12. WAVES

Wave motion is closely related to the phenomenon of vibration. Sound waves, earthquake waves, waves of stretched string and water waves are all produced by some source of vibrations. As a sound wave travels through some medium, such as air, the molecules of the medium vibrate back and forth, the water molecules vibrate up and down etc. As wave travels through the medium, the particles of the medium move in repetitive cycles. Therefore, the motion of the particle bears a strong resemblance to the period motion of a vibrating pendulum or a mass attached to a spring.

There are three physical quantities characterizing of the waves: **the wavelength, the frequency and the wave velocity.**

One wavelength is the minimum distance between two points on a wave that behave identically. For example, in the case of water waves, the wavelength is the distance between adjacent crests or between adjacent troughs.

Most waves are periodic in nature. **The frequency** of such periodic waves is the rate at which the disturbance repeats itself.

Waves travel, or propagate, with **a specific velocity**, which depends on the properties of the medium being disturbing. For example, sound waves travel through air at 20°C with a speed 344 $m.s^{-1}$, whereas the speed of sound in most solids is higher than 344 $m.s^{-1}$. For example, the speed of sound in iron is 5130 $m.s^{-1}$. A special class of waves that do not require a medium in order to propagate is electromagnetic waves, which travel through a vacuum with a speed of about $3 \times 10^8\ m.s^{-1}$.

12.1 Types of waves

On way to demonstrate wave motion is to flip on end of a long rope that is under tension and has its opposite end fixed, as is shown in Fig.12.1. In this manner a single pulse is formed that travels to the left with a definite speed. This type of disturbance is called **a traveling wave**. As we shall see later the **speed of the wave depends on the tension in the rope** and **on the properties of the rope**. Note that, as the wave travels along the rope, each segment of the rope that is disturbed moves in a direction perpendicular to the wave motion. A traveling wave such as this, in which the particles of the disturbed medium move perpendicular to the wave velocity is called **transverse wave**.

In another class of waves, called **longitudinal waves**, the particles of the medium undergo displacements in a direction parallel to the direction of wave motion.

Sound waves are longitudinal waves. The disturbance corresponds to a series hight and low pressure regions that travel through air or through any material medium with a certain velocity.

Example of **transverse wave is electromagnetic waves**, such as light, radio and television waves. At a given point in space, the electric and magnetic fields of an electromagnetic wave are perpendicular to the direction of the wave and to each other, as we shall see later.

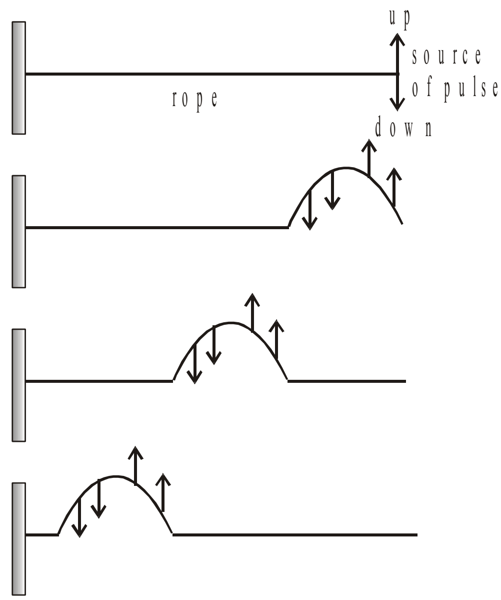

Fig.12. 1

Some waves in nature are neither transverse nor longitudinal, but a combination of the two. Surface water waves are a good example. When a water wave travels on the surface of deep water, water molecules at the surface move in nearly circular path (as is shown in Fig.12.2), where the water surface is drawn as a series of crests and through. Note that **the disturbance has both transverse and longitudinal components**. As the waves pass, water molecules at the crests move in the direction of the wave and molecules at the through move in the opposite direction. Hence, there is not displacement of a water molecule after the passage of any number of complete waves.

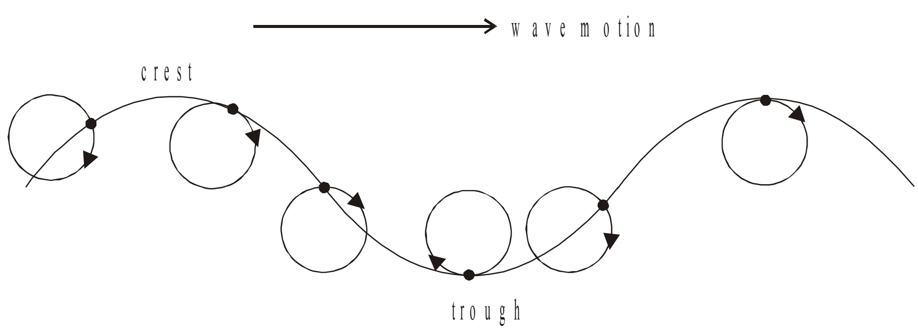

Fig.12. 2

12.2 One-dimensional waves

Let us now give a mathematical description of one-dimensional travelling wave. Consider a wave pulse travelling to the right on a long stretched string with constant *v* as is shown in Fig.12.3.

