Kinematics of point particle

1. *The position vector of a point particle depends on time according to the relation* $\vec{r} = \vec{i}\,A\cos bt + \vec{j}\,A\sin bt$*, kde* $A = 5\,\mathrm{m}$*,* $b = \pi/4\,\mathrm{s}^{-1}$*. Express its components, coordinates, magnitude and direction cosines at any time and at time* $t = 2\,\mathrm{s}$*.*

The position vector can be decomposed into components

$$\vec{r} = \vec{x} + \vec{y}\ ,$$

where the components of the position vector are

$$\vec{x} = \vec{i}\,A\cos bt\ ,$$

$$\vec{y} = \vec{j}\,A\sin bt\ ,$$

at time $t = 2\,\mathrm{s}$ the components of the position vector have the values

$$\vec{x} = \vec{i}\,5\,\mathrm{m}\ .\cos\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 0\,\vec{i}\ ,$$

$$\vec{y} = \vec{j}\,5\,\mathrm{m}\ .\sin\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 5\,\mathrm{m}\,\vec{j}\ .$$

The position vector can be written using coordinates

$$\vec{r} = x\vec{i} + y\vec{j}\ ,$$

where the coordinates of the position vector are

$$x = A\cos bt\ ,$$

$$y = A\sin bt\ ,$$

at time $t = 2\,\mathrm{s}$ the coordinates of the position vector have the values

$$x = 5\,\mathrm{m}\ .\cos\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 0\ ,$$

$$y = 5\,\mathrm{m}\ .\sin\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 5\,\mathrm{m}\ .$$

The magnitude of the position vector is constant

$$r = \sqrt{x^2 + y^2} = \sqrt{(A\sin bt)^2 + (A\cos bt)^2} = A = 5\,\mathrm{m}\ .$$

The direction cosines of the position vector are

$$\cos\alpha = \frac{x}{r} = \frac{A\cos bt}{A} = \cos bt\ ,$$

$$\cos\beta = \frac{y}{r} = \frac{A\sin bt}{A} = \sin bt\ ,$$

at time $t = 2\,\mathrm{s}$ the direction cosines have values

$$\cos\alpha = \cos\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 0\ ,$$

$$\cos\beta = \sin\left(\pi/4\,\mathrm{s}^{-1}\ .\ 2\,\mathrm{s}\right) = 1\ .$$

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2. *Two bodies that are $d = 100\,\mathrm{m}$ apart started moving in a straight line opposite each other. The first body is moving uniformly with velocity $v = 3\,\mathrm{m\,s^{-1}}$. The second body is moving uniformly accelerated with an initial velocity $v_0 = 7\,\mathrm{m\,s^{-1}}$ and acceleration $a = 4\,\mathrm{m\,s^{-2}}$. Find the time and place of their meeting.*

The distance travelled by the first body in uniform motion will be

$$s_1 = vt\ .$$

The distance travelled by the second body in uniformly accelerated motion will be

$$s_2 = v_0 t + \frac{at^2}{2}\ .$$

The bodies meet when the sum of the paths they have travelled equals their initial distance

$$s_1 + s_2 = d\ ,$$

$$vt + v_0 t + \frac{at^2}{2} = d\ .$$

By adding the numerical values, the quadratic equation can be obtained

$$3\,\mathrm{m\,s^{-1}}\ t + 7\,\mathrm{m\,s^{-1}}\ t + \frac{4\,\mathrm{m\,s^{-2}}\ t^2}{2} = 100\,\mathrm{m}\ ,$$

the time of the meeting of the bodies is thus the root of the quadratic equation

$$2t^2 + 10t - 100 = 0\ ,$$

which has two solutions

$$t_1 = 5\,\mathrm{s}\ ,$$

$$t_2 = -10\,\mathrm{s}\ .$$

The physical solution of the problem corresponds to the positive solution of the quadratic equation

$$t = 5\,\mathrm{s}\ .$$

The point at which the bodies meet will be distant from the first body

$$s_1 = vt = 3\,\mathrm{m\,s^{-1}}\ .\ 5\,\mathrm{s} = 15\,\mathrm{m}$$

and will be distant from the other body

$$s_2 = d - s_1 = 100\,\mathrm{m} - 15\,\mathrm{m} = 85\,\mathrm{m}\ .$$

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3. *The train starts from rest with a uniformly accelerated motion so that in time* $t_1 = 30\,\mathrm{s}$ *it passes a path* $s_1 = 90\,\mathrm{m}$. *What path will it pass, what will be its instantaneous velocity and what will be its average velocity in time* $t_2 = 60\,\mathrm{s}$*?*

