sqrt{1-x^2}} 2y\,dy\right)dx = \int_{-1}^{1}\left[y^2\right]_0^{\sqrt{1-x^2}}dx = \int_{-1}^{1}\left(1-x^2\right)dx = \left[x-\frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3},$$

$$\iint_{I} 2\,dx\,dy = \int_{0}^{2}\left(\int_{-1}^{1} 2\,dx\right)dy = \int_{0}^{2}\left[2x\right]_{-1}^{1}dy = \int_{0}^{2} 4\,dy = \left[4y\right]_0^2 = 8\,.$$



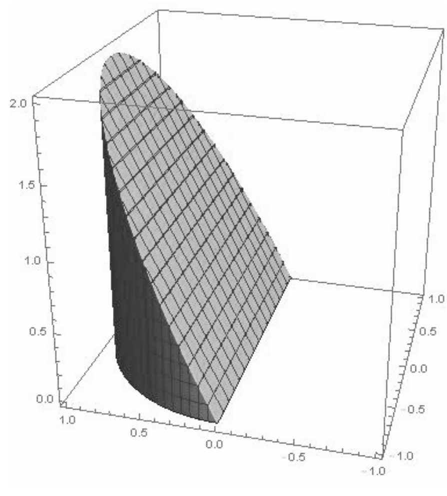


Fig. 3.8. Archimedean problem.

3.3 Triple integrals

Consider the following problem. Let a physical body occupy a bounded region D in space, $D \subset \mathbf{E}^3$, and let this body be non-homogeneous, meaning its mass density varies depending on the position of a point in the body. We can assume that the density is represented by a non-negative continuous function of three variables $\sigma(x, y, z)$ defined on D. We want to find the total mass of this non-homogeneous solid. The procedure for the estimation of this mass is analogous to that for computing the volume of a curvilinear cylinder. We divide region D into n sub-regions $D_1, D_2, \dots,$ D_n, not overlapping and such, that volumes $V(D_1), V(D_2), \dots, V(D_n)$ can be computed. Then we choose an arbitrary point from each sub-region $[\xi_i, \eta_i, \tau_i] \in D_i$, $i = 1, 2, \dots, n$ and form the sum

$$m = \sum_{i=1}^{n} \sigma(\xi_i, \eta_i, \tau_i) \,.V(D_i)\,.$$

This number is equal to the mass of a by parts homogeneous solid, depending on the division of the region D and the choice of points $[\xi_i, \eta_i, \tau_i]$. It is natural to consider this sum as an approximation of the desired mass of the given physical body. This idea leads to the concept of triple integrals for functions of three variables over space regions. In what follows we will discuss a simpler case, where the region D is a three dimensional interval, i. e. a rectangular parallelepiped, and the function $f(x, y, z)$ is not necessarily non-negative.

Let $I \subset \mathbf{E}^3$ be a three-dimensional interval, which is the Cartesian product of three closed intervals $\langle a, b\rangle$, $\langle c, d\rangle$ and $\langle e, h\rangle$, i.e. a prismatic region

$$I = \{[x, y, z] \in \mathbf{E}^3 \colon a \le x \le b,\ c \le y \le d,\ e \le z \le h\ \} = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle.$$

Let us take an arbitrary division of the interval $\langle a, b\rangle$

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

an arbitrary division of the interval $\langle c, d\rangle$

$$c = y_0 < y_1 < \dots < y_{m-1} < y_m = d$$

and an arbitrary division of the interval $\langle e, h\rangle$

$$e = z_0 < z_1 < \dots < z_{p-1} < z_p = h$$

where n, m and p are any natural numbers. These three divisions specify a division of the three-dimensional interval (prism) I consisting of $n.m.p$ three-dimensional sub-intervals (prisms)

$$I_{ijk} = \langle\, x_{i-1}, x_i\rangle \times \langle\, y_{j-1}, y_j\rangle \times \langle\, z_{k-1}, y_k\rangle,\ i = 1, \dots, n,\ j = 1, \dots, m,\ k = 1, \dots, p$$

such that $I = \bigcup\limits_{i,j,k=1}^{n,m,p} I_{ijk}$ and $V(I) = \sum\limits_{i,j,k=1}^{n,m,p} V(I_{ijk})\,.$

Now let $f(x, y, z)$ be any function of three variables, defined and bounded on I. In a similar way to above we can compute the sum $\sum_{i,j,k=1}^{n,m,p} f(\xi_i, \eta_j, \tau_k).V(I_{ijk})$ for an arbitrary division of the three-dimensional interval I and an arbitrary choice of points $[\xi_i, \eta_i, \tau_i] \in I_{ijk}$, $i = 1, 2, \dots, n$, $j = 1, \dots, m$, $k = 1, \dots, p$, while the volume of the three-dimensional sub-intervals equals $V(I_{ijk}) = \Delta x_i \, \Delta y_j \, \Delta z_k$.
This number is called the integral sum of function $f(x, y, z)$ over the rectangular region, namely the three-dimensional interval I.

If the limit of the integral sums exists, as the volume of the greatest three-dimensional sub-interval (prism) approaches zero, it is called the triple integral of function $f(x, y, z)$ on (over) region I and it is denoted by $\iiint\limits_I f(x, y, z)\, dx\, dy\, dz$. Therefore

$$\iiint\limits_I f(x, y, z)\, dx\, dy\, dz = \lim_{\max V(I_{ijk}) \to 0} \sum_{i,j,k=1}^{n,m,p} f(\xi_i, \eta_j, \tau_k).V(I_{ijk}) \, .$$

Function $f(x, y, z)$ is then called integrable on I.

Sufficient condition of integrability
If a bounded function of three variables possesses only a finite number of points of discontinuity on a three-dimensional interval $I \subset \mathbf{E}^3$, then it is integrable on this interval.

Corollary. Every function of three variables continuous on a three-dimensional interval $I \subset \mathbf{E}^3$ is integrable on I.

Example

1. Function $f(x, y, z) = c$, where c is an arbitrary constant, is integrable on any three dimensional interval I and $\iiint\limits_I f(x, y, z)\, dx\, dy\, dz = c.V(I)$.

The physical interpretation of a triple integral of function $\sigma(x, y, z) \geq 0$ on region I is the mass of a non-homogenenous prism I in $\mathbf{E}^3$ that is bounded by planes $x = a$, $x = b$, $y = c$, $y = d$, $z = e$, $z = h$, while its density varies according to a non-negative continuous function of three variables $\sigma(x, y, z)$ defined on I.

Fubini theorem (simple form)
If function $f(x, y, z)$ is continuous on rectangular region $I = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle$, then

$$\iiint\limits_I f(x, y, z)\, dx\, dy\, dz = \int_e^h \left(\int_c^d \left(\int_a^b f(x, y, z)\, dx \right) dy \right) dz \, .$$

In computing triple integrals over three-dimensional intervals by means of this theorem we can interchange the order of integrals appearing on the right hand side of the above equality. In fact, we have 6 possibilities how to rearrange them.

Three-dimensional intervals are space regions with measurable volumes. Any space region, whose volume is measurable is called a measurable region. All above considerations can be therefore rewritten for triple integrals defined on measurable regions. Some properties of triple integrals on measurable regions are given in the following.

Properties of triple integrals:

1. Linearity: Let functions $f_1, f_2, \ldots, f_k$ be integrable on a measurable region $M \subset \mathbf{E}^3$ and $c_1, c_2, \ldots, c_k$ are real numbers, then

$$\iiint_M \left(c_1 f_1(x,y,z) + c_2 f_2(x,y,z) + \ldots + c_k f_k(x,y,z)\right) dx\, dy\, dz =$$
$$= c_1 \iiint_M f_1(x,y,z)\, dx\, dy\, dz + c_2 \iiint_M f_2(x,y,z)\, dx\, dy\, dz + \ldots + c_k \iiint_M f_k(x,y,z)\, dx\, dy\, dz.$$

2. Additivity: Let function f be integrable on a measurable region $M \subset \mathbf{E}^3$ that is the union of a finite number of measurable regions $M_i \subset \mathbf{E}^3$ with no common interior points, then

$$M = \bigcup_{i=1}^{k} M_i \qquad \iiint_M f(x,y,z)\, dx\, dy\, dz = \sum_{i=1}^{k} \iiint_{M_i} f(x,y,z)\, dx\, dy\, dz\,.$$

3. Monotonicity: Let functions f, g be integrable on a measurable region $M \subset \mathbf{E}^3$ and let it hold for all points $X = [x, y, z] \in M$ that $f(x, y, z) \leq g(x, y, z)$, then

$$\iiint_M f(x,y,z)\, dx\, dy\, dz \leq \iiint_M g(x,y,z)\, dx\, dy\, dz\,.$$

4. Positivity: Let function f be integrable on a measurable region $M \subset \mathbf{E}^3$ and let the inequality $f(x, y, z) \geq 0$ hold for all $X = [x, y, z] \in M$, then

$$\iiint_M f(x,y,z)\, dx\, dy\, dz \geq 0\,.$$

5. Let function f be integrable on a measurable region $M \subset \mathbf{E}^3$, then function $|f|$ is also integrable on M and

$$\left|\iiint_M f(x,y,z)\, dx\, dy\, dz\right| \leq \iiint_M |f(x,y,z)|\, dx\, dy\, dz\,.$$

Using the above properties, triple integrals over measurable regions can be evaluated, namely triple integrals defined over three-dimensional intervals, prismatic regions $I = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle \subset \mathbf{E}^3$ and their unions. It is easy to show that for function $f(x, y, z) = r$, where r is an arbitrary real constant, the value of a triple integral of function f over region I is equal to the r-multiple of the region volume $V(I)$,

$$\iiint_I r\, dx\, dy\, dz = rV(I) = r(b-a)(d-c)(h-e)\,.$$

Examples

1. The volume of a prism $I = \langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle$ can be evaluated as triple integral

$$\iiint_I 1\,dx\,dy\,dz = \int_e^h \left(\int_c^d \left(\int_a^b dx \right) dy \right) dz = \int_e^h \left(\int_c^d [x]_a^b\,dy \right) dz = \int_e^h \left(\int_c^d (b-a)\,dy \right) dz =$$

$$= (b-a)\int_e^h [y]_c^d\,dz = (b-a)(d-c)\int_e^h dz = (b-a)(d-c)[z]_e^h = (b-a)(d-c)(h-e).$$

2. The triple integral of function $f(x, y, z) = 2(x + z)$ over $I = \langle 0, 1\rangle^3$ equals:

$$\iiint_I 2(x+z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^1 \left(\int_0^1 2(x+z)\,dx \right) dy \right) dz = \int_0^1 \left(\int_0^1 \left[(x^2 + 2zx)\right]_0^1 dy \right) dz =$$

$$= \int_0^1 \left(\int_0^1 (1+2z)\,dy \right) dz = \int_0^1 [y + 2yz]_0^1\,dz = \int_0^1 (1+2z)\,dz = \left[z + z^2\right]_0^1 = 2\,.$$

3. The triple integral of function $f(x, y, z) = x^3 + xy + yz$ over the three-dimensional interval $\langle 0, 1\rangle \times \langle 0, 2\rangle \times \langle 0, 3\rangle$ can be evaluated for example as follows:

$$\iiint_I (x^3 + xy + yz)\,dx\,dy\,dz = \int_0^3 \left(\int_0^2 \left(\int_0^1 (x^3 + xy + yz)\,dx \right) dy \right) dz =$$

$$= \int_0^3 \left(\int_0^2 \left[\frac{x^4}{4} + \frac{x^2 y}{2} + xyz \right]_0^1 dy \right) dz = \int_0^3 \left(\int_0^2 \left(\frac{1}{4} + \frac{y}{2} + yz \right) dy \right) dz =$$

$$= \int_0^3 \left[\frac{y}{4} + \frac{y^2}{4} + \frac{y^2 z}{2} \right]_0^2 dz = \int_0^3 \left(\frac{3}{2} + 2z \right) dz = \left[\frac{3z}{2} + z^2 \right]_0^3 = \frac{27}{2}\,.$$

4. Triple integral of function $f(x, y, z) = \sin(x + y + z)$ over three-dimensional interval $I = \langle 0, \pi\rangle^3$ equals:

$$\iiint_I \sin(x+y+z)\,dx\,dy\,dz = \int_0^\pi \left(\int_0^\pi \left(\int_0^\pi \sin(x+y+z)\,dx \right) dy \right) dz =$$

$$= \int_0^\pi \left(\int_0^\pi [-\cos(x+y+z)]_0^\pi\,dy \right) dz = \int_0^\pi \left(\int_0^\pi 2\cos(y+z)\,dy \right) dz =$$

$$= \int_0^\pi [2\sin(y+z)]_0^\pi\,dz = 4\int_0^\pi -\sin(z)\,dz = 4[\cos z]_0^\pi = -8\,.$$

In $\mathbf{E}^3$ we can distinguish six types of regular regions: *xyz*, *yxz*, *xzy*, *zxy*, *yzx*, *zyx*. These represent various combinations of variables for functions of two variables and functions of one variable determining the boundaries of a generalised solid T in $\mathbf{E}^3$ that is the domain of definition of function $f(x, y, z)$ integrated over this region.

For example, the sets

$$R_{xyz} = \{[x, y, z] \in \mathbf{E}^3\colon [x, y] \in M,\ z_1(x, y) \le z \le z_2(x, y)\}$$

or

$$R'_{xyz} = \{[x, y, z] \in \mathbf{E}^3\colon [x, y] \in M',\ z_1(x, y) \le z \le z_2(x, y)\}$$

where $z_1(x, y) \le z_2(x, y)$ are continuous and bounded functions on regular region

$$M = \{[x, y] \in \mathbf{E}^2\colon a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

or

$$M' = \{[x, y] \in \mathbf{E}^2\colon c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

are regular regions in $\mathbf{E}^3$ with respect to the plane xy, regular regions of type xyz

$$R_{xyz} = \{[x, y, z] \in \mathbf{E}^3 : a \le x \le b, g_1(x) \le y \le g_2(x), z_1(x, y) \le z \le z_2(x, y)\}$$

or of type yxz

$$R'_{xyz} = \{[x, y, z] \in \mathbf{E}^3 : c \le y \le d, h_1(y) \le x \le h_2(y), z_1(x, y) \le z \le z_2(x, y)\}.$$

Examples

1. Set $R = \{[x, y, z] \subset \mathbf{E}^3,\ x^2 + y^2 \le z^2 \le 4,\ z \ge 0\}$ can be described as a regular region of the type xyz, see in Fig. 3.9, left.

$$R_{xyz} = \left\{[x, y, z] \in \mathbf{E}^3 : -2 \le x \le 2, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}, \sqrt{x^2+y^2} \le z \le 4\right\}$$

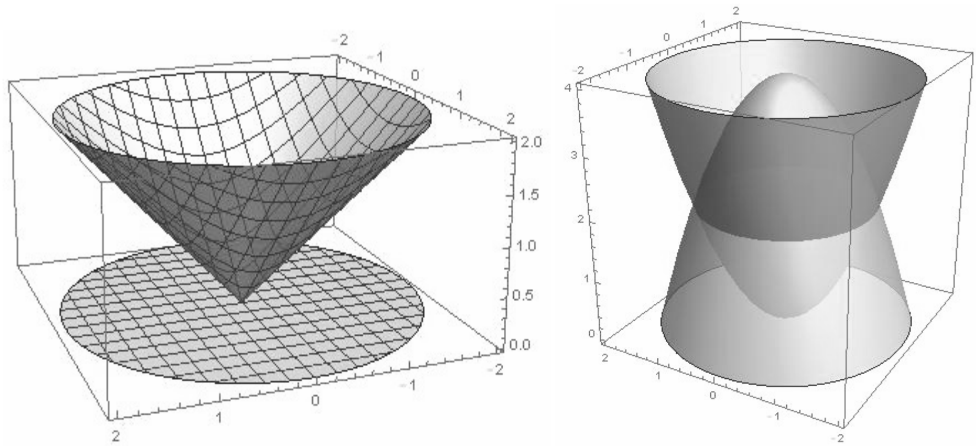


Fig. 3.9. Regular regions of types xyz and yxz.