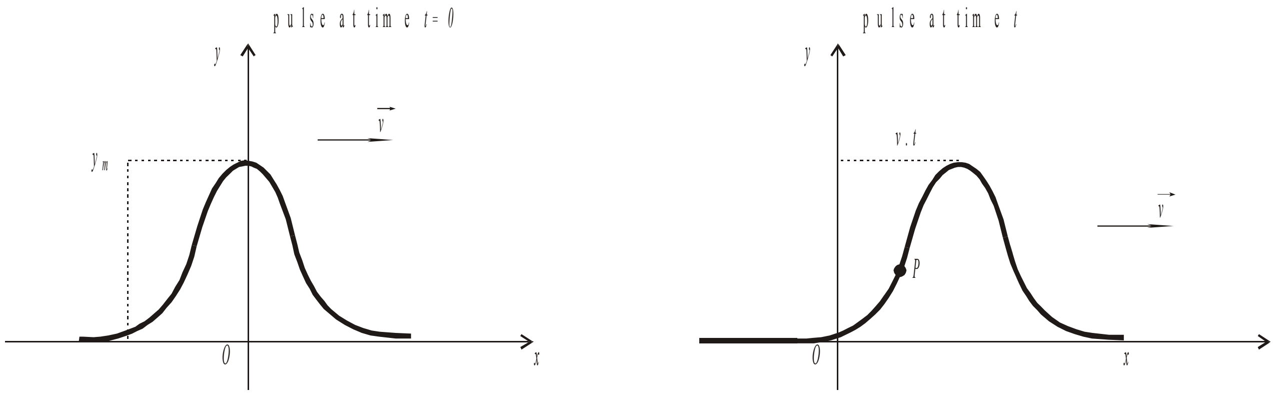

Fig.12. 3

Let us denote the pulse at time $t = 0$ as $y = f(x)$. That is, *y* is some definite function of *x*. **The maximum displacement**, y_m, is called **the amplitude of wave**. Since the speed of wave pulse is *v*, it travels to the right a distance vt in time *t*.

If the shape of the wave pulse does not change with time, we can represent the displacement at any time *t* measured in a stationary frame with the origin at *O* as

$$y = f(x - vt) \qquad (12.1)$$

if the wave is travelled to the right.. Similar, if the pulse travels to the left, its displacement is given by

$$y = f(x + vt) \qquad (12.2)$$

The displacement *y* is called **the wave function**. This function depends on the two variables *x* and *t*. Consider a point *P* on the string. As wave passes the point *P* the *y* component of this point will increase, reaches a maximum, and then decreases to zero. Therefore, **the wave function represents the *y* component of any point *P* at any time *t***. To find the velocity of the pulse, we can calculate how far the crest moves in a short time and divide this distance by the time interval. The crest of the pulse corresponds to that point for which *y* has its maximum. In order to follow the motion of the crest, x_0, must be substituted for $x - vt$. At the time $t = 0$, the crest is at $x = x_0$ and at a time $\mathrm{d}t$ later, the crest is at $x + vt$. Therefore, in time $\mathrm{d}t$, the crest has moved a distance $\mathrm{d}x = (x_0 + v\mathrm{d}t) - x_0 = v\mathrm{d}t$. From this follows, that the wave speed called **the phase velocity** is given by

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad (12.3)$$

Note:

The phase velocity must not be confused with the transverse velocity of a particle in medium.

12.3 Harmonic wave

A harmonic wave has a sinusoidal shape as is shown in Fig.12.4. Assume that at the time $t = 0$, the displacement of the curve is given by

$$y = A\sin\left(\frac{2\pi}{\lambda}x\right) \tag{12.4}$$

where the constant A is called **the amplitude of the wave**. It represents the maximum value of the displacement. The constant λ is called **the wavelength of the wave**. It equals the distance between two successive maxima or crests, or between any two adjacent points that have the same phase as is shown in Fig.12.5.

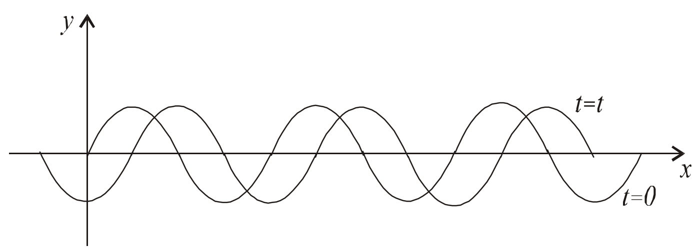

Fig.12. 4

To describe the same point P on the wave shape, the argument of the sine function must be the same.

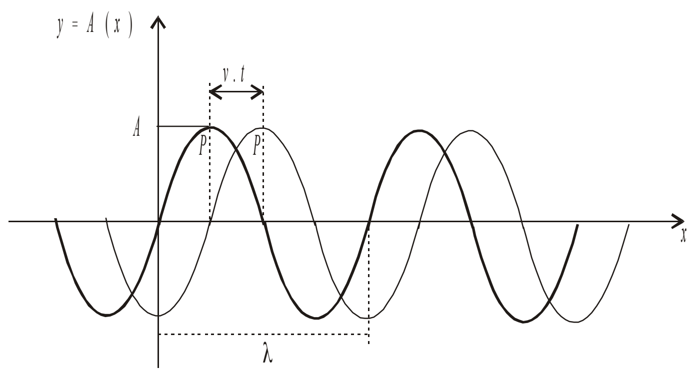

Fig.12. 5

Thus we must replace x by $(x - vt)$ in eq.(12.4). Then the displacement is given by

$$y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right] \tag{12.5}$$

As we can see the displacement repeats itself when x is increased by any integral multiple of λ. The time it takes the wave to travel a distance of one wavelength is called **the period *T***. The phase velocity, wavelength and period are related by

$$v = \frac{\lambda}{T} \tag{12.6}$$

or

$$\lambda = vT \tag{12.7}$$

From this equation we can express

$$v = \lambda f \tag{12.8}$$

where $f = \frac{1}{T}$. Substituting this expression into eq.(12.5) gives

$$y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \tag{12.9}$$

This form of wave function shows the periodic nature of *y*. It means, at any time *t*, *y* has the same value at the positions *x*, $x + \lambda$, $x + 2\lambda$, etc.

We can express the harmonic wave function in a convenient form by defining two other quantities called **the angular frequency** ω and **the wave number *k*** as

$$\omega = \frac{2\pi}{T} \ (\mathrm{s^{-1}}) \tag{12.10}$$

$$k = \frac{2\pi}{\lambda} \ (\mathrm{cm^{-1}}) \tag{12.11}$$

Using these definitions the equation (12.5) can be rewritten in the more general form

$$y = A\sin(kx - \omega t) \tag{12.12}$$

This equation assumes that the displacement *y* is zero at $x = 0$ and $t = 0$. **If the transverse displacement is not zero** at $x = 0$ and $t = 0$, we generally express the wave motion in the form

$$y = A\sin(kx - \omega t - \varphi), \tag{12.13}$$

where φ is called **phase constant**. This constant can be determined from the initial conditions.