For the path s_1 that the train passes in time t_1 in uniformly accelerated motion, the following holds

$$s_1 = \frac{at_1^2}{2}\ ,$$

from which the acceleration of the train can be calculated

$$a = \frac{2s_1}{t_1^2} = \frac{2\ .\ 90\,\mathrm{m}}{(30\,\mathrm{s})^2} = 0{,}2\,\mathrm{m\,s^{-2}}\ .$$

The path of the train at time t_2 will be

$$s_2 = \frac{at_2^2}{2} = \frac{0{,}2\,\mathrm{m\,s^{-2}}\ .\ (60s)^2}{2} = 360\,\mathrm{m}\ .$$

The instantaneous velocity of the train at time t_2 will be

$$v_2 = at_2 = 0{,}2\,\mathrm{m\,s^{-2}}\ .\ 60\,\mathrm{s} = 12\,\mathrm{m\,s^{-1}}\ .$$

The average speed of the train over time t_2 will be

$$v_p = \frac{s_2}{t_2} = \frac{360\,\mathrm{m}}{60\,\mathrm{s}} = 6\,\mathrm{m\,s^{-1}}\ .$$

4. *The position vector of a point particle has the form* $\vec{r} = (A_1 t^2 + B_1)\vec{i} + (A_2 t^2 + B_2)\vec{j}$*, where* $A_1 = 0{,}2\,\mathrm{m\,s^{-2}}$, $B_1 = 0{,}05\,\mathrm{m}$, $A_2 = 0{,}15\,\mathrm{m\,s^{-2}}$, $B_2 = -0{,}03\,\mathrm{m}$. *Find the magnitude and direction of the velocity and acceleration of the point particle at time* $t_1 = 2\,\mathrm{s}$. *Express the direction using the angle to the x-axis.*

The coordinates of the position vector are

$$x = A_1 t^2 + B_1 \ ,$$

$$y = A_2 t^2 + B_2 \ .$$

For the velocity vector it states

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

and the coordinates of the velocity vector will be

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{d(A_1 t^2 + B_1)}{dt} = 2A_1 t \ ,$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{d(A_2 t^2 + B_2)}{dt} = 2A_2 t \ .$$

The magnitude of the velocity vector will be

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2A_1 t)^2 + (2A_2 t)^2} = 2t\sqrt{A_1^2 + A_2^2} \ .$$

The magnitude of the velocity vector at time $t_1 = 2\,\mathrm{s}$ will be

$$v = 2 \ . \ 2\,\mathrm{s} \ . \ \sqrt{(0{,}2\,\mathrm{m\,s^{-2}})^2 + (0{,}15\,\mathrm{m\,s^{-2}})^2} = 1\,\mathrm{m\,s^{-1}} \ .$$

The direction cosine of the velocity vector will be constant

$$\cos\alpha_v = \frac{v_x}{v} = \frac{2A_1 t}{2t\sqrt{A_1^2 + A_2^2}} = \frac{A_1}{\sqrt{A_1^2 + A_2^2}}$$

and its value will be

$$\cos\alpha_v = \frac{0{,}2\,\mathrm{m\,s^{-2}}}{\sqrt{(0{,}2\,\mathrm{m\,s^{-2}})^2 + (0{,}15\,\mathrm{m\,s^{-2}})^2}} = 0{,}8 \ ,$$

which implies that the angle between the velocity vector and the x-axis will be

$$\alpha_v = \arccos 0{,}8 = 36{,}6^\circ \ .$$

For the acceleration vector it states

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \ ,$$

the coordinates of the acceleration vector will be

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{d(2A_1t)}{dt} = 2A_1 \ ,$$

$$a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{d(2A_2t)}{dt} = 2A_2 \ .$$