2. The solid generated as a sub-space in $\mathbf{E}^3$ bounded by two paraboloids with the equations $z = x^2 + y^2$, $z = 4 - (x^2 + y^2)$, sharing a common circle $x^2 + y^2 = 2$ in the plane $z = 2$, see in Fig. 3.9, right, can be described as a regular region of type yxz,

$$M_{yx} = \left\{[x, y] \in \mathbf{E}^2\colon -\sqrt{2} \le y \le \sqrt{2}, -\sqrt{2-y^2} \le x \le \sqrt{2-y^2}\right\},$$
$$R_{yxz} = \left\{[x, y, z] \in \mathbf{E}^3\colon [x, y] \in M_{yx},\ x^2 + y^2 \le z \le 4 - \left(x^2 + y^2\right)\right\}.$$

Fubini theorem

For function $f(x, y, z)$ continuous on a regular region R of type xyz it holds that

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iint_M \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dx\,dy =$$

$$= \int_a^b \left\{ \int_{g_1(x)}^{g_2(x)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dy \right\} dx .$$

If function $f(x, y, z)$ is continuous on a regular region R' of type yxz, then

$$\iiint_{R'} f(x,y,z)\,dx\,dy\,dz = \iint_{M'} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dx\,dy =$$

$$= \int_c^d \left\{ \int_{h_1(y)}^{h_2(y)} \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dx \right\} dy .$$

All properties of double integrals on region $\langle a, b\rangle \times \langle c, d\rangle$, or on a regular region M, analogously hold for triple integrals on region $\langle a, b\rangle \times \langle c, d\rangle \times \langle e, h\rangle$, or on regular regions R, R' in $\mathbf{E}^3$, or on any set that is a union of a finite number of regular regions.

These properties are:

0. Sufficient condition of integrability
1. Linearity
2. Additivity
3. Monotonicity
4. Positivity
5. $\left| \iiint_D f(x,y,z)\,dx\,dy\,dz \right| \le \iiint_D |f(x,y,z)|\,dx\,dy\,dz$

Examples

1. Triple integral of function $f(x, y, z) = 2z$ on a region defined by equalities $x = 0$, $y = 0$, $z = 1$, $x + y + z = 2$ is triple integral over a regular region of type xyz,

   $R = \{[x, y, z] \in \mathbf{E}^3\colon 0 \le x \le 1, 0 \le y \le 1-x, 1 \le z \le 2-x-y\}$,

   and its value is

$$\iiint_R x\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \left(\int_1^{2-x-y} x\,dz \right) dy\,dx = \int_0^1 \left(\int_0^{1-x} [xz]_0^{2-x-y}\,dy \right) dx = \int_0^1 \left(\int_0^{1-x} (2x - x^2 - xy)\,dy \right) dx =$$

$$\int_0^1 \left[2xy - x^2 y - \frac{xy^2}{2} \right]_0^{1-x} dx = \frac{1}{2} \int_0^1 (3x - 4x^2 + x^3)\,dx = \frac{1}{2} \left[\frac{3x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{5}{24} .$$

2. Triple integral of function $f(x, y, z) = x + y + z$ over a region defined by

$$R = \{[x, y, z] \in \mathbf{E}^3\colon x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + z \leq 1\}$$

can be evaluated as triple integral on regular region of type xyz,

$$\iiint_R (x+y+z)\,dx\,dy\,dz = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} (x+y+z)\,dz \right) dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \left[xz + yz + \frac{z^2}{2} \right]_0^{1-x-y} dy \right) dx = \int_0^1 \left(\frac{1}{2} \int_0^{1-x} (1 - x^2 - 2xy - y^2)\,dy \right) dx =$$

$$= \frac{1}{2} \int_0^1 \left[y - x^2 y - xy^2 - \frac{y^3}{3} \right]_0^{1-x} dx = \frac{1}{6} \int_0^1 (2 - 3x + x^3)dx = \frac{1}{6} \left[2x - \frac{3x^2}{2} + \frac{x^4}{4} \right]_0^1 = \frac{1}{8}.$$

3.4 Multiple integrals

Multiple integral of function $f(X)$ of more variables on an arbitrary measurable closed region $G \subset \mathbf{E}^n$, $n \geq 1$ is a generalisation of the concepts of double or triple integrals, therefore it can be defined in a similar way.

Region G is called measurable, if there exists its measure, a unique positive real number denoted as $\mu(G)$.

Let region G can be divided into n partial measurable not overlapping sub-regions G_i with measures denoted as $\mu(G_i)$, while

1. $G_i \cap G_j = \varnothing$, for all $i \neq j$, $i, j = 1, \ldots, n$
2. $\cup\, G_i = G$, for $i = 1, \ldots, n$.

If for any sequence of integral sums of function $f(X)$ on G with the norms $\mu(G_i)$ convergent to zero

$$\sum_{i=1}^{n} f(\mathrm{X}_i).\mu(G_i)$$

a unique proper limit exists, then this number is called multiple integral of function f on region G denoted

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} f(\mathrm{X}_i).\mu(G_i) = \int_G f(X)\,dX$$

and function $f(X)$ is said to be integrable on region $G \subset \mathbf{E}^n$.

Function f integrable on a measurable closed region $G \subset \mathbf{E}^n$ is bounded on this region.

Function f continuous (up to finite number of points) on a measurable closed region $G \subset \mathbf{E}^n$ is integrable on this region.

Properties of multiple integrals

1. Linearity: If functions $f_1, f_2, ..., f_k$ are integrable on a region $G \subset \mathbf{E}^n$ and $c_1, c_2, ..., c_k$ are real numbers, then

$$\int_G (c_1 f_1(X) + c_2 f_2(X) + ... + c_k f_k(X))\,dX =$$
$$= c_1 \int_G f_1(X)\,dX + c_2 \int_G f_2(X)\,dX + ... + c_k \int_G f_k(X)\,dX$$

2. Additivity: Let function f be integrable on region $G \subset \mathbf{E}^n$ and let $G = \bigcup_{i=1}^{k} G_i$,

   where $G_i \subset \mathbf{E}^n$ are measurable regions with no common interior points, then

$$\int_G f(X)dX = \sum_{i=1}^{k} \int_{G_i} f(X)dX .$$

3. Monotonicity: Let functions f, g be integrable on a measurable region $G \subset \mathbf{E}^n$ and let for all points $X \in G$ hold that $f(X) \le g(X)$, then

$$\int_G f(X)dX \le \int_G g(X)dX .$$

4. Positivity: Let function f be integrable on a measurable region $G \subset \mathbf{E}^n$ and let $f(X) \ge 0$ for all $X \in G$, then

$$\int_G f(X)dX \ge 0 .$$

5. Let function f be integrable on a measurable region $G \subset \mathbf{E}^n$, then function $|f(X)|$ is also integrable on G and

$$\left| \int_G f(X)dX \right| \le \int_G |f(X)|dX .$$

3.5 Transformations in the plane

Let $M^* \subset \mathbf{E}^2$ be a non-empty set. A mapping (transformation) Φ from set M^* to the set $\mathbf{E}^2$ is a rule by which every point $[u, v] \in M^*$ can be associated with a unique point $[x, y] \in \mathbf{E}^2$. Point $[x, y]$ is the image of point $[u, v]$ in the given mapping denoted

$$[x, y] = \Phi([u, v]),$$

while set M^* is the domain of this mapping Φ.

Transformation Φ is determined by two functions of two variables

$$x = \varphi_1(u, v)$$
$$y = \varphi_2(u, v)$$

defined on set M^*.

Transformation Φ: $M^* \to \mathbf{E}^2$ is called a one-to-one mapping, if for any two points $[u_1, v_1]$, $[u_2, v_2] \in M^*$ it holds that

$$[u_1, v_1] \neq [u_2, v_2] \Rightarrow \Phi([u_1, v_1]) \neq \Phi[u_2, v_2],$$

which means that the images of two different points are two different points.
For any one-to-one mapping Φ there exists a mapping Φ^{-1} from the set $M = \Phi(M^*)$ to the set M^* such, that for any $[x, y] \in M = \Phi(M^*)$ it holds

$$\Phi^{-1}([x, y]) = [u, v] \Leftrightarrow \Phi([u, v]) = [x, y].$$

Mapping Φ^{-1} is called the inverse mapping of the mapping Φ and it is determined by the equations

$$u = \varphi_1^{-1}(x, y)$$
$$v = \varphi_2^{-1}(x, y)\ .$$

Mapping Φ: $M^* \to \mathbf{E}^2$ determined by relations

$$[u, v] \to [x, y] = [\varphi_1(u, v), \varphi_2(u, v)]$$

is said to be continuous at the point $[u_0, v_0] \in M^*$, if functions φ_1, φ_2 are continuous at this point. In the case of functions φ_1, φ_2 continuous on set M^*, mapping Φ is said to be continuous on set M^*.

Any one-to-one continuous mapping maps a simple curve to a simple curve.

Mapping Φ: $M^* \to \mathbf{E}^2$, $M^* \neq 0$, $M^* \subset \mathbf{E}^2$ is said to be regular on M^*, if the following properties hold:

1. functions φ_1, φ_2 have continuous partial derivatives on set M^* with respect to both variables
2. for all $[u, v] \in M^*$

$$J(u,v) = \begin{vmatrix} \dfrac{\partial \varphi_1(u,v)}{\partial u} & \dfrac{\partial \varphi_1(u,v)}{\partial v} \\ \dfrac{\partial \varphi_2(u,v)}{\partial u} & \dfrac{\partial \varphi_2(u,v)}{\partial v} \end{vmatrix} \neq 0 \cdot$$

Determinant $J(u, v)$ is called the Jacobi functional determinant of mapping Φ, or the Jacobian in short. The sign of the Jacobian $J(u, v)$ of any regular mapping Φ on the set M^* is the same at all points $[u, v] \in M^*$.

Any regular mapping on M^*, Φ: $M^* \to \mathbf{E}^2$ is continuous on M^*.
A one-to-one mapping maps any regular region to a regular region, and any closed region to a closed region.

Let $\Omega^* \subset \mathbf{E}^2$ be a regular region and let mapping Φ: $\Omega^* \to \mathbf{E}^2$ be one-to-one and regular on Ω^*. If $G^* \subset \Omega^*$ is a measurable closed region, then $G = \Phi(G^*)$ is also a measurable closed region and

$$\mu(G) = \int_{G^*} |J(u,v)|\, du\, dv \cdot$$

3.6 Double integrals in polar coordinates

Transformation to polar coordinates is a useful technique leading to a considerably easier representations of many curves and regions described by complicated formulas in Cartesian coordinates. It is often used for the description of domains of integration, which are plane regions bounded by arcs of circles, and for evaluation of double integrals of functions defined by formulas, in which square roots of sums of squares of variables appear. This is namely the case of functions whose graphs are parts of quadratic surfaces.

Let P be a fixed point in the plane. The half-line $\vec{o}$ with the start point P and a revolution about point P in the positive (anti-clockwise) sense determine a polar coordinate system (P, $\vec{o}$, φ) in the plane. Point P is called the pole (origin) of the coordinate system, half-line $\vec{o}$ is the polar axis of this system.

Each point M in the plane can be attached an ordered pair of real numbers, $M = (\rho, \varphi)$, whose geometric interpretation is clear from Fig. 3. 10:

1. $\rho = |PM|$ is the distance of point M to the pole P,
2. $\varphi = |\sphericalangle(\vec{o}, \overrightarrow{PM})|$ is the size of positively oriented angle with the vertex in the pole P, formed by the polar axis $\vec{o}$ and half-line $\overrightarrow{PM}$.

Ordered pair of real numbers (ρ, φ) is called the polar coordinates of a point, number $\rho \in \langle 0, \infty)$ is the modul, while number $\varphi \in \langle 0, 2\pi)$ or $\varphi \in (-\pi, \pi\rangle$ is the polar angle.

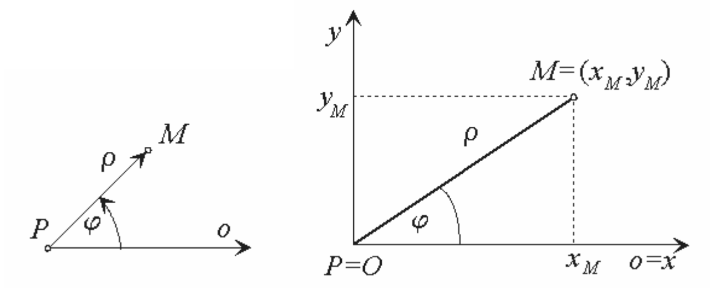


Fig. 3.10. Polar coordinates in plane.

Let both, polar (P; $\vec{o}$, φ) and Cartesian (O; x, y) coordinate systems be determined in the plane $\mathbf{E}^2$, while $P = O$, and polar axis $\vec{o}$ coincides with the positive part of the coordinate axis x, as in Fig. 3.10. These two systems can be mapped to one another, and the relations between the two pairs of coordinates of point $M \neq O$ are defined by the equations

$$x_M = \rho \cos \varphi, \quad y_M = \rho \sin \varphi, \quad 0 < \rho < \infty, \qquad 0 \le \varphi < 2\pi .$$

Because $x_M^2 + y_M^2 \neq 0$, the following equations hold

$$\cos\varphi = \frac{x_M}{x_M^2 + y_M^2}, \quad \sin\varphi = \frac{y_M}{x_M^2 + y_M^2} .$$

In the case $\rho = 0$, and therefore $x = y = 0$, the polar angle is not defined by the equations above, and for polar coordinates of the pole holds $P = (0,\ \varphi)$, for arbitrary number φ.

Mapping of the plane with the polar coordinate system to the plane with the Cartesian coordinate system is called the polar transformation of the plane.
Polar transformation is a one-to-one and regular mapping

$$\Phi : \Omega^* = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2\right\} \to \Omega = \left\{[x, y] \in \mathbf{E}^2\right\}$$

defined on the set $\Omega = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 0 < \rho < \infty,\ 0 \le \varphi < 2\pi\right\}$ and determined by the relations

$$x = \rho \cos \varphi,\ y = \rho \sin \varphi, \text{ while } \rho = \sqrt{x^2 + y^2},\ \varphi = \arctan\frac{y}{x},\ x \neq 0,$$

while $\varphi = \pi/2$ for $x = 0$, $y > 0$, and $\varphi = 3\pi/2$ for $x = 0$, $y < 0$.

The Jacobi determinant (Jacobian) of the polar transformation is

$$J(\rho,\ \varphi) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho > 0.$$

There are many important plane curves consisting of the points whose polar coordinates, in contrary to the Cartesian coordinates, satisfy a simple equation of the form $\rho = f(\varphi)$, e.g. the polar equation of a circle is $\rho = a \sin \varphi$, $a > 0$.

Examples

1. Graphs of some curves with polar equations
   $\rho = a(1 - \cos \varphi)$, $a > 0$, $0 \le \varphi \le 2\pi$, cardioid
   $\rho = \sqrt{a\cos 2\varphi}$, $a > 0$, $0 \le \varphi \le 2\pi$, leminscate of Bernoulli
   $\rho = a\varphi$, $a > 0$, $0 \le \varphi \le 2\pi$, Archimedean spiral
   $\rho = a\left|\sin n\varphi\right|$, $a > 0$, n - natural, $0 \le \varphi \le 2\pi$, $2n$ leaved rose
   are presented in Fig. 3.11, from left to right.