Notes:

1. For a wave travelling along the *x* axis to the left (value of *x* decreases) the equation of the wave is in the form

$$y = A\sin(kx + \omega t) \tag{12.14}$$

2. Let us consider a general wave of any shape and suppose the wave has a shape given by $y = f(x)$ at $t = 0$, where *y* is the displacement of the wave at *x* and $f(x)$ is some function of *x*, which gives the actual shape of the wave. If the wave is travelling to the right along the *x* axis the wave will have the same shape but it will have moved a distance vt. We must replace *x* by $x - vt$ to obtain displacement at the time *t*. Therefore, the displacement is given by equation $y(x,t) = f(x - vt)$.

Examples

A sinusoidal wave travelling in the positive *x* direction has amplitude $A = 15\ \mathrm{cm}$, wavelength $\lambda = 40\ \mathrm{cm}$ and frequency $f = 8\ \mathrm{Hz}$. The displacement of the wave at the time $t = 0$ and $x = 0$ is also $\varphi = 15^\circ$. Find

a. The wave number *k*, period *T*, angular frequency ω and phase velocity *v* of the wave.

b. Determine the phase constant φ and write a general expression for the wave function.

Solution:

This wave is shown in Fig.12.6

a. Using eqs.(12.8), (12.10) and (12.11) we find out the following values:

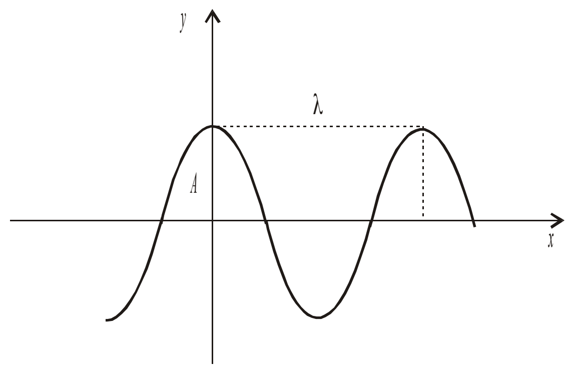

Fig.12. 6

$$k = \frac{2\pi}{\lambda} = 0.157\ \mathrm{cm^{-1}}$$

$$T = \frac{1}{f} = 0.125\ \mathrm{s}$$

$$\omega = 2\pi f = 50.8\ \mathrm{rad.s^{-1}}$$

$$v = f\lambda = 320\ \mathrm{cm.s^{-1}}$$

b. Since the amplitude $A = 15\ \mathrm{cm}$ then equation (12.13) gives

$$15\ \mathrm{cm} = 15\ \mathrm{cm} \cdot \sin(-\varphi)$$

at the time $t = 0$ and $x = 0$ or

$$\sin(-\varphi) = 1.$$

Since $\sin(-\varphi) = -\sin\varphi$ we see that $\varphi = -90^\circ = -\frac{\pi}{2}\ \mathrm{rad}$. Hence, the wave function is in the form

$$y = A\sin\left(kx - \omega t + \frac{\pi}{2}\right) = A\cos(kx - \omega t)$$

or

$$y = 15\cos(0.157x - 50.3t)$$

12.4 Superposition and interference of waves

If two or more travelling waves are moving through the medium the resultant wave function at any point is the algebraic sum of the wave functions of the individual waves. This principle is called **the superposition principle**.

One consequence of the superposition principle is observation that two travelling waves can pass through each other without being destroyed or even altered. For example, when sound waves from two sources move through air they also can pass through each other. The resultant sound one hears at a given point is the resultant of the both disturbances.

Let us consider two one-dimensional waves of equal frequencies ω, different amplitudes $A_1 \neq A_2$ and different phase constants $\varphi_1 \neq \varphi_2$. We shall assume suppose that the displacement direction of the first and the second wave are the same and waves travel down a positive direction of x axis

$$x_1 = A_1 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_1\right] \tag{12.15}$$

$$x_2 = A_2 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_2\right] \tag{12.16}$$

where v is the velocity of the waves. Using principle of superposition gives

$$x = x_1 + x_2 = A_1 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_1\right] + A_2 \sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi_2\right] \tag{12.17}$$

or

$$x = \left(A_1 \cos\varphi_1 + A_2 \cos\varphi_2\right)\sin\left[\omega\left(t - \frac{x}{v}\right)\right] + \left(A_1 \sin\varphi_1 + A_2 \sin\varphi_2\right)\cos\left[\omega\left(t - \frac{x}{v}\right)\right] \tag{12.18}$$

The terms in brackets are constants and we see that the resultant wave is also harmonic with the same frequency. To find the amplitude and phase of this wave we express it in standard form

$$x = A\sin\left[\omega\left(t - \frac{x}{v}\right) + \varphi\right] \tag{12.19}$$

or

$$x = A\cos\varphi\sin\left[\omega\left(t - \frac{x}{v}\right)\right] + A\sin\varphi\cos\left[\omega\left(t - \frac{x}{v}\right)\right] \tag{12.20}$$

where A and φ represent the amplitude and the phase angle of the resultant wave, respectively. Equation (12.18) and (12.20) must be identical for any x and t and we obtain the following expressions

$$A\cos\varphi = A_1 \cos\varphi_1 + A_2 \cos\varphi_2 \tag{12.21}$$

$$A\sin\varphi = A_1 \sin\varphi_1 + A_2 \sin\varphi_2 \tag{12.22}$$

Dividing of these two equations we find the phase angle φ

$$\tan\varphi = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\varphi_2} \tag{12.23}$$

To find the amplitude *A* we square equations (12.21) and (12.22) and then we add them

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\varphi_2 - \varphi_1) \tag{12.24}$$

From this equation follows:

1. If phase difference $\varphi_2 - \varphi_1 = 2n\pi$, where $n = 0, 1, 2, \ldots$ the amplitude of resultant wave will have its maximum $A = A_1 + A_2$. This case is called **constructive interference**.
2. If the phase difference $\varphi_2 - \varphi_1 = (2n+1)\pi$, where $n = 0, 1, 2, \ldots$ the amplitude of the resultant wave will have its minimum