The magnitude of the acceleration vector will be constant

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(2A_1)^2 + (2A_2)^2} = 2\sqrt{A_1^2 + A_2^2}$$

and its value will be

$$a = 2 \ . \ \sqrt{(0{,}2\,\mathrm{m\,s^{-2}})^2 + (0{,}15\,\mathrm{m\,s^{-2}})^2} = 0{,}5\,\mathrm{m\,s^{-2}} \ .$$

The direction cosine of the acceleration vector will be constant

$$\cos\alpha_a = \frac{a_x}{a} = \frac{2A_1}{2\sqrt{A_1^2 + A_2^2}} = \frac{A_1}{\sqrt{A_1^2 + A_2^2}}$$

and its value will be

$$\cos\alpha_a = \frac{0{,}2\,\mathrm{m\,s^{-2}}}{\sqrt{(0{,}2\,\mathrm{m\,s^{-2}})^2 + (0{,}15\,\mathrm{m\,s^{-2}})^2}} = 0{,}8 \ ,$$

which implies that the angle between the acceleration vector and the x-axis will be

$$\alpha_a = \arccos 0{,}8 = 36{,}6^\circ \ .$$

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5. *The wheel started to rotate from rest with a constant angular acceleration* $\alpha = 5\,\mathrm{s^{-2}}$*. How many times has the wheel rotated in the time* $t_1 = 10\,\mathrm{s}$ *since the start of the motion?*

The angular velocity at constant angular acceleration is

$$\omega = \int \alpha \, dt = \alpha t + c_1 \ .$$

If the starting angular velocity is zero, then the integration constant is zero

$$\omega(t = 0\,\mathrm{s}) = 0 \implies c_1 = 0$$

and the angular velocity will be

$$\omega = \alpha t \ .$$

The angular displacement at constant angular acceleration is

$$\varphi = \int \omega \, dt = \int \alpha t \, dt = \frac{\alpha t^2}{2} + c_2 \ .$$

If the starting angular displacement is zero, then the integration constant is zero

$$\varphi(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0$$

and the angular displacement will be

$$\varphi = \frac{\alpha t^2}{2} \ .$$

The angular distance of one revolution is 2π, so the number of revolutions of the wheel will be

$$n = \frac{\varphi}{2\pi} = \frac{\alpha t^2}{4\pi}$$

and the number of revolutions of the wheel in time t_1 will be

$$n_1 = \frac{\alpha t_1^2}{4\pi} = \frac{5\,\mathrm{s}^{-2} \ . \ (10\,\mathrm{s})^2}{4\pi} = 39{,}8 \ .$$

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6. *The magnitude of the train speed after leaving the station gradually increased from zero to $v_1 = 20\,\mathrm{m\,s^{-1}}$ at time $t_1 = 180\,\mathrm{s}$. The track is curved with radius of curvature $R = 800\,\mathrm{m}$. Calculate the magnitude of tangential, normal and total acceleration at time $t_2 = 120\,\mathrm{s}$.*

Tangential acceleration indicates the change in magnitude of the velocity

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} \ ,$$

for constant a_t it is

$$a_t = \frac{v_1}{t_1} = \frac{20\,\mathrm{m\,s^{-1}}}{180\,\mathrm{s}} = 0{,}111\,\mathrm{m\,s^{-2}} \ .$$

The magnitude of the velocity at time t_2 will be

$$v_2 = a_t t_2 = \frac{v_1 t_2}{t_1} \ .$$

The normal acceleration indicates the change in direction of the velocity

$$a_n = \frac{v^2}{R} \; ,$$

at time t_2 the normal acceleration will be

$$a_n = \frac{v_1^2 t_2^2}{R t_1^2} = \frac{(20\,\mathrm{m\,s^{-1}})^2 \,.\, (120\,\mathrm{s})^2}{800\,\mathrm{m} \,.\, (180\,\mathrm{s})^2} = 0{,}222\,\mathrm{m\,s^{-2}} \; .$$