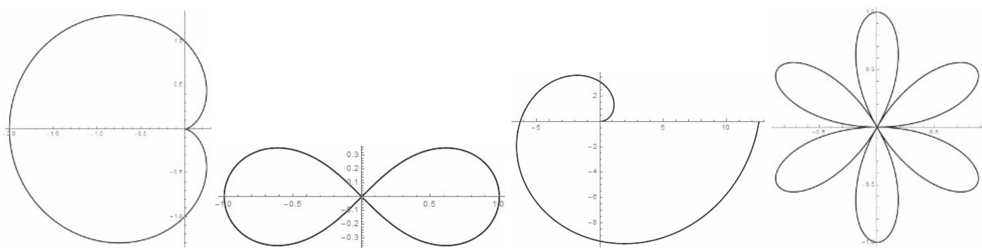


Fig. 3.11. Graphs of curves determined in polar coordinates.

Domain of integration is often a plane region not bounded by lines, but by curves, as for instance: combinations of line segments, arcs of circles, or parts of other conic sections, like ellipses or hyperbolas. Especially in these cases it is simpler to describe the regions in polar coordinates.

Examples

1. Regular region described in Cartesian coordinates
   $R = \left\{[x,\ y] \in \mathbf{E}^2 : 1 \le x^2 + y^2 \le 9\right\}$ in Fig. 3.12, left,
   can be rewritten in polar coordinates as
   $R^* = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 1 \le \rho \le 3, 0 \le \varphi \le 2\pi\right\}$,
   and region $R = \left\{[x,\ y] \in \mathbf{E}^2 : x^2 - 2y + y^2 \ge 0 \wedge x^2 + y^2 \le 4, y \ge 0\right\}$ in Fig. 3.12, middle, as $R^* = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 2\sin\varphi \le \rho \le 2, 0 \le \varphi \le \pi\right\}$.
2. Closed set of points in $\mathbf{E}^2$ bounded by two circles $x^2 + y^2 = 2$, $x^2 + y^2 = 6$ and two lines $y = x$, $y = -x$ for $x > 0$ can be easily described in polar coordinates as the region $R^* = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : \sqrt{2} \le \rho \le \sqrt{6}, -\dfrac{\pi}{4} \le \varphi \le \dfrac{\pi}{4}\right\}$, see in Fig. 3.12, right.

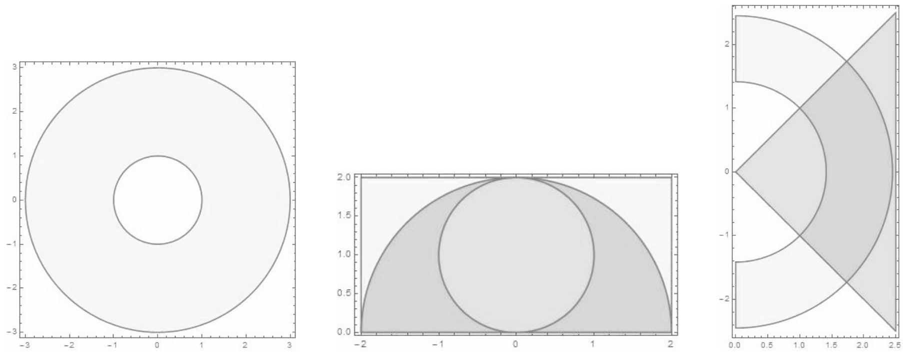


Fig. 3.12. Planar regions described in Cartesian coordinates.

3. Below regions described in polar coordinates are sketched in Fig. 3.13.

$$R = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 0 \le \rho \le 2\sin\varphi,\ \pi/2 \le \varphi \le \pi,\right\}$$
$$R = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 0 \le \rho \le 4\cos\varphi, 0 \le \varphi \le \pi/2,\right\}$$
$$R = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 0 \le \rho \le 4\varphi, 0 \le \varphi \le \pi/2\right\}$$
$$R = \left\{(\rho,\ \varphi) \in \boldsymbol{R}^2 : 0 \le \rho \le 1 + \cos\varphi, 0 \le \varphi \le 2\pi\right\}$$

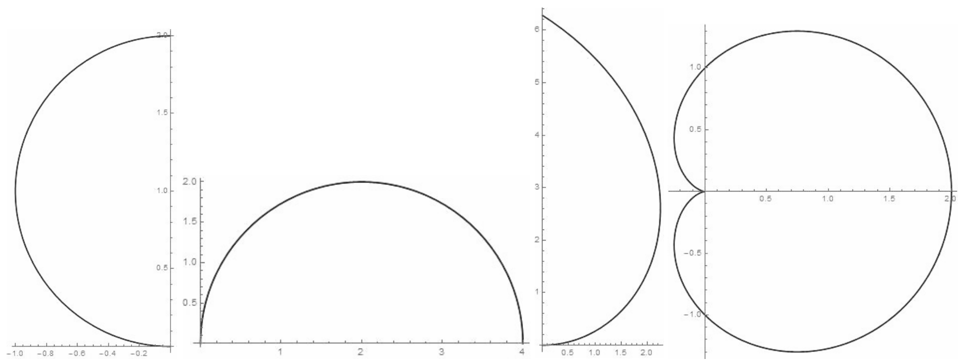


Fig. 3.13. Planar regions described in polar coordinates.

The rule for change of variable in the definite integral plays an extremely important role in practical integration. It states: If φ is a function defined on an interval I such that the derivative of φ is continuous and different from zero on the interval I and f is a function continuous on $\varphi(I)$, then

$$\int_{\varphi(I)} f(x)dx = \int_I f(\varphi(t))\varphi'(t)\,dt\,.$$

Change of variables in double integrals is used not only in the case of too complicated integrands (functions to be integrated), but also in the case of too complicated domains of integration. The rule for change of variables in the double integrals is essentially more complicated, therefore we will restrict all considerations to the final formula and to the case of the transformation from Cartesian coordinates to polar ones.

Let $\Omega^* \subset \mathbf{E}^2$ be a regular region and let the mapping Φ: $\Omega^* \to \mathbf{E}^2$ be a regular and one-to-one transformation on Ω^*given by formulas

$$x = \varphi_1(u, v),\ y = \varphi_2(u, v),$$

where φ_1 and φ_2 are real functions of two variables. Let $G^* \subset \Omega^*$ be a measurable closed region and let function $f(x, y)$ be continuous on the closed region $G = \Phi(G^*)$. Then the formula for change of variables in double integrals has the form

$$\iint_G f(x,y)\,dx\,dy = \iint_{G^*} f(\varphi_1(u,v),\varphi_2(u,v))\left|J(u,v)\right|du\,dv\,,$$

where $J(u, v)$ is the Jacobian of transformation Φ.

Remark. Note that functions φ_1, φ_2, f and their partial derivatives must be continuous on their respective domains of definition.

Applying the general formula to the transformation Φ from Cartesian coordinates $[x, y]$ to polar coordinates (ρ, φ) given by the formulas

$$x = \varphi_1(\rho, \varphi) = \rho \cos \varphi$$

$$y = \varphi_2(\rho, \varphi) = \rho \sin \varphi$$

with the Jacobian

$$J(u,v) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho > 0$$

the formula for polar transformation in double integrals can be obtained

$$\iint_R f(x,y)dxdy = \iint_{R^*} f(\rho\cos\varphi,\rho\sin\varphi)\rho d\rho d\varphi\,,$$

where $R^* = \Phi^{-1}(R)$. Regions R and R^* are domains of integration in the plane $\mathbf{E}^2$, while ρ and φ can be interpreted as Cartesian coordinates in the plane.
The presented change of variables may substantially simplify the given integral, as it may lead for instance to constant limits of integration in the transformed integral.

Examples

1. The integral $\iint\limits_R (x^2+y^2)\,dxdy$ over the region $R=\left\{[x,\ y]\in \mathbf{E}^2 : x^2+y^2\le 4\right\}$ can be easily evaluated in polar coordinates as an integral on the region

$R^*=\left\{(\rho,\ \varphi)\in \boldsymbol{R}^2 : 0\le\rho\le 2,\, 0\le\varphi\le 2\pi\right\}$,

$$\iint\limits_R (x^2+y^2)dxdy=\iint\limits_{R^*}(\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)\rho\,d\rho\,d\varphi=\int\limits_0^2\int\limits_0^{2\pi}\rho^3 d\varphi\,d\rho=$$

$$\int\limits_0^2 \rho^3[\varphi]_0^{2\pi}\,d\rho=\frac{\pi}{2}\left[\rho^4\right]_0^2=8\pi\,.$$

2. The evaluation of the integral $\iint\limits_R (x^2+y^2)dxdy$ on more complicated region described in the Cartesian coordinates as $R=\left\{[x,\ y]\in\mathbf{E}^2 : 1\le x^2+y^2\le 4, y\ge|x|\right\}$ can be simplified by means of transformation to polar coordinates, over the integration region $R^*=\left\{(\rho,\ \varphi)\in\boldsymbol{R}^2 : 1\le\rho\le 2, \dfrac{\pi}{4}\le\varphi\le\dfrac{3\pi}{4}\right\}$,

$$\iint\limits_R (x^2+y^2)dxdy=\iint\limits_{R^*}(\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)\rho\,d\rho\,d\varphi=\int\limits_1^2\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\rho^3 d\varphi\,d\rho=$$

$$\int\limits_1^2 \rho^3[\varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\,d\rho=\frac{\pi}{2}\left[\frac{\rho^4}{4}\right]_1^2=\frac{15\pi}{8}.$$

3. The integral $\iint\limits_R x\,dxdy$, while R is the set bounded by the curve $x^2+y^2-2y=0$, can be evaluated in polar coordinates. The boundary of the set R is a circle with the unit radius and shifted centre, $x^2+(y-1)^2=1$, therefore in polar coordinates it is described as $R^*=\left\{(\rho,\ \varphi)\in\boldsymbol{R}^2 : 0\le\rho\le 2\sin\varphi,\, 0\le\varphi\le\pi\right\}$, see Fig. 3.13, left. Transformation to the integral yields

$$\iint\limits_R x\,dxdy=\iint\limits_{R^*}\rho^2\cos\varphi\,d\rho\,d\varphi=\int\limits_0^{\pi}\int\limits_0^{2\sin\varphi}\rho^2\cos\varphi\,d\rho\,d\varphi=\int\limits_0^{\pi}\cos\varphi\left[\frac{\rho^3}{3}\right]_0^{2\sin\varphi}d\varphi=$$

$$=\int\limits_0^{\pi}\frac{8}{3}\sin^3\varphi\cos\varphi\,d\varphi=\left|\begin{array}{c}\sin\varphi=t\\ \cos\varphi\,d\varphi=dt\\ \varphi=0\Rightarrow t=0\\ \varphi=\pi\Rightarrow t=0\end{array}\right|=\frac{8}{9}\int\limits_0^0 t^3dt=0\,.$$

3.7 Transformations in the space

Let $T^* \subset \mathbf{E}^3$ be a non-empty set. A mapping (transformation) Φ from the set T^* to the set $\mathbf{E}^3$ attaching to every point $[u, v, w] \in M^*$ a unique image, point $[x, y, z] \in \mathbf{E}^3$ is determined by three functions of three variables

$$x = \varphi_1(u, v, w)$$
$$y = \varphi_2(u, v, w)$$
$$z = \varphi_3(u, v, w)$$

while set T^* is the domain of this mapping Φ.

Mapping Φ: $T^* \to \mathbf{E}^3$ determined by the relations

$$[u, v, w] \to [x, y, z] = [\varphi_1(u, v, w), \varphi_2(u, v, w), \varphi_3(u, v, w)]$$

is said to be continuous at the point $[u_0, v_0, w_0] \in T^*$, if functions φ_1, φ_2 and φ_3 are continuous at this point. When all three functions φ_1, φ_2, φ_3 are continuous on set T^*, mapping Φ is said to be continuous on set T^*.

Any one-to-one continuous mapping of the space maps a regular surface patch to a regular surface patch.

Mapping Φ: $T^* \to \mathbf{E}^3$, $T^* \neq 0$, $T^* \subset \mathbf{E}^3$ is said to be regular on T^*, if the following properties hold:

1. functions φ_1, φ_2, φ_3 have continuous partial derivatives on set T^* with respect to all three variables
2. for all $[u, v, w] \in T^*$

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial \varphi_1(u,v,w)}{\partial u} & \dfrac{\partial \varphi_1(u,v,w)}{\partial v} & \dfrac{\partial \varphi_1(u,v,w)}{\partial w} \\ \dfrac{\partial \varphi_2(u,v,w)}{\partial u} & \dfrac{\partial \varphi_2(u,v,w)}{\partial v} & \dfrac{\partial \varphi_2(u,v,w)}{\partial w} \\ \dfrac{\partial \varphi_3(u,v,w)}{\partial u} & \dfrac{\partial \varphi_3(u,v,w)}{\partial v} & \dfrac{\partial \varphi_3(u,v,w)}{\partial w} \end{vmatrix} \neq 0 \, .$$

Determinant $J(u, v, w)$ is called the Jacobi functional determinant (the Jacobian) of mapping Φ. The sign of the Jacobian $J(u, v, w)$ of any regular mapping Φ on the set T^* is the same at all points $[u, v, w] \in T^*$.

Any regular mapping on T^*, Φ: $T^* \to \mathbf{E}^3$ is continuous on T^*.

A one-to-one mapping maps any regular region to a regular region, and any closed region to a closed region.

Let $\Omega^* \subset \mathbf{E}^3$ be a regular region and let mapping Φ: $\Omega^* \to \mathbf{E}^3$ be one-to-one and regular on Ω^*. If $G^* \subset \Omega^*$ is a measurable closed region, then $G = \Phi(G^*)$ is also a measurable closed region and

$$\mu(G) = \int_{G^*} |J(u,v,w)| \, du \, dv \, dw \, .$$

3.8 Triple integrals in cylindrical coordinates

Similarly to the two-dimensional plane, the rectangular coordinate system in three dimensional space is not appropriate to all types of problems. There are circumstances in which other systems are more convenient. In some problems concerning triple integrals over some special types of domains of integration we will use cylindrical coordinates, which can be regarded as a simple three dimensional extension of the polar coordinate system.

Let a plane π and perpendicular line p be given, while a polar coordinate system $(P; \vec{o}, \varphi)$ be determined in the plane π such, that the pole P is the intersection point of plane π and line p. Plane π with the polar coordinate system and coordinate axis p with origin at the point P determine a cylindrical coordinate system $(P; \vec{o}, \varphi, p)$ in the three dimensional space.

Any point M in the space is attached a unique triple of real numbers, its cylindrical coordinates $M = (\rho, \varphi, z)$, while

1. ρ, φ are polar coordinates of the orthogonal view M_1 of point M to plane π in polar coordinate system $(P; \vec{o}, \varphi)$
2. z is an oriented distance of point M and plane π
   and it holds $\rho \in (0, \infty)$, $\varphi \in [0, 2\pi)$, or $\varphi \in (-\pi, \pi]$, and $z \in (-\infty, \infty)$.

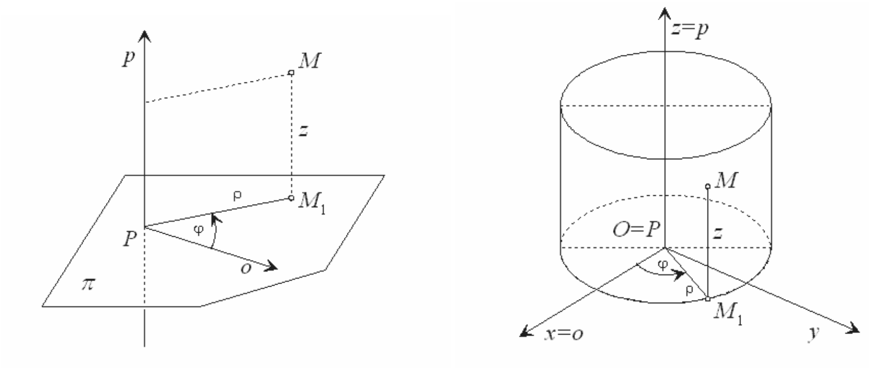


Fig. 3.14. Cylindrical coordinate system in space.