   $A = A_2 - A_1$ if $A_2 > A_1$

   or

   $A = A_1 - A_2$ if $A_2 < A_1$

   This case is called **destructive interference**.
3. The displacement at any point *x* and any time *t* is described by the function $y(x,t)$ of two variables *x* and *t* as

$$y(x,t) = A \sin\omega\left(t - \frac{x}{v}\right) \tag{12.25}$$

If we choose the fixed point x_0, the function $y(x_0, t)$ describe the oscillating motion at this point x_0 and thus for $x = x_0$ we can write the function in form

$$y(t) = A \sin\left(\omega t - \omega\frac{x_0}{v}\right) = A\sin(\omega t - \varphi) \tag{12.26}$$

where $\varphi = \omega\frac{x_0}{v}$ is constant which represents **the phase delay** of the point x_0 with respect to the point in origin. Thus for any point *x* we have the relationship

$$\varphi = \omega\frac{x}{v} = \frac{2\pi}{T}\frac{x}{v} = \frac{2\pi x}{\lambda} \tag{12.27}$$

Then the phase difference $\varphi_2 - \varphi_1$ of two waves of different phase angle φ_1 and φ_2 is equal

$$\varphi_2 - \varphi_1 = \frac{2\pi}{\lambda}(x_2 - x_1) \tag{12.28}$$

or

$$x_2 - x_1 = \frac{\lambda}{2\pi}(\varphi_2 - \varphi_1) \tag{12.29}$$

where $x_2 - x_1$ represents **the path difference.** Therefore, from eq.(12.29) we have

$$d = x_2 - x_1 = \frac{\lambda}{2\pi}(\varphi_2 - \varphi_1) \tag{12.30}$$

Because for **a constructive interference** the phase difference equals $\varphi_2 - \varphi_1 = 2n\pi$ then

$$d = 2n\frac{\lambda}{2} \tag{12.31}$$

Condition for the phase difference of **a destructive interference** is $\varphi_2 - \varphi_1 = (2n+1)\pi$. Threfore,

$$d = (2n+1)\frac{\lambda}{2} \tag{12.32}$$

12.5 Standing wave

If we vibrate one end of a rope and the other end is kept fixed (Fig.12.7) a wave will travel down to the fixed end and be reflected back. There will be waves travelling in both directions and these waves will interfere. If vibrations of the rope will be at just the right frequency these two waves will interfere and **standing wave** will be produced. **This standing wave is the result of two waves travelling in opposite directions**. The points of **destructive interference, called nodes and of constructive interference, called antinodes** remain in fixed points.

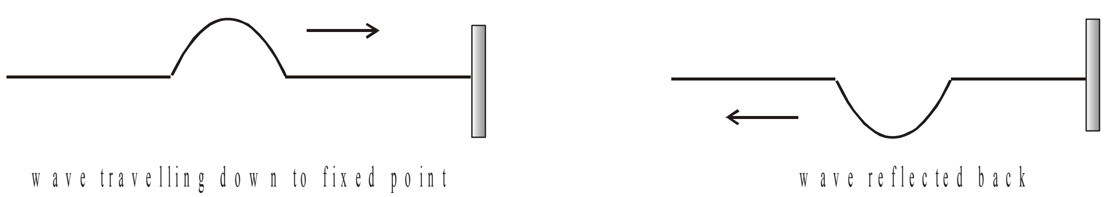

Fig.12. 7

Each of waves can be described in terms of an equation for the displacement $y(x,t)$ of a linear travelling wave as a function of position *x* and time *t*. We assume that the amplitudes, frequencies and wavelengths are the same. Then

$$y_1 = A_0 \sin(kx - \omega t) \tag{12.33}$$

$$y_2 = A_0 \sin(kx + \omega t) \tag{12.34}$$

Using the identity $\sin\alpha + \sin\beta = A_0 \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ we have

$$y = y_1 + y_2 = 2A_0 \sin(kx)\cos(\omega t) \tag{12.35}$$

From this equation we can see that **a particle at any point at position of x vibrate in simple harmonic motion** due to factor $\cos(\omega t)$. **We also see that all particles vibrate with the same frequency** $f = \frac{\omega}{2\pi}$.

The amplitude of the motion equals $A = 2A_0 \sin(kx)$ and depends on x. From this expression follows that

1. The amplitude of a standing wave has maximum equal to $2A_0$ when $\sin(kx) = 1$. From condition is valid for $kx = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$ or $kx = (2n+1)\frac{\lambda}{2}$. Solving it for x gives $x = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \ldots$. These positions accord with **the positions, which is called the antinodes**.
2. The amplitude of a standing wave equals zero, when $kx = n\pi$, where $n = 0, 1, 2, \ldots$ is integer that is at points $x = 0, \frac{\lambda}{2}, \lambda, \frac{3}{2}\lambda, \ldots$ what are **the positions called the nodes**.

Standing wave on the string fixed at one end is shown in Fig.12.8.