The total acceleration is the vector sum of the tangential and normal accelerations

$$\vec{a} = \vec{a}_t + \vec{a}_n \; ,$$

the magnitude of the total acceleration will be

$$a = \sqrt{a_t^2 + a_n^2} \; ,$$

thus the total acceleration at time t_2 will be

$$a = \sqrt{\left(\frac{v_1}{t_1}\right)^2 + \left(\frac{v_1^2 t_2^2}{R t_1^2}\right)^2}$$

and its value will be

$$a = \sqrt{\left(\frac{20\,\mathrm{m\,s^{-1}}}{180\,\mathrm{s}}\right)^2 + \left(\frac{(20\,\mathrm{m\,s^{-1}})^2 \,.\, (120\,\mathrm{s})^2}{800\,\mathrm{m} \,.\, (180\,\mathrm{s})^2}\right)^2} = 0{,}248\,\mathrm{m\,s^{-2}} \; .$$

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7. *The point particle started moving in a circle with constant angular acceleration* $\alpha = 0{,}25\,\mathrm{s^{-2}}$. *At what time from the start of the motion will the angle between particle's acceleration and particle's velocity be* $\gamma = 45°$*?*

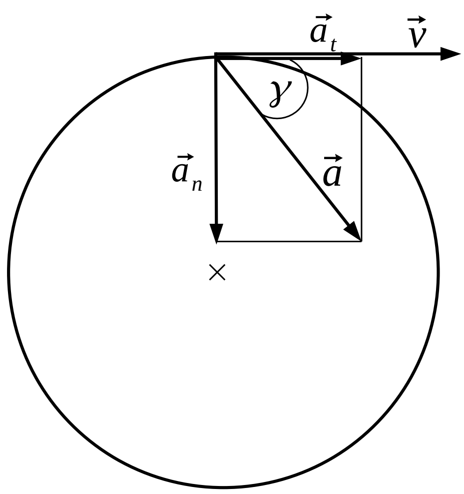

Fig. 1

In circular motion the tangential acceleration is

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}(R\omega)}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t} = R\alpha \; .$$

The angular velocity at constant angular acceleration is

$$\omega = \alpha t \; .$$

The normal acceleration can be expressed as

$$a_n = \frac{v^2}{R} = \frac{(\omega R)^2}{R} = \frac{\alpha^2 t^2 R^2}{R} = \alpha^2 t^2 R \; .$$

The angle between the velocity and acceleration (Fig. 1) is

$$\tan\gamma = \frac{a_n}{a_t} = \frac{\alpha^2 t^2 R}{R\alpha} = \alpha t^2 \; ,$$

which implies for time

$$t = \sqrt{\frac{\tan\gamma}{\alpha}} = \sqrt{\frac{\tan 45^\circ}{0{,}25\,\mathrm{s}^{-2}}} = 2\,\mathrm{s} \; .$$

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8. *A wheel with radius* $R = 0{,}1\,\mathrm{m}$ *rotates such that the dependence of the angle of rotation on time is given by the function* $\varphi = A + Bt + Ct^3$ *, where* $B = 2\,\mathrm{s}^{-1}$, $C = 1\,\mathrm{s}^{-3}$. *For points that lie on the circumference of the wheel, calculate their velocity, angular velocity, angular acceleration, tangential acceleration and normal acceleration at time* $t_1 = 2\,\mathrm{s}$.

The magnitude of the angular velocity can be calculated using the definition

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} \; ,$$

which implies

$$\omega = \frac{\mathrm{d}(A + Bt + Ct^3)}{\mathrm{d}t} = B + 3Ct^2 = \left[2\,\mathrm{s}^{-1} + 3 \; . \; (1\,\mathrm{s}^{-3}) \; . \; (2\,\mathrm{s})^2\right] = 14\,\mathrm{s}^{-1} \; .$$

The magnitude of the angular acceleration can be calculated using the definition

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} \; ,$$

which implies

$$\alpha = \frac{\mathrm{d}(B + 3Ct^2)}{\mathrm{d}t} = 6Ct = 6 \; . \; (1\,\mathrm{s}^{-3}) \; . \; (2\,\mathrm{s}) = 12\,\mathrm{s}^{-2} \; .$$

The magnitude of the velocity is

$$v = \omega R = (B + 3Ct^2)R = \left[2\,\mathrm{s}^{-1} + 3\,.\,(1\,\mathrm{s}^{-3})\,.\,(2\,\mathrm{s})^2\right]\,.\,0{,}1\,\mathrm{m} = 1{,}4\,\mathrm{m\,s}^{-1}\ .$$