Pole P is the orthographic view of an arbitrary point M on coordinate axis z to plane π, therefore cylindrical coordinates of all points on the axis z are represented by a triple of real numbers $(0, \varphi, z)$, for an arbitrary number φ.

Let the Cartesian coordinate system $(O; x, y, z)$ and the cylindrical coordinate system $(P; \vec{o}, \varphi, p)$ be given. These two coordinate systems are said to be related, if:

1. plane π determining the cylindrical coordinate system $(P; \vec{o}, \varphi, p)$ coincides with the coordinate plane xy of the Cartesian coordinate system $(O; x, y, z)$
2. the polar coordinate system $(P; \vec{o}, \varphi)$ and the Cartesian coordinate system $(O; x, y)$ in the plane π are the related coordinate systems
3. axis p of the cylindrical coordinate system $(P; \vec{o}, \varphi, p)$ coincides with the coordinate axis z of the Cartesian coordinate system $(O; x, y, z)$.

Let $[x, y, z]$ be the Cartesian coordinates and (ρ, φ, z) be the cylindrical coordinates of point M not on the coordinate axis z, then their relation can be determined by the following equations

$$x = \varphi_1(\rho, \varphi) = \rho \cos \varphi, \quad y = \varphi_2(\rho, \varphi) = \rho \sin \varphi, \quad z = \varphi_3(\rho, \varphi) = z$$

and because $x^2 + y^2 > 0$

$$\rho = \sqrt{x^2 + y^2}$$

$$\varphi = \arccos \frac{x}{\sqrt{x^2 + y^2}}, \ y \geq 0, \quad \varphi = 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}}, \quad y < 0.$$

All points with the constant first cylindrical coordinate $\rho = a$, $a > 0$ are points on the cylindrical surface of revolution with a basic circle in the plane π. The centre of this circle is at the point P and its radius equals a. The coordinate axis p is the axis of this cylindrical surface of revolution, while all generatrices – lines on the surface, are parallel to this axis, see in Fig. 3.14, on the right.

Transformation of space $\mathbf{E}^3$, in which a cylindrical coordinate system is transformed to the Cartesian orthogonal coordinate system is called a cylindrical transformation of the space. The Jacobi determinant (the Jacobian) of the cylindrical transformation is

$$J(\rho, \varphi, z) = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho > 0 .$$

Therefore, the cylindrical transformation of the space is a regular transformation at all points in $\mathbf{E}^3$, but the origin of the coordinate system.
The cylindrical coordinates are suitable mainly to describe solids enclosed by the cylindrical surfaces.

Examples

1. Equation $\rho = 1$, for $0 \leq \varphi \leq 2\pi$, $0 \leq z \leq 1$ in cylindrical coordinates represents a patch of the cylindrical surface of revolution with radius 1 and axis z of a unit height. This can be described in the Cartesian coordinates as a set of points whose coordinates satisfy the relations $x^2 + y^2 = 1$ and $0 \leq z \leq 1$.
2. Plane of symmetry of coordinate planes xz and yz passing through the coordinate axis z can be represented in cylindrical coordinates by the equation $\varphi = \pi/2$, while the equation $z = a$, $a \in \boldsymbol{R}$ represents a plane perpendicular to the coordinate axis z.
3. The equation $z = \sqrt{\rho}$ in the cylindrical coordinates represents a paraboloid of revolution with vertex at origin and axis in the coordinate axis z, while the equation $z = \rho$ describes the positive part of a conical surface of revolution with the vertex at origin and generatrices forming angle $\pi/2$ with coordinate planes xy, which are depicted in Fig. 3. 15 on the left and in the middle.

4. Inequalities $1 \leq \rho \leq 3$, $0 \leq \varphi \leq \pi/2$, $1 \leq z \leq 2$ describe a part of a cylinder of revolution with axis in coordinate axis z, radius 3 and height 1, illustrated in Fig. 3.15, right.

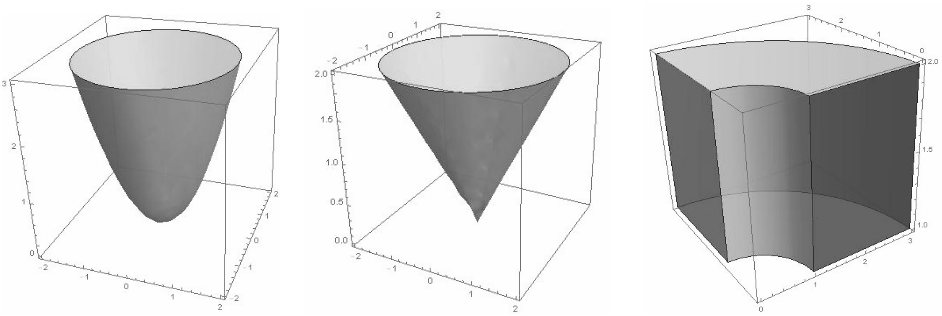


Fig. 3.15. Space regions described in cylindrical coordinates.

A change of variables in triple integrals is used in a similar way as in double integrals not only in the case of complicated integrand functions, but even more frequently for complicated domains of integration. Domain of integration is often a space region not bounded by planes but by several surfaces, which could be, for instance, parts of cylindrical or spherical surfaces. Especially in these cases it is simpler to describe the regions by means of cylindrical instead of Cartesian coordinates.

Let $\Omega^* \subset \mathbf{E}^3$ be a regular region and let mapping Φ: $\Omega^* \to \mathbf{E}^3$ be a regular and one-to-one transformation on Ω^*given by formulas

$$x = \varphi_1(u, v, w),\ y = \varphi_2(u, v, w),\ z = \varphi_3(u, v, w)$$

where φ_1, φ_2 and φ_3 are real functions of three variables. Let $T^* \subset \Omega^*$ be a measurable closed region and let function $f(x, y, z)$ be continuous on a closed region $T = \Phi(G^*)$. Then the formula for the change of variables in triple integrals has the form

$$\iiint_T f(x, y, z)\, dxdydz = \iiint_{T^*} f(\varphi_1(u,v,w), \varphi_2(u,v,w), \varphi_3(u,v,w)) \left| J(u,v,w) \right| du\, dvdw,$$

where $J(u, v, w)$ is the Jacobian of transformation Φ.

Remark. Note that functions φ_1, φ_2, φ_3, f and all their partial derivatives must be continuous on their respective domains of definition.

Applying the general formula to the transformation Φ from Cartesian coordinates $[x, y, z]$ to cylindrical coordinates (ρ, φ, z) given by the formulas

$$x = \varphi_1(\rho, \varphi, z) = \rho \cos \varphi, \quad y = \varphi_2(\rho, \varphi, z) = \rho \sin \varphi, \quad z = \varphi_3(\rho, \varphi, z) = z$$

with the the Jacobian $J(u,v,w) = \rho > 0$, the formula for cylindrical transformation in triple integrals can be obtained

$$\iiint_T f(x, y, y) dxdydz = \iiint_{T^*} f(\rho \cos \varphi, \rho \sin \varphi, z) \rho d\rho d\varphi\, dz,$$

where $T^* = \Phi^{-1}(T)$. Regions T and T^* are domains of integration in the space $\mathbf{E}^3$, while ρ, φ and z can be interpreted as Cartesian coordinates in the space.

Examples

1. $\iiint\limits_T z\,dxdydz$, where T is bounded by surfaces $x^2 + y^2 = 4$, $z = 1$, $z = 3$ (illustrated in Fig. 3.16, left) can be represented in the cylindrical coordinates as the integral $\iiint\limits_{T^*} \rho\, z\, d\rho d\varphi\, dz$, while region T is thus transformed to the region $T^* = \{(\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 0 \le \rho \le 2, 0 \le \varphi \le 2\pi, 1 \le z \le 3\}$, which yields the evaluation

$$\iiint\limits_T z\,dxdydz = \iiint\limits_{T^*} \rho\, z\, d\rho\, d\varphi\, dz = \int_0^2 \int_0^{2\pi} \int_1^3 \rho\, z dz\, d\varphi\, d\rho = \int_0^2 \int_0^{2\pi} \rho \left[\frac{z^2}{2}\right]_1^3 d\varphi\, d\rho =$$

$$= \int_0^{2\pi} \int_0^2 4\rho\, d\rho d\varphi = \int_0^{2\pi} 2\left[\rho^2\right]_0^2 d\varphi = \int_0^{2\pi} 8\, d\varphi = 8[\varphi]_0^{2\pi} = 16\pi .$$

2. Integral $\iiint\limits_T (x^2 + y^2)\, dxdydz$ on set $T = \{[x,\ y, z] \in \mathbf{E}^3 : x^2 + y^2 \le 2z, y \le 0, z \le 2\}$ illustrated in Fig. 3.16, middle, is represented in the cylindrical coordinates as the integral $\iiint\limits_{T^*} \rho^3 d\rho d\varphi\, dz$, on the region described in the cylindrical coordinates

$$T^* = \left\{ (\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 0 \le \rho \le 2, \pi \le \varphi \le 2\pi, \frac{\rho^2}{2} \le z \le 2 \right\},$$

leading to the evaluation

$$\iiint\limits_T (x^2 + y^2)\, dxdydz = \iiint\limits_{T^*} \rho^3 d\rho d\varphi\, dz = \int_\pi^{2\pi} \int_0^2 \int_{\frac{\rho^2}{2}}^2 \rho^3 dz\, d\rho d\varphi = \int_\pi^{2\pi} \int_0^2 \rho^3 [z]_{\frac{\rho^2}{2}}^2 d\rho d\varphi =$$

$$= \int_\pi^{2\pi} \int_0^2 \left(2\rho^3 - \frac{\rho^5}{2} \right) d\rho d\varphi = \int_\pi^{2\pi} \left[\frac{\rho^4}{2} - \frac{\rho^6}{12} \right]_0^2 d\varphi = \frac{8}{3} \int_\pi^{2\pi} d\varphi = \frac{8}{3} [\varphi]_\pi^{2\pi} = \frac{8\pi}{3} .$$

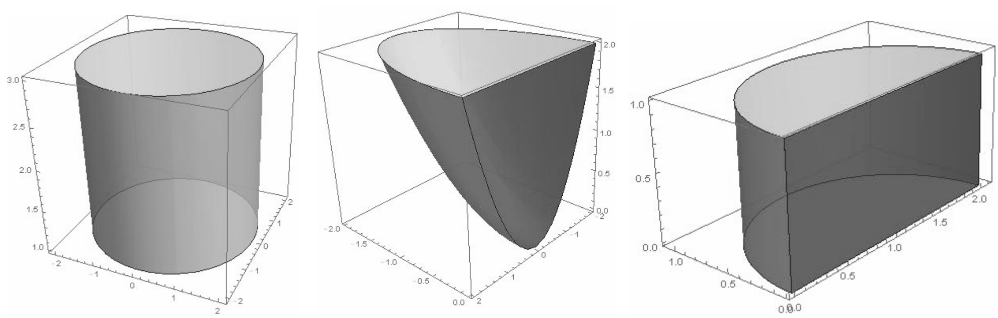


Fig. 3.16. Integration regions in space.

3. Evaluation of integral $\iiint\limits_T z(x^2+y^2)\,dxdydz$ on the set T bounded by planes $z = 0$, $z = 1$ and by surface $x^2 + y^2 = 2x$, while $y \geq 0$, illustrated in Fig. 3.16, right, can be simplified by transformation to the cylindrical coordinates, leading to integration on the set $T^* = \{(\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 0 \leq \rho \leq 2\cos\varphi,\ -\pi \leq \varphi \leq \pi,\ 0 \leq z \leq 1\}$, which yields the following integration

$$\iiint\limits_T z(x^2+y^2)\,dxdydz = \iiint\limits_{T^*} \rho^3 z\, d\rho d\varphi\, dz = \int_0^1 \int_{-\pi}^{\pi} \int_0^{2\cos\varphi} \rho^3 z d\rho\, d\varphi\ dz =$$

$$= \int_0^1 \int_{-\pi}^{\pi} z\left[\frac{\rho^4}{4}\right]_0^{2\cos\varphi} d\varphi\, dz = \int_0^1 \int_{-\pi}^{\pi} 4z\cos^4\varphi\, d\varphi\, dz = \int_0^1 4z\left[\frac{3\varphi}{8} - \frac{\sin 2\varphi}{4} + \frac{\sin 4\varphi}{32}\right]_{-\pi}^{\pi} d\varphi =$$

$$= 3\pi\int_0^1 z\,dz = \frac{3\pi}{2}\left[z^2\right]_0^1 = \frac{3\pi}{2}.$$

3.9 Triple integrals in spherical coordinates

Another coordinate system that proved to be useful for the description of special sets in space $\mathbf{E}^3$ is the spherical coordinate system. The most fruitful and generally known application of the spherical coordinate system can be seen in cartography and geodesy. The position on the globe as a model of the Earth can be easily determined by two angles, known as azimuth (longitude) and elevation, which describe the position on the globe related to the set of defined meridians and parallels. These angles simply represent the position as two specific angles of revolution. One of them is the angle of revolution about the axis of the globe measured in the counter-clockwise orientation from the prime meridian passing through London to the actual position, called azimuth (longitude). The other, elevation, is the angle of revolution from the plane passing through the globe's centre perpendicularly to the axis of the globe. The equator located in this plane is the parallel circle with the largest radius, i.e. radius equal to the radius of the Earth estimated as r = 6378 km.

Let S be a fixed point in the plane π, and let half-line $\overrightarrow{p_1}$ with the start point S and counter-clockwise revolution about point S in the plane π determine the polar coordinate system $(S;\ \overrightarrow{p_1},\ \varphi)$ in this plane. Let line $\overrightarrow{p_2}$ be passing through the point S perpendicularly to the plane π, $\overrightarrow{p_2} \perp \pi$. Plane π with the polar coordinate system and half-line $\overrightarrow{p_2}$ form the spherical coordinate system $(S;\ \overrightarrow{p_1},\ \varphi,\ \overrightarrow{p_2})$ in the space. Half-lines $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ are coordinate axes and pole S is the origin of this spherical coordinate system.

Any point M in the space can be attached a unique triple of real numbers, $M = (\rho,\ \varphi,\ \zeta)$, whose geometric interpretation can be understood from Fig. 3. 17.

The following relations are valid:

1. $\rho = |SM|$, therefore ρ is the distance of points M and S

2. $\varphi = |\sphericalangle(\overrightarrow{p_1}, \overrightarrow{SM_1})|$, therefore φ is the oriented angle with the vertex at the point S, with the starting arm formed by half-line $\overrightarrow{p_1}$ and the end arm by half-line $\overrightarrow{SM_1}$, while M_1 is the orthographic view of point M in the plane π

3. $\zeta = |\sphericalangle(\overrightarrow{p_2}, \overrightarrow{SM})|$, therefore ζ is the angle formed by half-lines $\overrightarrow{p_2}$ and $\overrightarrow{SM}$.

The ordered triple of real numbers (ρ, φ, ζ) determines the spherical coordinates of point, where $\rho \in (0, \infty)$, $\varphi \in [0, 2\pi)$, or $\varphi \in (-\pi, \pi]$, $\zeta \in [0, \pi)$.

All points M on line p are attached spherical coordinates in the form (ρ, φ, ζ), where φ is an arbitrary number, and $\zeta = 0$ for all points on half-line $\overrightarrow{p_2}$, while $\zeta = \pi$ for all points on the opposite half-line.