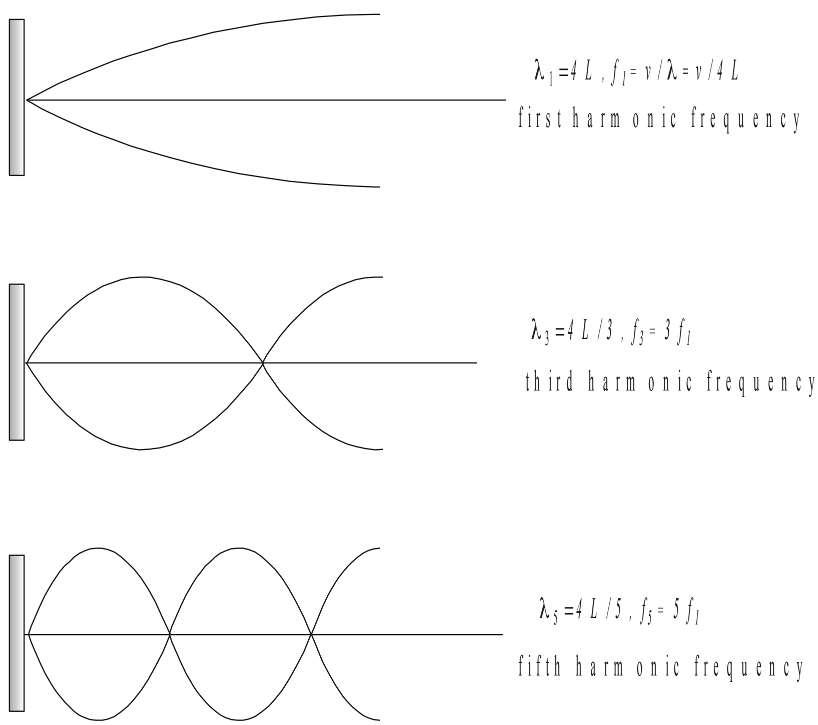

Fig.12. 8

When we consider the string fixed at its two ends, the function given by equation (12.35) is zero at $x = 0$ and $x = L$, where L is the length of the string. This satisfies the first condition $y = 0$ at $x = 0$. The second condition is if

$$kL = n\pi \tag{12.36}$$

where *n* is integer. Since the wave number equals

$$k = \frac{2\pi}{\lambda} \tag{12.37}$$

then equation (12.36) will be in form

$$2\pi \frac{L}{\lambda} = n\pi \tag{12.38}$$

or

$$\lambda = \frac{2L}{n}. \tag{12.39}$$

Situation is shown in Fig.12.9.

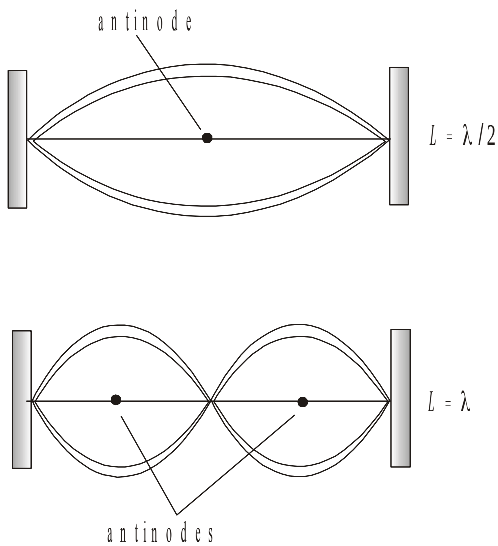

Fig.12. 9

12.6 The linear wave equation

Let us consider the general form of function of wave moving to the right $y(x,t) = f(x - vt)$, where *f* is the function of *x* and *t*. Let the quantity $(x - vt)$ be represented by *z*, where $z = x - vt$. Then the first derivate of $y(x,t)$ equals

$$\frac{\partial y}{\partial t} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial f}{\partial z}(-v) \tag{12.40}$$

and second derivate with respect to *t* equals

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\frac{\partial f}{\partial t}\right) = -v\frac{\partial^2 f}{\partial t^2}\frac{\partial z}{\partial t} = v^2\frac{\partial^2 f}{\partial z^2} \tag{12.41}$$

Similarly

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial z} \tag{12.42}$$

Since $\frac{\partial z}{\partial x} = 1$ and

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} \tag{12.43}$$

Comparing eq(12.41) and eq(12.43) gives

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{12.44}$$

where v is the wave velocity. This equation is one-dimensional linear equation. It applies in general to various types of waves moving through non-dispersive media. For the waves on string, y represents the vertical displacement, for sound waves, y corresponds to variation in the pressure or density of a gas in the case of electromagnetic waves, y corresponds to electric or magnetic field components.

12.7 The velocity of waves on strings

The velocity of mechanical waves depends only on the properties of the medium through which the disturbance travels.

At first we shall take an interest in the derivation of the expression for the speed of a pulse traveling on a stretched string with the velocity v.

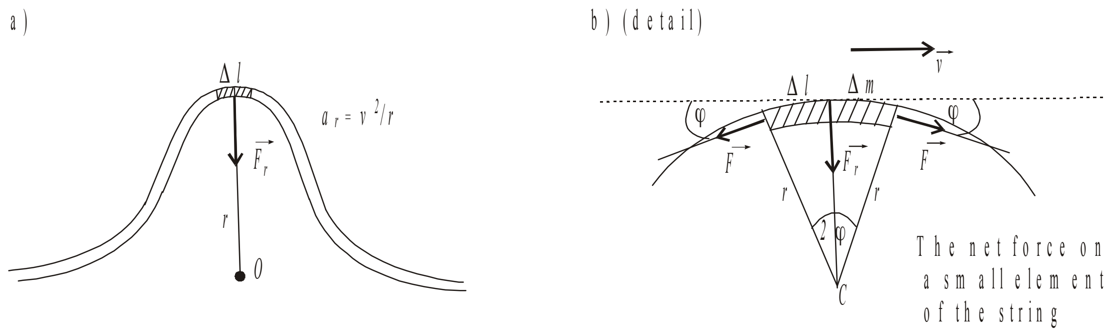

Fig.12. 10

Consider a pulse moving to the right with a uniform speed v, as is shown in Fig.12.10a,b. The force F acts on each side of segment of length Δl and mass Δm. Mass of this element is given by $\Delta m = \mu \Delta l$, where μ is the mass per unit length of the spring (linear density). We resolve this force into two perpendicular components. The horizontal components cancel and each vertical component $F \sin\varphi$ act radially toward the center of arc. Therefore, the total radial force equals

$$F_r = 2F \sin\varphi \tag{12.45}$$

Since the segment is small, φ is small and we can use the approximation

$$\sin\varphi = \varphi \qquad (12.46)$$

Hence, the force equals

$$F_r = 2F\varphi \qquad (12.47)$$

If we apply Newton`s second law to this segment, the radial component of motion gives

$$F_r = \Delta m \frac{v^2}{r} \qquad (12.48)$$

Inserting eq.(12.48) into eq.(12.47) gives

$$2F\varphi = \frac{2\mu r\varphi v^2}{r} \qquad (12.49)$$

where we used expression $\Delta m = l\mu = r\mu\varphi$ (see Fig.12.10). Solving for *v* gives

$$v = \sqrt{\frac{F}{\mu}} \qquad (12.50)$$

Remarks:

1. The derivation of velocity is based on the assumption that the pulse hight is small relative to the length of the string
2. We were able to use the approximation $\sin\varphi = \varphi$
3. The tension force *F* is the same at all points on the string
4. The velocity of longitudinal wave has similar form for a longitudinal wave traveling along a solid rod

   $$v = \sqrt{\frac{E}{\rho}} \qquad (12.51)$$

   where *E* is the elastic modulus of the material (Young`s modulus) and ρ is its volume density
5. For longitudinal wave traveling in a liquid or gas the velocity of the way is given by

   $$v = \sqrt{\frac{B}{\rho_0}} \qquad (12.52)$$

   where *B* is the bulk modulus of a matter defined as $B = \rho_0 \frac{\mathrm{d}p}{\mathrm{d}\rho}$ and ρ_0 is its density. (The derivation of this expression will be provided later.)

12.8 Sound waves

The sound waves are longitudinal waves. They travel through any medium (gas, solid or liquid) with a speed that depends on the properties of the medium. As sound waves travel through a medium, **the**

particles in the medium vibrate to produce the density and pressure changes along the direction of motion of the wave. This is in contrast to a traverse wave where the particle motion is perpendicular to the direction of wave motion.

The displacement of longitudinal wave involves the longitudinal displacements of individual molecules from their equilibrium positions. This is result in a series of high and low pressure regions.

There are three categories of longitudinal mechanical waves

1. **Audible waves**. These waves lie within the range of sensitivity of the human ear. This region is between 20 Hz and 20 000 Hz.
2. **Infrasonic waves** are ones with frequencies **below the audible range**. Such a wave is earthquake wave.
3. **Ultrasonic waves** are longitudinal waves with frequencies **above the audible range**. For example, they can be generated by inducing vibrations in quartz crystal with an applied alternating electric field.

12.9 Velocity of sound waves

We know that a one-dimensional sinusoidal wave travelling along x axis is represented by the relation

$$y(x,t) = A\sin(kx - \omega t) \tag{12.53}$$

where A is the amplitude of the wave, k is the wave number equals $2\pi/\lambda$ and $\omega = 2\pi f$ is the angular frequrncy.

The longitudinal sound wave is parallel to x axis and represents the displacement of a volume element from its equilibrium position.

We describe the pressure variation in a sound wave. We know that a pressure change p causes the fractional change in volume

$$p = -B\frac{\Delta V}{V} \tag{12.54}$$

where B is **the bulk modulus** and p is **the pressure difference** from the normal pressure in the absence of a wave. The negative sign in this equation means that the volume ΔV decreases if the pressure increases.

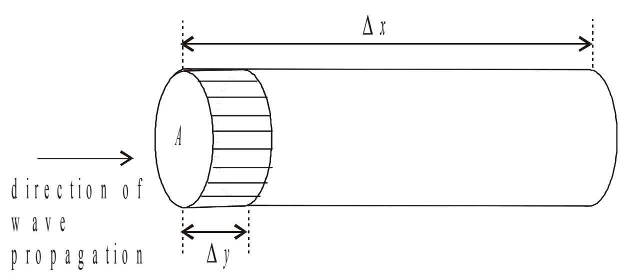

Fig.12. 11

From the Fig.12.11 we can see that

$$p = -B\frac{A\Delta y}{A\Delta x} \quad (12.55)$$

where $\Delta V = A\Delta y$ is the change in the volume and Δy is the change in the thickness of this layer as it is compressed or expanded. Taking the limit of $\Delta x \to 0$ gives

$$p = -B\frac{\partial y}{\partial x} \quad (12.56)$$

where it is used the partial derivate of *y* function since this function depends on the *x* and *t*.

For a sinusoidal wave the displacement, *y*, is given by eq.(12.53). Then we have

$$p = -BAK\cos(kx - \omega t) \quad (12.57)$$

From this equation we see that the pressure varies sinusoidally as well, but is out of phase from the displacement by 90°, as is shown in Fig.12.12.

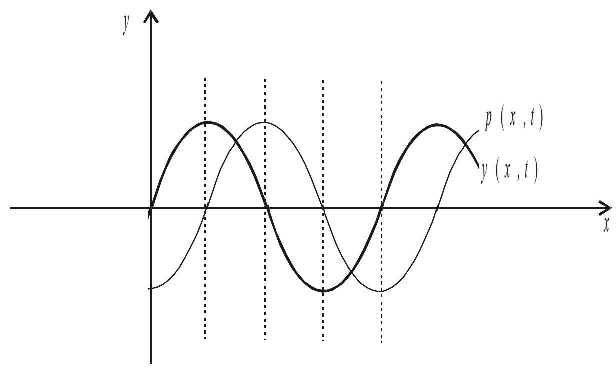

Fig.12. 12

The quantity *kBA* is called **the pressure amplitude p_{max}**. It represents the maximum and minimum amounts by which the pressure varies from the normal pressure. Using this fact we can express the change of pressure in the form

$$p = -p_{\max}\cos(kx - \omega t) \quad (12.58)$$

Now we shall derivate the expression for the velocity of the wave in the air. For simplicity we consider the wave propagated in a gas within a cylindrical tube, as is shown in Fig.12.13.