The magnitude of the tangential acceleration is

$$a_t = \alpha R = 12\,\mathrm{s}^{-2}\ 0{,}1\,\mathrm{m} = 1{,}2\,\mathrm{m\,s}^{-2}\ .$$

The magnitude of the normal acceleration is

$$a_n = \frac{v^2}{R} = \frac{(1{,}4\,\mathrm{m\,s}^{-1})^2}{0{,}1\,\mathrm{m}} = 19{,}6\,\mathrm{m\,s}^{-2}\ .$$

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9. *A point particle moves in a straight line so that its acceleration increases uniformly with time, and in time $t_1 = 10\,\mathrm{s}$ it increases from zero to $a_1 = 5\,\mathrm{m\,s}^{-2}$. What is the speed of the point particle at time $t_2 = 20\,\mathrm{s}$ and what is the path of the point paricle traveled in this time when it was initially at rest?*

The acceleration of the material point increases uniformly

$$a = kt\ ,$$

the acceleration from zero to a_1 increases in time t_1, that is

$$a_1 = kt_1 \implies k = \frac{a_1}{t_1}\ ,$$

therefore the acceleration will be

$$a = \frac{a_1}{t_1}t\ .$$

The speed of the point particle is

$$v = \int a\,dt = \int \frac{a_1}{t_1}t\,dt = \frac{a_1 t^2}{2t_1} + c_1\ ,$$

if the speed is initially zero

$$v(t = 0\,\mathrm{s}) = 0 \implies c_1 = 0\ ,$$

the speed at time t_2 will be

$$v_2 = \frac{a_1 t_2^2}{2t_1} = \frac{5\,\mathrm{m\,s}^{-2}\,.\,(20\,\mathrm{s})^2}{2\,.\,10\,\mathrm{s}} = 100\,\mathrm{m\,s}^{-1}\ .$$

The path of the point particle is

$$s = \int v\,dt = \int \frac{a_1 t^2}{2t_1}\,dt = \frac{a_1 t^3}{6t_1} + c_2\ ,$$

if the initial path is zero

$$s(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0\ ,$$

the path at time t_2 will be

$$s_2 = \frac{a_1 t_2^3}{6t_1} = \frac{5\,\mathrm{m\,s^{-2}}\ .\ (20\,\mathrm{s})^3}{6\ .\ 10\,\mathrm{s}} = 667\,\mathrm{m\,s^{-1}}\ .$$

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10. *The particle moves on a circle with angular deceleration that increases with time according to the relation* $\alpha = kt$ *, where* $k = -6\,\mathrm{rad\,s^{-3}}$*. The initial angular velocity was* $\omega_0 = 30\,\mathrm{rad\,s^{-1}}$*. What angle does the particle go around in time* $t_1 = 5\,\mathrm{s}$*?*

The angular deceleration of the particle increases uniformly

$$\alpha = kt\ .$$

The angular velocity of the particle is

$$\omega = \int \alpha\,dt = \int kt\,dt = \frac{kt^2}{2} + c_1\ ,$$

because the initial angular velocity of the particle was ω_0

$$\omega(t = 0\,\mathrm{s}) = \omega_0 \implies c_1 = \omega_0\ ,$$

the angular velocity will be

$$\omega = \frac{kt^2}{2} + \omega_0\ .$$

The angular displacement of the particle is

$$\varphi = \int \omega\,dt = \int \left(\frac{kt^2}{2} + \omega_0\right)\,dt = \frac{kt^3}{6} + \omega_0 t + c_2\ ,$$

because the initial angular displacement of the particle was zero

$$\varphi(t = 0\,\mathrm{s}) = 0 \implies c_2 = 0\ ,$$

the angular path will be

$$\varphi = \frac{kt^3}{6} + \omega_0 t\ .$$

The angular displacement of the particle at time t_1 will be

$$\varphi_1 = \frac{kt_1^3}{6} + \omega_0 t_1 = \frac{-6\,\mathrm{rad\,s^{-3}}\ .\ (5\,\mathrm{s})^3}{6} + 30\,\mathrm{rad\,s^{-1}}\ .\ 5\,\mathrm{s} = 25\,\mathrm{rad}\ .$$