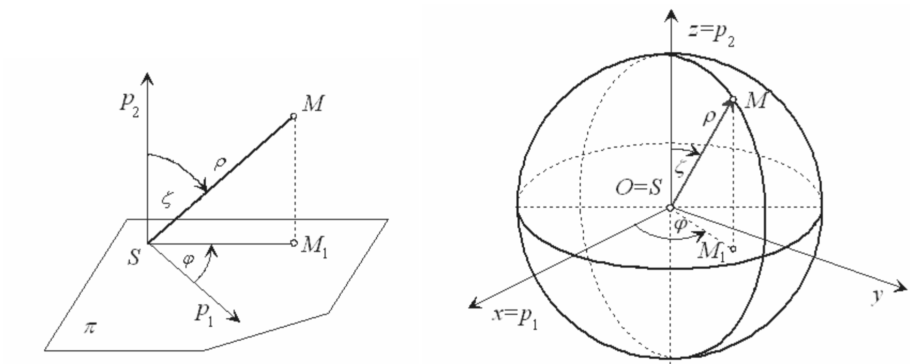


Fig. 3.17. Spherical coordinate system in space.

The spherical coordinate system $(S; \overrightarrow{p_1}, \varphi, \overrightarrow{p_2})$ is related to the Cartesian coordinate system $(O; x, y, z)$ in the space, if:

1. plane π determining the spherical coordinate system $(S; \overrightarrow{p_1}, \varphi, \overrightarrow{p_2})$ coincides with the coordinate plane xy of the Cartesian coordinate system $(O; x, y, z)$
2. point S coincides with the origin O and oriented half-line $\overrightarrow{p_1}$ coincides with the positive part of the coordinate axis x
2. oriented half-line $\overrightarrow{p_2}$ coincides with the positive part of the coordinate axis z.

Similarly to polar and cylindrical coordinates, the relation between spherical coordinates of a point in the space and its related Cartesian coordinates can be determined by the means of three continuous functions of three variables.

Let $[x, y, z]$ be the Cartesian coordinates and (ρ, φ, ζ) be the spherical coordinates of point M that is not on coordinate axis z. Then their relationships can be represented by the following equations

$$x = \rho \cos \varphi \sin \zeta, \quad y = \rho \sin \varphi \sin \zeta, \quad z = \rho \cos \zeta$$

and because $x^2 + y^2 + z^2 > 0$,

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\varphi = \arccos \frac{x}{\sqrt{x^2 + y^2}}, \; y \geq 0, \;\; \varphi = 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}}, \;\; y < 0$$

$$\zeta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

All points with the constant first spherical coordinate $\rho = a$, $a > 0$ are points on the sphere with the centre at the origin S and radius a. The spherical coordinates are appropriate mainly to describe solids enclosed by spheres.

Examples

1. The equation $\rho = 1$, for $0 \leq \varphi \leq 2\pi$, $\pi/2 \leq \zeta \leq \pi$ in spherical coordinates represents a half-sphere with the radius 1 in the half-space determined by the negative part of coordinate axis z. This can be described in Cartesian coordinates as the set of points whose coordinates satisfy the equation $z = -\sqrt{1-(x^2 + y^2)}$.

2. Equation $\varphi = \pi$ in spherical coordinates is the equation of the coordinate half-plane xz with a boundary line in coordinate axis z and determined by the negative part of coordinate axis x, defined in the Cartesian coordinates by equations $x = -\rho \sin \zeta$, $y = 0$, $z = \rho \cos \zeta$, for $0 \leq \rho < \infty$, $0 \leq \zeta \leq \pi$, and equation $\zeta = 0$ defines the positive half-line in coordinate axis z with the Cartesian coordinates $x = 0$, $y = 0$, $z = \rho$.

3. The region defined in spherical coordinates as

$$T^* = \left\{ (\rho, \; \varphi, z) \in \boldsymbol{R}^3 : 0 \leq \rho \leq 2, 0 \leq \varphi \leq 2\pi, \zeta = -\frac{\pi}{4} \right\}$$

is illustrated in Fig. 3.18, on the left, a patch of the cylindrical surface of revolution; the region defined as $T^* = \left\{ (\rho, \; \varphi, z) \in \boldsymbol{R}^3 : 1 \leq \rho \leq 2, \varphi = \pi, 0 \leq \zeta \leq \pi \right\}$ illustrated in Fig. 3.18, in the middle, is a half-annulus in the coordinate plane xz
and the region represented in spherical coordinates as

$$T^* = \left\{ (\rho, \; \varphi, z) \in \boldsymbol{R}^3 : 0 \leq \rho \leq 2, 0 \leq \varphi \leq 2\pi, \frac{\pi}{4} \leq \zeta \leq \frac{3\pi}{4} \right\}$$

illustrated in Fig. 3.18, on the right, is a ball with the centre at the origin S and radius 2, with removed double cylinder of revolution with spherical cap bounded by circles with radii 2 in the planes $z = \pm\sqrt{2}$.

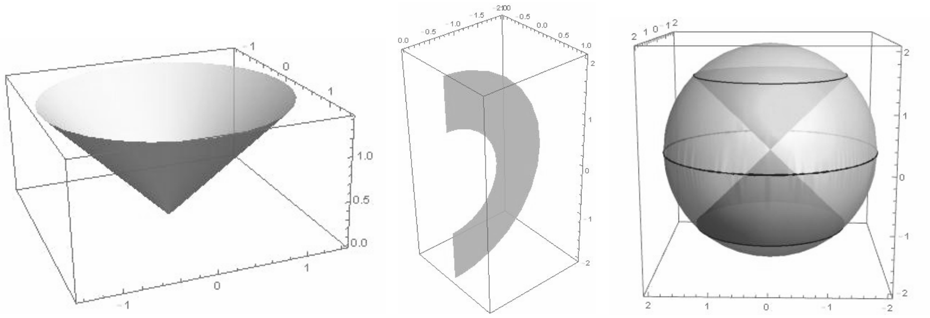


Fig. 3.18. Regions in space determined in spherical coordinates.

Transformation of the space with the spherical coordinate system to the space with the orthogonal Cartesian coordinate system represents so called spherical transformation of the space. The spherical transformation is a one-to-one and regular mapping defined on the set

$$\Omega^* = \{(\rho, \varphi, \zeta) \in \boldsymbol{R}^3 : 0 \le \rho < \infty, 0 \le \varphi < 2\pi, 0 < \zeta < \pi\}.$$

The Jacobi determinant (the Jacobian) of the spherical transformation is represented as follows

$$J(\rho,\ \varphi, \zeta) = \begin{vmatrix} \cos\varphi\sin\zeta & -\rho\sin\varphi\sin\zeta & \rho\cos\varphi\cos\zeta \\ \sin\varphi\sin\zeta & \rho\cos\varphi\sin\zeta & \rho\sin\varphi\cos\zeta \\ \cos\zeta & 0 & -\rho\sin\zeta \end{vmatrix} =$$

$$= \cos\zeta \begin{vmatrix} -\rho\sin\varphi\sin\zeta & \rho\cos\varphi\cos\zeta \\ \rho\cos\varphi\sin\zeta & \rho\sin\varphi\cos\zeta \end{vmatrix} - \rho\sin\zeta \begin{vmatrix} \cos\varphi\sin\zeta & -\rho\sin\varphi\sin\zeta \\ \sin\varphi\sin\zeta & \rho\cos\varphi\sin\zeta \end{vmatrix} =$$

$$= -\rho^2 \sin\zeta\cos^2\zeta - \rho^2\sin^3\zeta = -\rho^2\sin\zeta < 0\,.$$

If $T^* \subset \Omega^*$ is a closed measurable region, and function f of the three variables is continuous on the set $T = \Phi(T^*)$, then it holds

$$\iiint\limits_T f(x, y, y)\,dxdydz = \iiint\limits_{T^*} f(\rho\cos\varphi\sin\zeta, \rho\sin\varphi\sin\zeta, \rho\cos\zeta)\rho^2\sin\zeta\ d\rho d\varphi\, dz\,.$$

Examples

1. To evaluate the triple integral $\iiint\limits_T (x^2 + y^2 + z^2)dxdydz$ on the set T determined by the relations $x \ge 0$, $y \ge 0$, $z \ge 0$, $x^2 + y^2 + z^2 = 1$ is quite simple after transformation to spherical coordinates, receiving thus integral $\iiint\limits_{T^*} \rho^4 \sin\zeta\ d\rho d\varphi\, d\zeta$ over region $T^* = \left\{ (\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 0 \le \rho \le 1, 0 \le \varphi \le \dfrac{\pi}{2}, 0 \le \zeta \le \dfrac{\pi}{2} \right\}$, leading to the evaluation of the following integral

$$\iiint_T (x^2+y^2+z^2)\,dxdydz = \iiint_{T^*} \rho^4 \sin\zeta\, d\rho d\varphi\, d\zeta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^4 \sin\zeta\, d\rho\, d\varphi d\zeta =$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin\zeta \left[\frac{\rho^5}{5}\right]_0^1 d\varphi d\zeta = \frac{1}{5}\int_0^{\frac{\pi}{2}} \sin\zeta [\varphi]_0^{\frac{\pi}{2}} d\zeta = \frac{\pi}{10}\int_0^{\frac{\pi}{2}} \sin\zeta\, d\zeta = \frac{\pi}{10}[-\cos\zeta]_0^{\frac{\pi}{2}} = \frac{\pi}{10}.$$

2. The integral $\iiint_T (x^2+y^2)\,dxdydz$ on the set defined in the Cartesian coordinates $T = \left\{[x,y,z] \in \mathbf{E}^3 : 4 \le x^2+y^2+z^2 \le 9, z \ge 0\right\}$ can be simplified by transformation to the spherical coordinates, leading to an integration on the set $T^* = \left\{ (\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 2 \le \rho \le 3, 0 \le \varphi \le 2\pi, 0 \le \zeta \le \frac{\pi}{2} \right\}$, therefore

$$\iiint_T (x^2+y^2)\,dxdydz = \iiint_{T^*} \rho^4 \sin^3\zeta\, d\rho d\varphi d\zeta =$$

$$= \int_0^{2\pi}\int_2^3\int_0^{\frac{\pi}{2}} \rho^4 \sin\zeta (1-\cos^2\zeta)\, d\zeta\, d\rho\, d\varphi = \int_0^{2\pi}\int_2^3\int_0^{\frac{\pi}{2}} \rho^4 (\sin\zeta - \sin\zeta\cos^2\zeta)\, d\zeta\, d\rho\, d\varphi =$$

$$= \int_0^{2\pi}\int_2^3 \rho^4 \left[-\cos\zeta + \frac{1}{3}\cos^3\zeta\right]_0^{\frac{\pi}{2}} d\rho\, d\varphi = \int_0^{2\pi}\int_2^3 \rho^4 \left[0 - \left(-1 + \frac{1}{3}\right)\right] d\rho\, d\varphi =$$

$$= \frac{2}{3}\int_0^{2\pi}\int_2^3 \rho^4\, d\rho\, d\varphi = \frac{2}{3}\int_0^{2\pi}\left[\frac{\rho^5}{5}\right]_2^3 d\varphi = \frac{422}{15}\int_0^{2\pi} d\varphi = \frac{422}{15}[\varphi]_0^{2\pi} = \frac{844\pi}{15}.$$

3. The integral $\iiint_T z\,dxdydz$ on the set $T = \left\{[x,y,z] \in \mathbf{E}^3 : x^2+y^2+z^2 \le z\right\}$ can be simplified by transformation to the sphericaél coordinates, leading to integration on the set $T^* = \left\{ (\rho,\ \varphi, \zeta) \in \boldsymbol{R}^3 : 0 \le \rho \le \cos\zeta, 0 \le \varphi \le 2\pi, 0 \le \zeta \le \frac{\pi}{2} \right\}$ as follows

$$\iiint_T z\,dxdydz = \iiint_{T^*} \rho^3 \sin\zeta\cos\zeta\, d\rho d\varphi d\zeta = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\cos\zeta} \rho^3 \sin\zeta\cos\zeta\, d\rho\, d\zeta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\zeta\cos\zeta \left[\frac{\rho^4}{4}\right]_0^{\cos\zeta} d\zeta\, d\varphi = \frac{1}{4}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\zeta\cos^5\zeta\, d\zeta\, d\varphi =$$

$$= \frac{1}{4}\int_0^{2\pi}\left[-\frac{1}{6}\cos^6\zeta\right]_0^{\frac{\pi}{2}} d\varphi = -\frac{1}{24}\int_0^{2\pi} d\varphi = -\frac{1}{24}[\varphi]_0^{2\pi} = -\frac{\pi}{12}.$$

Region T is ball bounded by sphere with Cartesian equation $x^2 + y^2 + z^2 - z = 0$, which can be rewritten as $x^2 + y^2 + (z - 0.5)^2 = 0.25$, with radius 0.5 and its centre at the point [0, 0, 0.5] on the coordinate axis z. Equation of this sphere in spherical coordinates is $\rho = \cos\zeta$, for $\zeta \in \langle 0, \pi/2\rangle$.

3.10 Applications of multiple integrals

Double integrals were primarily introduced for the calculation of volume. The most natural application of double integrals, which can be easily visualized geometrically, is that if $f(x, y)$ is a non-negative (continuous) function defined on a region $R \subset \mathbf{E}^2$, then the double integral of function f over region R represents the volume of a solid S bounded by the graph of f, surface $z = f(x, y)$ form above, by the coordinate plane xy from below and laterally by a cylindrical surface generated by vertical lines passing through all points of the boundary of R,

$$V(S) = \iint_R f(x,y)\,dx\,dy .$$

Examples

1. Volume of a tetrahedron bounded by coordinate planes and plane $z = 4 - 4x - 2y$ in Fig. 3.19, left, can be evaluated as a double integral over the region

$$R = \left\{[x,y] \in \mathbf{E}^2 : 0 \le x \le 1,\, 0 \le y \le 2-2x\right\},$$

$$V = \iint_R f(x,y)\,dx\,dy = \int_0^1 \int_0^{2-2x} (4-4x-2y)\,dy\,dx = \int_0^1 \left[4y - 4xy - y^2\right]_0^{2-2x} dx =$$

$$= \int_0^1 \left[4(2-2x) - 4x(2-2x) - (2-2x)^2\right] dx = 4\int_0^1 (x^2 - 2x + 1)\,dx = 4\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{4}{3}.$$

2. Volume of a solid in the first octant that is bounded from above by the paraboloid $z = 12 - (x^2 + 3y^2)$ and laterally by the surfaces $y = x^2$ and $y = 2 - x^2$ illustrated in Fig. 3.19, middle, equals to the double integral over the region

$$R = \left\{[x,y] \in \mathbf{E}^2 : 0 \le x \le 1,\, x^2 \le y \le 2 - x^2\right\},$$

$$V = \iint_R f(x,y)\,dx\,dy = \int_0^1 \int_{x^2}^{2-x^2} [12 - (x^2 + 3y^2)]\,dy\,dx = \int_0^1 \left[12y - x^2 y - y^3\right]_{x^2}^{2-x^2} dx =$$

$$= \int_0^1 \left[24 - 12x^2 - 2x^2 + x^4 - (2-x^2)^3 - (12x^2 - x^4 - x^6)\right] dx =$$

$$= \int_0^1 (16 - 14x^2 - 4x^4 + 2x^6)\,dx = \left[16x - \frac{14x^3}{3} - \frac{4x^5}{5} + \frac{2x^7}{7}\right]_0^1 = \frac{1136}{105}.$$

3. The solid bounded by the surface $z = e^{y-x}$, plane $x + y = 1$ and coordinate planes, Fig. 3.19, right, has its volume represented by the double integral

$$V = \int_0^1 \int_0^{1-x} e^{y-x}\,dy\,dx = \int_0^1 \left[e^{y-x}\right]_0^{1-x} dx = \int_0^1 (e^{1-2x} - e^{-x})\,dx =$$

$$= \left[-\frac{e^{1-2x}}{2} + e^{-x}\right]_0^1 = -\frac{e^{-1}}{2} + e^{-1} - \left(-\frac{e}{2} + 1\right) = \frac{e^{-1}}{2} + \frac{e}{2} - 1 = \frac{(e-1)^2}{2e}.$$

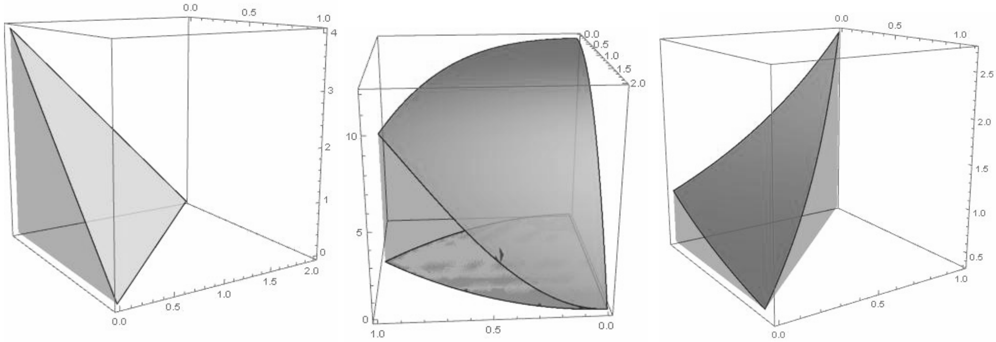


Fig. 3.19. Solids bounded by surface patches and planes.