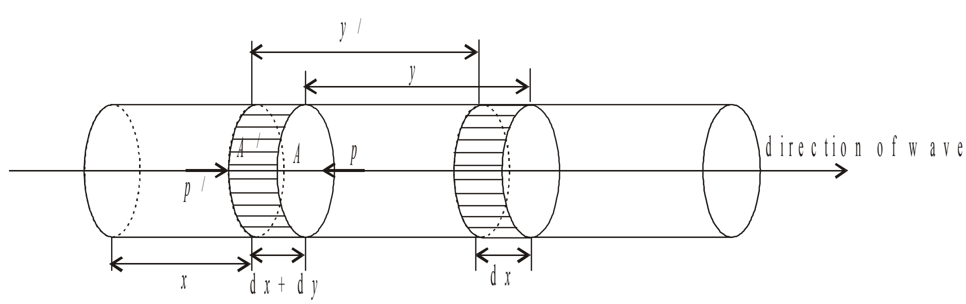

Fig.12. 13

Let us denote p_0 and ρ_0 the equilibrium pressure and density in a gas. If the pressure of the gas is disturbed, a volume element $A\mathrm{d}x$ is set in motion because the pressure *p* and p' on one side and the other are different. As a result, section *A* is displaced the amount *y* and section A' the amount y'. The thickness of the volume element after deformation is

$$\mathrm{d}x + (y' - y) = \mathrm{d}x + \mathrm{d}y \tag{12.59}$$

The mass within the undisturbed volume element is

$$\mathrm{d}m = \rho_0 A\mathrm{d}x \tag{12.60}$$

and mass of the disturbed element is

$$\mathrm{d}m = \rho A(\mathrm{d}x + \mathrm{d}y) \tag{12.61}$$

From these equations we have

$$\rho_0 A\mathrm{d}x = \rho A(\mathrm{d}x + \mathrm{d}y) \tag{12.62}$$

or

$$\rho_0 = \rho\left(1 + \frac{\partial y}{\partial x}\right)^{-1} \tag{12.63}$$

Since $\frac{\partial y}{\partial x}$ is small we can use two terms of Binomial expansion and so we have

$$\rho = \rho_0\left(1 - \frac{\partial y}{\partial x}\right) \tag{12.64}$$

or

$$\rho - \rho_0 = -\rho_0 \frac{\mathrm{d}y}{\mathrm{d}x} \tag{12.65}$$

The pressure is related to the gas density by the equation of state

$$p = p_0 \frac{V_0}{V} \tag{12.66}$$

or

$$p = p_0 \frac{\rho}{\rho_0} \tag{12.67}$$

From this we see that the pressure is function of density, i. e. $p = f(\rho)$.

Applying the Taylor`s expansion to this function and using the fact that the change in density is small we give

$$p = p_0 + (\rho - \rho_0)\frac{\mathrm{d}p}{\mathrm{d}\rho} \tag{12.68}$$

where we used the first and second terms of Taylor`s expansion only.

Using the definition of bulk in form

$$B = \rho_0 \frac{\mathrm{d}p}{\mathrm{d}\rho} \tag{12.69}$$

we have

$$p = p_0 + B\frac{\rho - \rho_0}{\rho_0} \tag{12.70}$$

Inserting eq.(12.65) into eq.(12.70) gives

$$p = p_0 - B\frac{\mathrm{d}y}{\mathrm{d}x} \tag{12.71}$$

This expression relates the pressure at any point in the gas to the deformation at the same point. We take a first derivate of *p* with respect to *x* and we have

$$\frac{\partial p}{\partial x} = - B\frac{\partial^2 y}{\partial x^2} \tag{12.72}$$

since the $p_0 = \text{const.}$ (p_0 is independent on *x*).

Now we write the equation of motion for the volume element $\mathrm{d}V$ of the gas. The mass of this element equals

$$\mathrm{d}m = \rho_0 \mathrm{d}V = \rho_0 A\mathrm{d}x$$

and its acceleration is given by

$$a = \frac{\partial^2 y}{\partial t^2}$$

The gas at the left of our volume element pushes to the right with the force pA and the gas at the right pushes to the left within force $p'A'$. Since $A \approx A'$, therefore, the resultant force acting on this element along positive *x* direction equals

$$\frac{\partial^2 y}{\partial t^2} = A\mathrm{d}x\rho_0 = - A(p - p')$$

or

$$\frac{\partial^2 y}{\partial t^2}\rho_0 = - \frac{\mathrm{d}p}{\mathrm{d}x} \tag{12.73}$$

where $(p - p') = \mathrm{d}p$ is the infinitesimal change in pressure. Comparing eq.(12.73) with eq.(12.72) gives

$$\frac{\partial^2 y}{\partial t^2}\rho_0 = B\frac{\partial^2 y}{\partial x^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} = \frac{B}{\rho_0}\frac{\partial^2 y}{\partial x^2}$$

You can see that this equation is linear wave equation for a harmonic wave propagating through gas. If we compare this equation with general form of linear wave equation given by eq.(12.44) we get

$$v = \sqrt{\frac{B}{\rho_0}} \qquad (12.74)$$

From this equation follows that the velocity of the wave is inversely proportional to the density of the gas

12.10Intensity of harmonic sound wave

The intensity, *I*, of sound is defined as the energy transmitted by a sound wave per unit time across unit area.

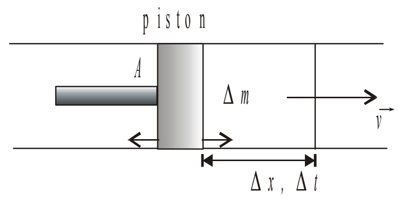

Fig.12. 14

Consider a layer of air of mass Δm and width Δx in front of piston oscillating with an angular frequency ω as is shown in Fig.12.14. The piston transmits energy to the layer of air in tube, causing the layer of width Δx and mass Δm to oscillate with amplitude $A_{\max}$. Since the average kinetic energy equals the average potential energy in the simple harmonic motion, the average total energy of the mass Δm equals its maximum kinetic energy. Therefore

$$\Delta E = \frac{1}{2}\Delta m v^2 = \frac{1}{2}\Delta m(\omega A_{\max})^2 = \frac{1}{2}(\rho A \Delta x)(\omega A_{\max})^2 \qquad (12.75)$$

where $A\Delta x$ is the volume of the layer and $A_{\max}$ is the displacement amplitude. Inserting these values into definition of *I* we have

$$I = \frac{\Delta E}{A\Delta t} = \frac{\frac{1}{2}\rho A \Delta x(\omega A_{\max})^2}{A\Delta t} = \frac{1}{2}\rho v(\omega A_{\max})^2 \qquad (12.76)$$

where $v = \frac{\Delta x}{\Delta t}$ is the velocity of the disturbance to the right.