4. The relationship between the volumes of other basic geometric solids (Fig. 3.20) that were discovered by Archimedes (287 BC - 212 BC) are the following:

   A2. Paraboloid has a volume of three-halves of the inscribed cone and one half of the circumscribed cylinder.

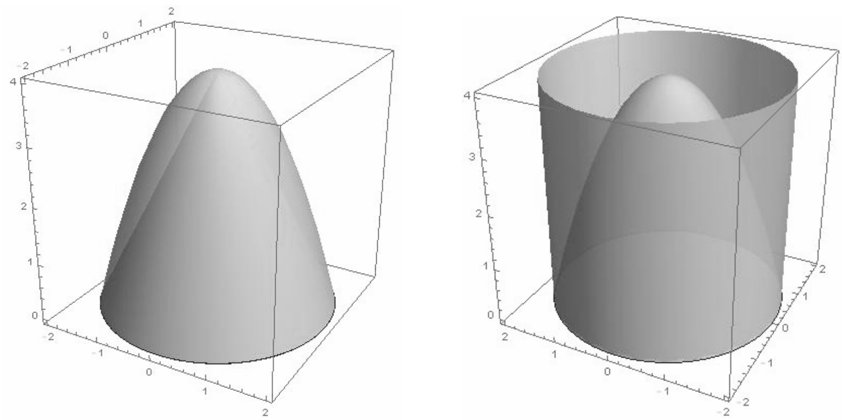


Fig. 3.20. Archimedean problems A2.

Consider the paraboloid $z = 4 - (x^2 + y^2)$ and the inscribed cone $z = 4 - 2\sqrt{x^2 + y^2}$ with a common disc $x^2 + y^2 \le 4$ in the coordinate plane xy. Their volumes can be evaluated by means of a transformation to polar coordinates over the region

$$R^* = \left\{[\rho,\varphi] \in \boldsymbol{R}^2 : 0 \le \rho \le 2,\, 0 \le \varphi \le 2\pi\right\},$$

$$V(P) = \iint_R \left(4 - x^2 - y^2\right) dxdy = \int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4 - x^2 - y^2\right) dy\right) dx = \iint_{R^*} (4 - \rho^2)\rho\, d\rho\, d\varphi =$$

$$\int_0^{2\pi}\int_0^{2} (4\rho - \rho^3)\, d\rho\, d\varphi = \int_0^{2\pi}\left[2\rho^2 - \frac{\rho^4}{4}\right]_0^2 d\varphi = \int_0^{2\pi} 4 d\varphi = 8\pi\,,$$

$$V(CI)=\iint_R\left(4-2\sqrt{x^2+y^2}\right)dx\,dy=\iint_{R^*}\rho(4-2\rho)\,d\rho\,d\varphi=\int_0^{2\pi}\int_0^{2}(4\rho\text{ - }2\rho^2)\,d\rho\,d\varphi=$$

$$=\int_0^{2\pi}\left[2\rho^2-\frac{2\rho^3}{3}\right]_0^2 d\varphi=\frac{8}{3}\int_0^{2\pi}d\varphi=\frac{16\pi}{3}\,,$$

$$V(CS)=\iint_R 4\,dx\,dy=\int_{-2}^{2}\left(\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}4\,dy\right)dx=\int_0^{2\pi}\int_0^{2}4\rho\,d\rho\,d\varphi=[\varphi]_0^{2\pi}.\left[2\rho^2\right]_0^2=16\pi\,\cdot$$

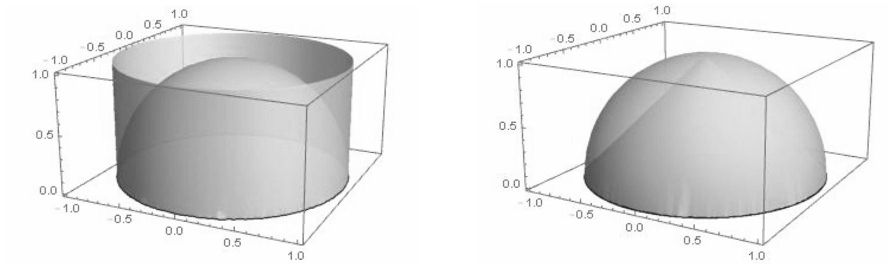


Fig. 3.21. Archimedean problems A3.

A3. Semi-sphere has a volume of two-thirds of the circumscribed cylinder and double the volume of the inscribed cone.

The volume of a semi-sphere with the centre at the origin and radius *r* can be calculated as a double integral $V(S)=\iint_R\sqrt{r^2-x^2-y^2}\,dxdy$ over the region

$$R=\left\{[x,y]\in\mathbf{E}^2:-r\le x\le r,-\sqrt{r^2-x^2}\le y\le\sqrt{r^2-x^2}\right\}.$$

Using the transformation to polar coordinates calculation can be simplified to an integration over region $R^*=\left\{[\rho,\varphi]\in\boldsymbol{R}^2:0\le\rho\le r,\,0\le\varphi\le 2\pi\right\}$,

$$V(S)=\iint_{R^*}\sqrt{r^2-\rho^2}\,\rho\,d\rho\,d\varphi=\int_0^{2\pi}\int_0^{r}\sqrt{r^2-\rho^2}\,\rho\,d\rho\,d\varphi=2\pi\left[-\frac{\sqrt{\left(r^2-\rho^2\right)^3}}{3}\right]_0^{r}=\frac{2\pi r^3}{3}\,.$$

Volume of a cylinder circumscribed to the above semi-sphere equals to the integral

$$V(CC)=\iint_R r\,dxdy\,,\ R=\left\{[x,y]\in\mathbf{E}^2:-r\le x\le r,-\sqrt{r^2-x^2}\le y\le\sqrt{r^2-x^2}\right\}.$$

Using the transformation to polar coordinates and integration over the region

$$R^*=\left\{[\rho,\varphi]\in\boldsymbol{R}^2:0\le\rho\le r,\,0\le\varphi\le 2\pi\right\},$$

the circumscribed cylinder volume equals

$$V(CC)=\iint_{R^*}r\rho\,d\rho\,d\varphi=\int_0^{2\pi}\int_0^{r}r\rho\,d\rho\,d\varphi=\int_0^{2\pi}\left[-\frac{r\rho^2}{2}\right]_0^{r}d\varphi=\frac{1}{2}\int_0^{2\pi}r^3d\varphi=\pi r^3\,.$$

The volume of a cone inscribed to the above semi-sphere is

$$V(CI)=\iint_R\left(r-\sqrt{x^2+y^2}\right)dx\,dy,$$

$$R=\left\{[x,y]\in\mathbf{E}^2:-r\le x\le r,-\sqrt{r^2-x^2}\le y\le\sqrt{r^2-x^2}\right\}.$$

Transforming to polar coordinates as above we obtain

$$V(CI)=\iint_{R^*}(r-\rho)\rho\,d\rho\,d\varphi=\int_0^{2\pi}\int_0^r(r\rho-\rho^2)d\rho\,d\varphi=$$

$$=\int_0^{2\pi}\left[\frac{r\rho^2}{2}-\frac{\rho^3}{3}\right]_0^r d\varphi=\frac{1}{6}\int_0^{2\pi}r^3d\varphi=\frac{\pi r^3}{3}.$$

From the geometric meaning of double integrals it easily follows that if $f(x, y)$ and $g(x, y)$ are two continuous functions defined on a region $R\subset\mathbf{E}^2$ and such that for each $[x, y]\in R$ holds $f(x, y)\le g(x, y)$, then the volume of the solid S enclosed by the graph of function $f(x, y)$ from above, by the graph of function $g(x, y)$ from below and laterally by a cylindrical surface generated by vertical lines passing through all points of the boundary of R is

$$V(S)=\iint_R\big(g(x,y)-f(x,y)\big)dx\,dy.$$

Examples

1. The volume of a solid bounded by the conical surface $z^2 = x^2 + y^2$ and by the cylindrical surface $x^2 + y^2 = 4$ in Fig. 3.22, left, can be calculated, due to its symmetry with respect to the coordinate plane $z = 0$, as the double integral $V=2\iint_R\sqrt{x^2+y^2}\,dx\,dy$, $R=\left\{[x,y]\in\mathbf{E}^2:-2\le x\le 2,-\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\right\}$.

   This integral can be simplified using polar coordinates to an integral on the region

   $$R^*=\left\{[\rho,\varphi]\in\boldsymbol{R}^2:0\le\rho\le 2,0\le\varphi\le 2\pi\right\},$$

   $$V=2\int_0^2\int_0^{2\pi}\rho^2\,d\varphi\,d\rho=2\int_0^2\rho^2[\varphi]_0^{2\pi}d\rho=4\pi\int_0^2\rho^2d\rho=\frac{4\pi}{3}\left[\rho^3\right]_0^2=\frac{32\pi}{3}.$$

2. The solid with the boundary in the conical surface $z=\sqrt{x^2+y^2}$ and hemisphere $z=\sqrt{4-\left(x^2+y^2\right)}$ depicted in Fig. 3.22, in the middle, has the volume equal to the value of the double integral $V=\iint_R\left(\sqrt{4-\left(x^2+y^2\right)}-\sqrt{x^2+y^2}\right)dx\,dy$ on the region

   $$R=\left\{[x,y]\in\boldsymbol{E}^2:-\sqrt{2}\le x\le\sqrt{2},-\sqrt{2-x^2}\le y\le\sqrt{2-x^2}\right\}.$$

   Simplification using polar coordinates yields an integration over the simple transformed region $R^*=\left\{[\rho,\varphi]\in\boldsymbol{R}^2:0\le\rho\le\sqrt{2},0\le\varphi\le 2\pi\right\}$, therefore

$$V = \int_0^{\sqrt{2}}\int_0^{2\pi}\left(\sqrt{4-\rho^2}-\rho\right)\rho\, d\varphi\, d\rho = \int_0^{\sqrt{2}}\rho\left(\sqrt{4-\rho^2}-\rho\right)[\varphi]_0^{2\pi}d\rho =$$

$$= 2\pi\int_0^{\sqrt{2}}\left(\rho\sqrt{4-\rho^2}-\rho^2\right)d\rho = \pi\int_0^{\sqrt{2}}2\rho\sqrt{4-\rho^2}\, d\rho - 2\pi\int_0^{\sqrt{2}}\rho^2\, d\rho =$$

$$= \frac{2\pi}{3}\left[-\sqrt{\left(4-\rho^2\right)^3}\right]_0^{\sqrt{2}} - \frac{2\pi}{3}\left[\rho^3\right]_0^{\sqrt{2}} = \frac{2\pi}{3}\left(-\sqrt{2^3}+8-\sqrt{2^3}\right) = \frac{8\pi}{3}\left(2-\sqrt{2}\right).$$

3. By means of double integration the volume of a solid in the first octant given by inequalities $x + y \le z \le 2$, the tetrahedron in Fig. 3.22, right, can be calculated as

$$V = \int_0^2\int_0^{2-x}(2-x-y)\, dy\, dx = \int_0^2\left[(2-x)y-\frac{y^2}{2}\right]_0^{2-x}dx = \int_0^2\frac{(2-x)^2}{2}\,dx =$$

$$= \int_0^2\left(2-2x+\frac{x^2}{2}\right)dx = \left[2x-x^2+\frac{x^3}{6}\right]_0^2 = \frac{4}{3}.$$

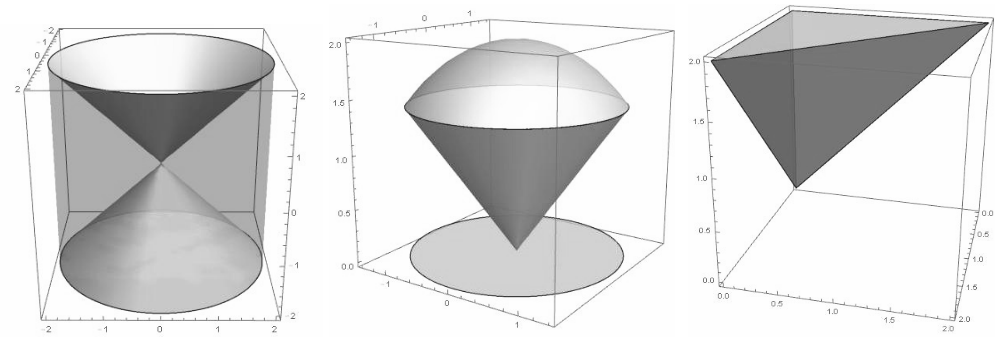


Fig. 3.22. Volumes of solids.

Double integrals can be also used for the calculation of the area, both of a surface patch, or a plane region. For the purpose of the area of a region R in the plane xy we consider the solid consisting of points between the plane $z = 1$ and region R in the plane $z = 0$. The volume V of this solid is equal to the value of the double integral of function f over region R, and simultaneously this volume equals to the product of the area of the region R and the height of this solid, $V = A(R)\,.\,1$, from which immediately follows that

$$A(R) = \iint_R 1\, dx\, dy\,.$$

Examples

1. Area of a region enclosed by curves $y^2 = 9 - x$ and $y = 3 - x$ in Fig. 3.23, left,

$$R = \left\{[x, y] \in \mathbf{E}^2 : 3 - y \le x \le 9 - y^2, -2 \le y \le 3\right\}$$

equals to the value of the double integral

$$A(R)=\int_{-2}^{3}\int_{3-y}^{9-y^2}1\,dx\,dy=\int_{-2}^{3}[x]_{3-y}^{9-y^2}dy=\int_{-2}^{3}\left(6+y-y^2\right)dy=\left[6y+\frac{y^2}{2}-\frac{y^3}{3}\right]_{-2}^{3}=\frac{125}{6}.$$

2. The area of the region R bounded by graphs of functions $y = \sin x$ and $y = \cos x$ on interval $0 \le x \le \pi/4$ equals to the double integral

$$A(R)=\int_{0}^{\frac{\pi}{4}}\int_{\sin x}^{\cos x}1\,dy\,dx=\int_{0}^{\frac{\pi}{4}}[y]_{\sin x}^{\cos x}dx=\int_{0}^{\frac{\pi}{4}}(\cos x-\sin x)dx=[\sin x+\cos x]_{0}^{\frac{\pi}{4}}=\sqrt{2}-1.$$

3. Transformation to polar coordinates can be used for the calculation of the area of the region enclosed by the three-level rose $\rho = \sin 3\varphi$ (see Fig. 3.23, middle). One sixth of the entire region lies between the lines through origin forming the polar angles $\varphi = 0$ and $\varphi = \pi/6$ with the polar axis, Fig. 3.22, right. Using the symmetry, area of the region can be evaluated as

$$A(R)=6\int_{0}^{\frac{\pi}{6}}\int_{0}^{\sin 3\varphi}\rho\,d\rho\,d\varphi=6\int_{0}^{\frac{\pi}{6}}\left[\frac{\rho^2}{2}\right]_{0}^{\sin 3\varphi}d\varphi=3\int_{0}^{\frac{\pi}{6}}\sin^2 3\varphi\,d\varphi=$$

$$=\frac{3}{2}\int_{0}^{\frac{\pi}{6}}(1-\cos 6\varphi)\,d\varphi=\frac{3}{2}\left[\varphi-\frac{1}{6}\sin 6\varphi\right]_{0}^{\frac{\pi}{6}}=\frac{\pi}{4}.$$

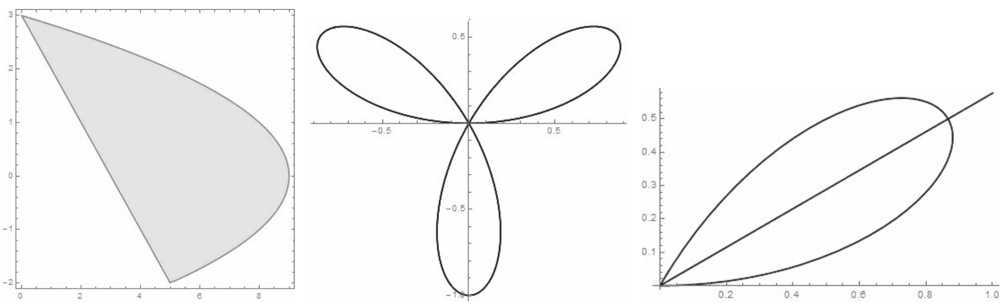


Fig. 3.23. Integration regions.

If a function $f(x, y)$ has continuous first partial derivatives on a closed region $R \subset \mathbf{E}^2$, then the area of a surface patch S given by the equation $z = f(x, y)$, $[x, y] \in R$ equals

$$A(S)=\iint_R\sqrt{1+\left(f_x'(x,y)\right)^2+\left(f_y'(x,y)\right)^2}\,dx\,dy.$$

This formula can be used for the calculation of a surface area of any surface patch determined as graph of function of two variables differentiable on a closed measurable region in $\mathbf{E}^2$.

Examples

1. The area of a paraboloid patch $z = x^2 + y^2$ over the region

   $R = \left\{[x, y] \in \mathbf{E}^2 : -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\right\}$, Fig. 3. 24, left,

   equals to the value of the double integral over this region from function

   $\sqrt{1+\left(f'_x(x,y)\right)^2+\left(f'_y(x,y)\right)^2}$, while

   $f(x,y) = x^2 + y^2,\ f'_x(x,y) = 2x,\ f'_y(x,y) = 2y$, therefore

   $A(S) = \int\limits_{-1}^{1} \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+4x^2+4y^2}\,dy\,dx$. This integral can be simplified by means of polar transformation to double integral over the region

   $R^* = \left\{[\rho, \varphi] \in \boldsymbol{R}^2 : 0 \le \rho \le 1, 0 \le \varphi \le 2\pi\right\}$,

   $$A(S) = \int\limits_0^1 \int\limits_0^{2\pi} \sqrt{1+4\rho^2}\,\rho\,d\varphi\,d\rho = \frac{1}{8}[\varphi]_0^{2\pi} \int\limits_0^1 8\rho\sqrt{1+4\rho^2}\,d\rho = \frac{\pi}{4}\int\limits_0^1 8\rho\sqrt{1+4\rho^2}\,d\rho =$$

   $$= \frac{\pi}{4}\left[\frac{2}{3}\sqrt{\left(1+4\rho^2\right)^3}\right]_0^1 = \frac{\pi}{6}\left(5\sqrt{5}-1\right).$$

2. The area of the cylindrical surface patch $x^2 + z^2 = 4$ above the rectangular region $R = \langle 0, 1\rangle \times \langle 0, 4\rangle \subset \mathbf{E}^2$ illustrated in Fig. 3.24, right, equals to

   $$A(S) = \int\limits_0^1 \int\limits_0^4 \sqrt{1+\frac{x^2}{4-x^2}}\,dydx = [y]_0^4 \int\limits_0^1 \frac{2}{\sqrt{4-x^2}}\,dx = 4\left[\arcsin\frac{x}{2}\right]_0^1 = \frac{8\pi}{3}.$$

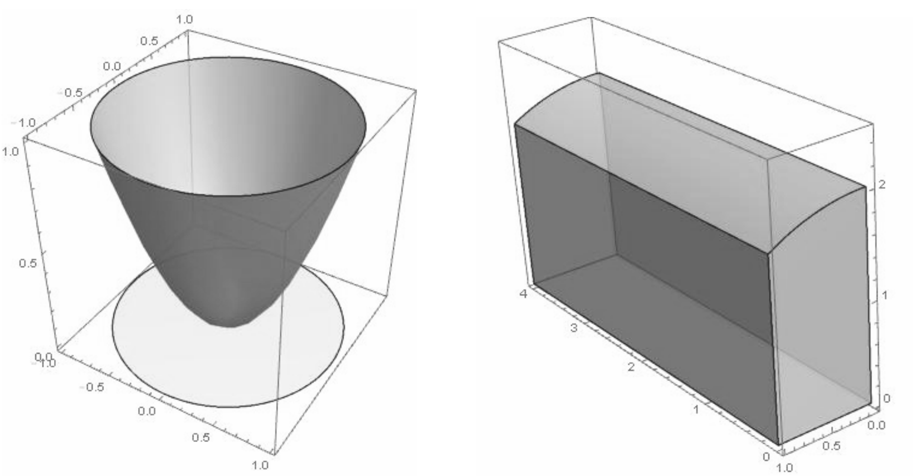


Fig. 3.24. Surface patches as graphs of functions $f(x, y)$ on set R.

Let us consider a thin plate (lamina) in $\mathbf{E}^2$ occupying a regular region R. The plate is supposed to be sufficiently thin so that the mass density is a function of only two variables, namely x and y, denoted as $\sigma(x, y)$. Then the total mass M of this plate is

$$M = \iint_R \sigma(x,y)\,dx\,dy\,.$$

By means of the double integration we can also find other physical characteristics of the plate, for example its static moments about the coordinate axes and the coordinates of its centre of mass (centre of gravity).
The static (first) moments about coordinate axes of the plate are

$$S_x = \iint_R y\,.\,\sigma(x,y)\,dx\,dy, \quad S_y = \iint_R x\,.\,\sigma(x,y)\,dx\,dy\,.$$

Denoting $T = [x_T, y_T] \in \mathbf{E}^2$ the centre of mass of the plate, then the coordinates of T are computed as follows

$$x_T = \frac{S_y}{M} = \frac{\iint_R x.\sigma(x,y)\,dx\,dy}{\iint_R \sigma(x,y)\,dx\,dy}, \quad y_T = \frac{S_x}{M} = \frac{\iint_R y.\sigma(x,y)\,dx\,dy}{\iint_R \sigma(x,y)\,dx\,dy}\,.$$

In the special case of a homogeneous region (lamina) R, $\sigma(x, y)$ = const., the centre of gravity is called the centroid of the region, and its coordinates are consequently calculated as

$$x_T = \frac{1}{A(R)}\iint_R x\,dx\,dy, \quad y_T = \frac{1}{A(R)}\iint_R y\,dx\,dy\,.$$

Examples

1. The lamina bounded by coordinate axis x, line $x = 1$ and curve $y = \sqrt{x}$ with density $\sigma(x, y) = x + y$ illustrated in Fig. 3. 25, left, has a mass equal to the value of the double integral over the region $R = \{[x,y] \in \mathbf{E}^2 : 0 \le x \le 1,\, 0 \le y \le \sqrt{x}\}$,

$$M = \int_0^1\int_0^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^1 \left(\sqrt{x^3} + \frac{x}{2}\right) dx = \left[\frac{2}{5}\sqrt{x^5} + \frac{x^2}{4}\right]_0^1 = \frac{13}{20}\,.$$

Coordinates of the centre of gravity of this lamina are

$$x_T = \frac{\int_0^1\int_0^{\sqrt{x}} x(x+y)\,dy\,dx}{\frac{13}{20}} = \frac{20}{13}\int_0^1\int_0^{\sqrt{x}} \left(x^2 + xy\right)dy\,dx = \frac{20}{13}\int_0^1 \left[x^2y + \frac{xy^2}{2}\right]_0^{\sqrt{x}} dx =$$

$$= \frac{20}{13}\int_0^1 \left(\sqrt{x^5} + \frac{x^2}{2}\right)dx = \frac{20}{13}\left[\frac{2}{7}\sqrt{x^7} + \frac{x^3}{6}\right]_0^1 = \frac{20}{13}.\left(\frac{2}{7} + \frac{1}{6}\right) = \frac{190}{273}\,,$$

$$y_T = \frac{\int_0^1 \int_0^{\sqrt{x}} y(x+y)\,dy\,dx}{\frac{13}{20}} = \frac{20}{13}\int_0^1 \int_0^{\sqrt{x}} \left(xy + y^2\right) dy\,dx = \frac{20}{13}\int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_0^{\sqrt{x}} dx =$$

$$= \frac{20}{13}\int_0^1 \left(\frac{x^2}{2} + \frac{\sqrt{x^3}}{3}\right) dx = \frac{20}{13}\left[\frac{x^3}{6} + \frac{2\sqrt{x^5}}{15}\right]_0^1 = \frac{20}{13}.\left(\frac{1}{6} + \frac{2}{15}\right) = \frac{6}{13}.$$

2. The mass of a rectangular lamina with vertices [0, 0], [0, 2], [3, 0], [?, ?] in $\mathbf{E}^2$, Fig. 3.25, middle, and density $\sigma(x, y) = xy^2$ can be calculated as the double integral over the region $R = \{[x, y] \in \boldsymbol{E}^2 : 0 \le x \le 3, 0 \le y \le 2\}$,

$$M = \int_0^3 \int_0^2 xy^2 dydx = \left[\frac{y^3}{3}\right]_0^2 . \left[\frac{x^2}{2}\right]_0^3 = \frac{8}{3} . \frac{9}{2} = 12 .$$

Coordinates of its centre of gravity are

$$x_T = \frac{\int_0^3 \int_0^2 x^2 y^2\, dy\, dx}{12} = \frac{1}{12}\left[\frac{y^3}{3}\right]_0^2 . \left[\frac{x^3}{3}\right]_0^3 dx = \frac{1}{12} . \frac{8}{3} . 9 = 2 ,$$

$$y_T = \frac{\int_0^3 \int_0^2 xy^3\, dy\, dx}{12} = \frac{1}{12}\left[\frac{y^4}{4}\right]_0^2 . \left[\frac{x^2}{2}\right]_0^3 dx = \frac{1}{12} . 4 . \frac{9}{2} = \frac{3}{2} .$$

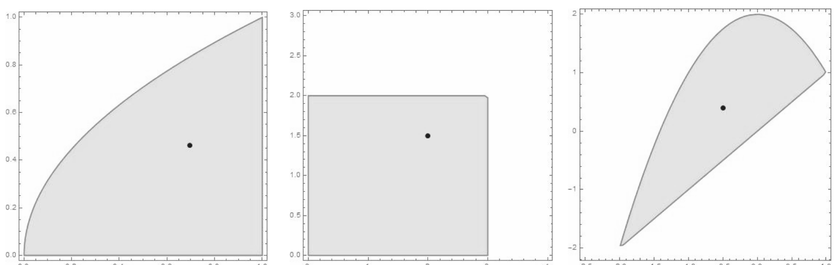


Fig. 3.25. Laminas with centre of gravity in plane $\mathbf{E}^2$.

3. The centroid of a planar homogeneous region enclosed between line $y = x$ and parabola $y = 2 - x^2$ in Fig. 3.25, right, can be found as

$$x_T = \frac{\int_{-2}^{1} \int_{x}^{2-x^2} xdydx}{\int_{-2}^{1} \int_{x}^{2-x^2} 1dydx} = \frac{\int_{-2}^{1} x[y]_x^{2-x^2} dx}{\int_{-2}^{1} [y]_x^{2-x^2} dx} = \frac{\int_{-2}^{1} \left(2x - x^2 - x^3\right) dx}{\int_{-2}^{1} \left(2 - x - x^2\right) dx} = \frac{\left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_{-2}^{1}}{\left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}} = -\frac{1}{2} ,$$

$$y_T=\frac{\int\limits_{-2}^{1}\int\limits_{x}^{2-x^2} y\,dy\,dx}{\int\limits_{-2}^{1}\int\limits_{x}^{2-x^2} 1\,dy\,dx}=\frac{\int\limits_{-2}^{1}\left[\frac{y^2}{2}\right]_x^{2-x^2}dx}{\int\limits_{-2}^{1}[y]_x^{2-x^2}dx}=\frac{\int\limits_{-2}^{1}\left(\frac{(2-x^2)^2}{2}-\frac{x^2}{2}\right)dx}{\int\limits_{-2}^{1}\left(2-x-x^2\right)dx}=\frac{\left[2x-\frac{5x^3}{6}+\frac{x^5}{10}\right]_{-2}^{1}}{\left[2x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-2}^{1}}=\frac{2}{5}\,.$$

If $R\subset \boldsymbol{E}^3$ is a regular region (of any type), then its volume $V(R)$ is computed by the triple integral

$$V(R)=\iiint\limits_R 1\,dx\,dy\,dz\,.$$

Examples

1. Volume of region from Fig. 3.9, right, described as regular region of type *yxz*

$$M_{yx}=\left\{[x,y]\in \mathbf{E}^2\!: -\sqrt{2}\le y\le\sqrt{2}, -\sqrt{2-y^2}\le x\le\sqrt{2-y^2}\right\},$$
$$R_{yxz}=\left\{[x,y,z]\in \mathbf{E}^3\!: [x,y]\in M_{yx}, x^2+y^2\le z\le 4-\left(x^2+y^2\right)\right\},$$

can be calculated as the triple integral

$$V(R_{yxz})=\iiint\limits_{R_{yxz}} 1\,dx\,dy\,dz=\int\limits_{-\sqrt{2}}^{\sqrt{2}}\int\limits_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int\limits_{x^2+y^2}^{4-(x^2+y^2)} 1\,dz\,dx\,dy$$

transformed to cylindrical coordinates to an integral over the region

$$T^*=\left\{(\rho,\ \varphi,z)\in \boldsymbol{R}^3:0\le\rho\le\sqrt{2},\, 0\le\varphi\le 2\pi,\, \rho^2\le z\le 4-\rho^2\right\}$$

$$V(R_{yxz})=\iiint\limits_{T^*}\rho\,d\rho\,d\varphi\,dz=\int\limits_0^{\sqrt{2}}\int\limits_0^{2\pi}\int\limits_{\rho^2}^{4-\rho^2}\rho\,dz\,d\varphi\,d\rho=\int\limits_0^{\sqrt{2}}\int\limits_0^{2\pi}\left(4\rho-2\rho^3\right)d\varphi\,d\rho=$$

$$=4\pi\int\limits_0^{\sqrt{2}}\left(2\rho-\rho^3\right)d\rho=4\pi\left[\rho^2-\frac{\rho^4}{4}\right]_0^{\sqrt{2}}=4\pi(2-1)=4\pi\,.$$

2. The space region illustrated in Fig. 3.18, on the right and described in spherical coordinates as $T^*=\left\{(\rho,\ \varphi,z)\in \boldsymbol{R}^3:0\le\rho\le 2,\, 0\le\varphi\le 2\pi,\, \frac{\pi}{4}\le\zeta\le\frac{3\pi}{4}\right\}$ has volume equal to the triple integral

$$V(T^*)=\iiint\limits_{T^*}\rho^2\sin\zeta\,d\rho\,d\varphi\,d\zeta=\int\limits_0^{2}\int\limits_0^{2\pi}\int\limits_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\rho^2\sin\zeta\,d\zeta\,d\varphi\,d\rho=$$

$$=\left[-\cos\zeta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\,.\,[\varphi]_0^{2\pi}\,.\left[\frac{\rho^3}{3}\right]_0^{2}=\frac{2}{\sqrt{2}}\,.\,2\pi\,.\,\frac{8}{3}=\frac{16\sqrt{2}\,\pi}{3}\,.$$

3. Solid S given by inequalities $3x^2 + 3y^2 \le z \le 1 - x^2 - y^2$ is bounded by two paraboloids with common axis in coordinate axis z sharing one circle with equation $3x^2 + 3y^2 = 1 - x^2 - y^2$, which yields $x^2 + y^2 = ¼$. Their common circle with radius ½ is located in the plane $z = ¾$. Due to the symmetric form of this solid, see Fig. 3.26, left, its volume $V(S)$ can be calculated over the region S_4

$$S_4 = \left\{ [x, y, z] \in \boldsymbol{E}^3 : 0 \le x \le \frac{1}{2},\, 0 \le y \le \sqrt{\frac{1}{4} - x^2},\, 3x^2 + 3y^2 \le z \le 1 - \left(x^2 + y^2\right) \right\},$$

$$V(S) = 4\iiint_{S_4} 1\,dx\,dy\,dz = 4\int_0^{\frac{1}{2}} \int_0^{\sqrt{\frac{1}{4}-x^2}} \int_{3x^2+3y^2}^{1-\left(x^2+y^2\right)} 1\,dz\,dy\,dx = 4\int_0^{\frac{1}{2}} \int_0^{\sqrt{\frac{1}{4}-x^2}} \left(1 - 4x^2 - 4y^2\right) dy\,dx =$$

$$= 4\int_0^{\frac{1}{2}} \left[\left(1 - 4x^2\right) y - \frac{4y^3}{3} \right]_0^{\sqrt{\frac{1}{4}-x^2}} dx = \frac{4}{3}\int_0^{\frac{1}{2}} \left(1 - 4x^2\right)\sqrt{1 - 4x^2}\,dx =$$

$$= \frac{4}{3}\int_0^{\frac{1}{2}} \frac{\left(1 - 4x^2\right)^2}{\sqrt{1 - 4x^2}}\,dx = \left[\frac{x\left(5 - 8x^2\right)\sqrt{1 - 4x^2}}{6} + \frac{\arcsin 2x}{4} \right]_0^{\frac{1}{2}} = \frac{\pi}{8}$$

Using cylindrical coordinates, the calculation is much simpler

$$V(S) = \iiint_{S^*} \rho\,d\rho\,d\varphi\,dz = \int_0^{\frac{1}{2}} \int_0^{2\pi} \int_{3\rho^2}^{1-\rho^2} \rho\,dz\,d\varphi\,d\rho = \int_0^{\frac{1}{2}} \rho\left(1 - 4\rho^2\right).\left[\varphi\right]_0^{2\pi} d\rho =$$

$$= 2\pi\left[\frac{\rho^2}{2} - \rho^4 \right]_0^{\frac{1}{2}} = 2\pi\left(\frac{1}{8} - \frac{1}{16} \right) = \frac{\pi}{8}.$$

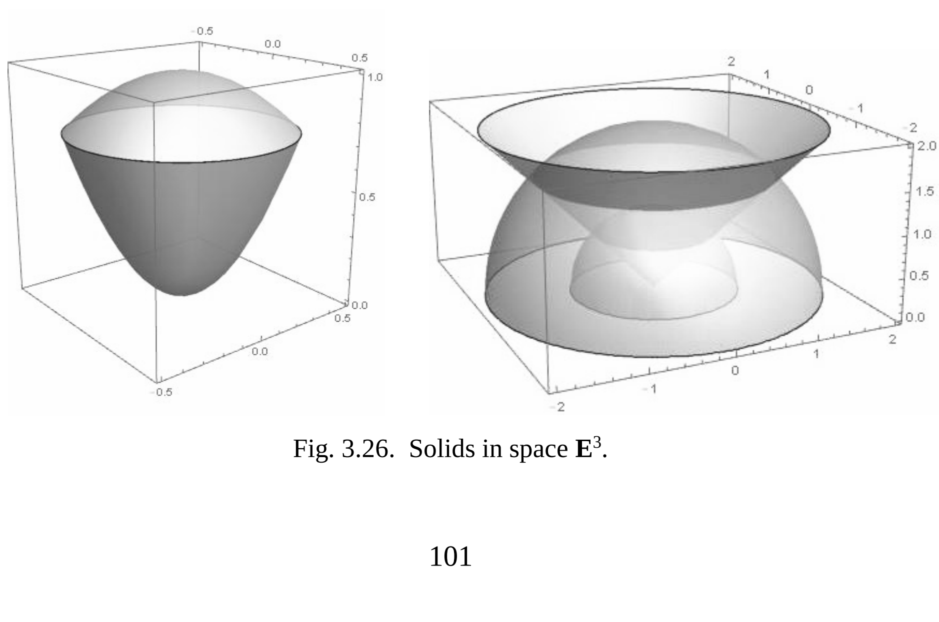


Fig. 3.26. Solids in space $\mathbf{E}^3$.

If a physical body occupies a regular region $R \subset \mathbf{E}^3$ and its point density is given by the function $\sigma(x, y, z)$, then the total mass of the body is

$$M = \iiint_R \sigma(x, y, z)\, dx\, dy\, dz \,.$$

If we denote by $T = [x_T,\ y_T,\ z_T] \in \mathbf{E}^3$ the centre of mass of the body R, then the coordinates of T are computed as follows

$$x_T = \frac{1}{M} \iiint_R x.\sigma(x, y, z)\, dx\, dy\, dz$$

$$y_T = \frac{1}{M} \iiint_R y\ .\sigma(x, y, z)\, dx\, dy\, dz$$

$$z_T = \frac{1}{M} \iiint_R z.\, \sigma(x, y, z)\, dx\, dy\ dz \,.$$

By analogy with a 2D case, for a homogeneous body R we speak about its centroid.

Examples

1. The total mass of the region from Fig. 3.9, right

$$M_{yx} = \left\{ [x, y] \in \boldsymbol{E}^2 \!: -\sqrt{2} \le y \le \sqrt{2}, -\sqrt{2 - y^2} \le x \le \sqrt{2 - y^2} \right\},$$

$$R_{yxz} = \left\{ [x, y, z] \in \boldsymbol{E}^3 \!: [x, y] \in M_{xy}, x^2 + y^2 \le z \le 4 - \left(x^2 + y^2\right) \right\},$$

with the point density $\sigma(x, y, z) = z$ can be calculated as the triple integral

$$M = \iiint_{R_{yxz}} z\, dx\, dy\, dz = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{x^2+y^2}^{4-\left(x^2+y^2\right)} z\, dz\, dx\, dy \,,$$

and transformed to cylindrical coordinates as an integral over the region

$$T^* = \left\{ (\rho,\ \varphi, z) \in \boldsymbol{R}^3 : 0 \le \rho \le \sqrt{2}, 0 \le \varphi \le 2\pi, \rho^2 \le z \le 4 - \rho^2 \right\},$$

$$V(R_{yxz}) = \iiint_{T^*} z\rho\, d\rho\, d\varphi\, dz = \int_0^{\sqrt{2}} \int_0^{2\pi} \int_{\rho^2}^{4-\rho^2} z\rho\, dz\, d\varphi\, d\rho = \int_0^{\sqrt{2}} \int_0^{2\pi} \rho \left[\frac{z^2}{2} \right]_{\rho^2}^{4-\rho^2} d\varphi\, d\rho =$$

$$= \frac{1}{2} \int_0^{\sqrt{2}} \int_0^{2\pi} \rho \left(16 - 8\rho^2 + \rho^4 - \rho^4\right) d\varphi\, d\rho = \frac{1}{2} \int_0^{\sqrt{2}} \int_0^{2\pi} \left(16\rho - 8\rho^3\right) d\varphi\, d\rho =$$

$$= 4 \int_0^{\sqrt{2}} \left(2\rho - \rho^3\right) [\varphi]_0^{2\pi}\, d\rho = 8\pi \int_0^{\sqrt{2}} \left(2\rho - \rho^3\right) d\rho = 8\pi \left[\rho^2 - \frac{\rho^4}{4} \right]_0^{\sqrt{2}} = 8\pi\left(2 - 1\right) = 8\pi \,.$$

Coordinates of the region centre of gravity are

$$x_T = \frac{1}{8\pi}\iiint\limits_{R_{yxz}} x.z\,dxdydz = \frac{1}{8\pi}\iiint\limits_{T^*} \rho^2\cos\varphi.\,z\,d\rho\,d\varphi\,dz = 0\,,$$

$$y_T = \frac{1}{8\pi}\iiint\limits_{R} y.z\,dxdydz = \frac{1}{8\pi}\iiint\limits_{T^*} \rho^2\sin\varphi.\,z\,d\rho\,d\varphi\,dz = 0\,,$$

$$z_T = \frac{1}{8\pi}\iiint\limits_{R} z^2 dxdydz = \frac{1}{8\pi}\int_0^{\sqrt{2}}\int_0^{2\pi}\int_{\rho^2}^{4-\rho^2} z^2\rho\,dz\,d\varphi\,d\rho = \frac{1}{8\pi}\int_0^{\sqrt{2}}\int_0^{2\pi}\rho\left[\frac{z^3}{3}\right]_{\rho^2}^{4-\rho^2} d\varphi\,d\rho =$$

$$= \frac{1}{24\pi}\int_0^{\sqrt{2}}\int_0^{2\pi}\rho\left(64-48\rho^2+12\rho^4-2\rho^6\right)d\varphi\,d\rho =$$

$$= \frac{1}{12\pi}\int_0^{\sqrt{2}}\left(32\rho-24\rho^3+6\rho^5-\rho^7\right)[\varphi]_0^{2\pi}\,d\rho =$$

$$= \frac{1}{6}\int_0^{\sqrt{2}}\left(32\rho-24\rho^3+6\rho^5-\rho^7\right)d\rho = \frac{1}{6}\left[16\rho^2-6\rho^4+\rho^6-\frac{\rho^8}{8}\right]_0^{\sqrt{2}} = \frac{7}{3}\,.$$

2. The total mass of the solid S from Fig. 3.26, left, with the constant point density function $\sigma(x, y, z) = k$, $k \in R$, can be calculated as

$$M = \iiint\limits_{S^*} k.\,\rho\,d\rho\,d\varphi\,dz = k\iiint\limits_{S^*}\rho\,d\rho\,d\varphi\,dz = k.V(S) = \frac{k\pi}{8}\,,$$

Coordinates of its centre of gravity, centroid T, are

$$x_T = \frac{8}{k\pi}\iiint\limits_{R_{yxz}} x\,dxdydz = \frac{8}{k\pi}\iiint\limits_{T^*}\rho^2\cos\varphi\,d\rho\,d\varphi\,dz = 0\,,$$

$$y_T = \frac{8}{k\pi}\iiint\limits_{R} y\,dxdydz = \frac{8}{k\pi}\iiint\limits_{T^*}\rho^2\sin\varphi\,d\rho\,d\varphi\,dz = 0\,,$$

$$z_T = \frac{8}{k\pi}\iiint\limits_{R} k.z\,dx\,dy\,dz = \frac{8}{k\pi}\int_0^{\frac{1}{2}}\int_0^{2\pi}\int_{3\rho^2}^{1-\rho^2} k.z.\rho\,dz\,d\varphi\,d\rho = \frac{8}{\pi}\int_0^{\frac{1}{2}}\int_0^{2\pi}\rho\left[\frac{z^2}{2}\right]_{3\rho^2}^{1-\rho^2} d\varphi\,d\rho =$$

$$= \frac{4}{\pi}\int_0^{\frac{1}{2}}\int_0^{2\pi}\rho\left(1-2\rho^2+\rho^4-9\rho^4\right)d\varphi\,d\rho = \frac{4}{\pi}\int_0^{\frac{1}{2}}\left(\rho-2\rho^3-8\rho^5\right)[\varphi]_0^{2\pi}\,d\rho =$$

$$= 8\int_0^{\frac{1}{2}}\left(\rho-2\rho^3-8\rho^5\right)d\rho = 4\left[\rho^2-\rho^4-\frac{4\rho^6}{3}\right]_0^{\frac{1}{2}} = 4\left(\frac{1}{4}-\frac{1}{16}-\frac{1}{24}\right) = \frac{7}{12}\,.$$

3. The volume of a homogeneous solid T bounded by the spheres $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + z^2 = 4$ and by the conical surface of revolution $x^2 + y^2 = z^2$, while $z \geq 0$ (Fig. 3.26, right) equals to the value of the triple integral over the region described in spherical coordinates as

$$T^* = \left\{ (\rho,\ \varphi, \zeta) \in \boldsymbol{R}^3 : 1 \leq \rho \leq 2,\ 0 \leq \varphi \leq 2\pi,\ 0 \leq \zeta \leq \frac{\pi}{4} \right\},$$

$$V(T) = \iiint_T 1\, dx\, dy\, dz = \iiint_{T^*} \rho^2 \sin\zeta\, d\rho\, d\varphi\, d\zeta = \int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^2 \sin\zeta\, d\zeta\, d\varphi\, d\rho =$$

$$= \left[-\cos\zeta\right]_0^{\frac{\pi}{4}} . \left[\varphi\right]_0^{2\pi} . \left[\frac{\rho^3}{3}\right]_1^2 = \left(1 - \frac{1}{\sqrt{2}}\right) . 2\pi . \left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7.(2-\sqrt{2})\pi}{3}.$$

Coordinates of the centroid of this solid are

$$x_T = \frac{3}{7.(2-\sqrt{2})\pi} \iiint_T x\, dxdydz = \frac{2}{3\sqrt{2}\pi} \iiint_{T^*} \rho^3 \cos\varphi \sin^2\zeta\, d\rho\, d\varphi\, d\zeta = 0,$$

$$y_T = \frac{3}{7.(2-\sqrt{2})\pi} \iiint_T y\, dxdydz = \frac{2}{3\sqrt{2}\pi} \iiint_{T^*} \rho^3 \sin\varphi \sin^2\zeta\, d\rho\, d\varphi\, d\zeta = 0,$$

$$z_T = \frac{3}{7.(2-\sqrt{2})\pi} \iiint_T z\, dxdydz = \frac{3}{7.(2-\sqrt{2})\pi} \int_1^2 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \rho^3 \sin\zeta \cos\zeta\, d\zeta\, d\varphi\, d\rho =$$

$$= \frac{3}{7.(2-\sqrt{2})\pi} \int_1^2 \int_0^{2\pi} \rho^3 \left[\frac{1}{2}\sin^2\zeta\right]_0^{\frac{\pi}{4}} d\varphi\, d\rho = \frac{3}{28.(2-\sqrt{2})\pi} \int_1^2 \int_0^{2\pi} \rho^3\, d\varphi\, d\rho =$$

$$= \frac{3}{28.(2-\sqrt{2})\pi} \int_1^2 \rho^3 \left[\varphi\right]_0^{2\pi} d\rho = \frac{3}{14.(2-\sqrt{2})} \int_1^2 \rho^3\, d\rho =$$

$$= \frac{3}{14.(2-\sqrt{2})} \left[\frac{\rho^4}{4}\right]_1^2 = \frac{45}{56.(2-\sqrt{2})}.$$

Finally, we can summarize that the main purpose of this section on multiple integrals lies primarily in the usefulness of the general theory of double and triple integrals for calculations of geometric and physical characteristics of planar or spatial objects, representing physical objects as 2D laminae in $\mathbf{E}^2$, or 3D solids in $\mathbf{E}^3$.

The geometric characteristics of these objects are given by the areas of the surface patches and by the volumes of the solids. The physical characteristics comprise the total mass, the static moments about the coordinate axes, and the coordinates of the centres of gravity (or centroids in homogeneous case) of these object.

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