From this equation we see that the intensity of the harmonic sound wave is proportional to the square of the amplitude (maximum of displacement) and to the square of the frequency. It is convenient to use a logarithmic scale, where **the sound level** β is defined by the equation

$$\beta = 10\log\left(\frac{I}{I_0}\right) \qquad (12.77)$$

where the constant I_0 is equal the reference intensity taken to be at the threshold of hearing equals $I_0 = 10^{-12}\ \mathrm{W/m^2}$ and I is the intensity in W/m^2 at a level β measured in decibels (dB). For example, the threshold of pain is

$$\beta = 10\log\frac{1}{10^{-12}} = 10\log\left(10^{12}\right) = 120\ \mathrm{dB} \tag{12.78}$$

Nearby jet plain $\beta = 150\ \mathrm{dB}$, rock concert $\beta = 120\ \mathrm{dB}$ and normal conversation $\beta = 50\ \mathrm{dB}$.

12.11Doppler effect

When a car is moving the frequency of the sound you hear is higher as the car approaches you and lower as it moves away from you.

We shall have a look at this effect in detail. We shall assume that air is at rest in our reference system as is shown in Fig.12.15. The distance between two successive wave picks is λ. If the frequency of the source is f that the time between emissions of successive wave peaks is equal to the period

$$T = \frac{1}{f} = \frac{\lambda}{v} \tag{12.79}$$

where v is the velocity of sound wave in air.

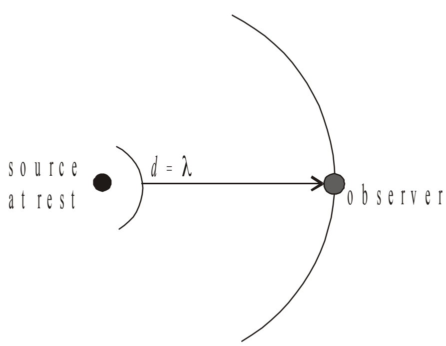

Fig.12. 16

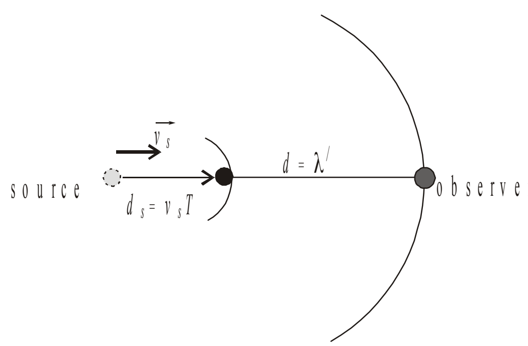

Fig.12. 15

1. In Fig.12.16 the source is moving with a velocity v_s **toward stationary observer**. In a time T the first wave peak is moved at distance $d = \lambda = vT$. In this time the source has moved a distance $d_s = v_s T$.

The distance

$$\lambda' = d - d_s = \lambda - v_s T = \lambda\left(1 - \frac{v_s}{v}\right) \tag{12.80}$$

is the distance between two successive wave peaks. Then the change in wavelength is

$$\Delta\lambda = \lambda' - \lambda = -v_s\frac{\lambda}{v} \tag{12.81}$$

From this equation we can see that $\Delta\lambda$ is proportional to the speed of the source. A new frequency

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda\left(1 - \frac{v_s}{v}\right)} \tag{12.82}$$

Since $f = \frac{v}{\lambda}$ then

$$f' = \frac{f}{1 - \frac{v_s}{v}} \tag{12.83}$$

This expression is valid for the source moving toward stationary observer. From this follows

$$f' > f \tag{12.84}$$

since the denominator is less of 1.

If the denominator will move away from observer (or receiver) the new wavelength will be $\lambda' = d + d_s$ and the change in wavelength will be

$$\Delta\lambda = \lambda' - \lambda = v_s \frac{\lambda}{v} \tag{12.85}$$

and new frequency

$$f' = \frac{f}{1 + \frac{v_s}{v}} \tag{12.86}$$

From this equation we can see that

$$f' < f \tag{12.87}$$

due to **the denominator which is greater of 1**.

2. Doppler effect also occurs when the source is at rest and the observer is in motion. The wavelength λ is not changed in this case but the wave velocity with respect to observer is changed.

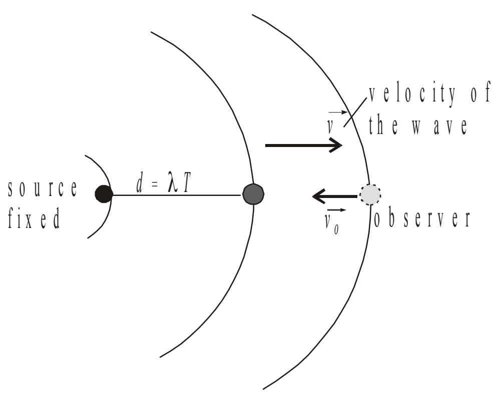

Fig.12. 17

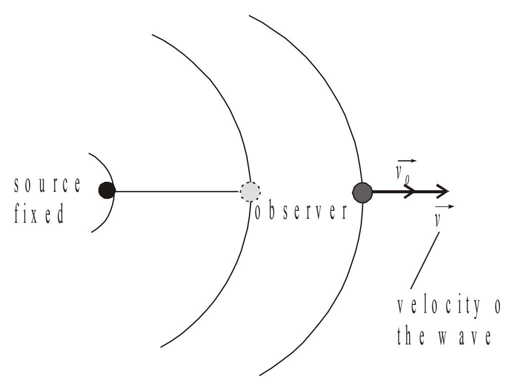

Fig.12. 18

Let the **observer moves toward the source** with a velocity v_o (Fig.12.16). The velocity of the wave relative to the observer is

$$v' = v + v_o \tag{12.88}$$

and new frequency

$$f' = \frac{v'}{\lambda} = \frac{v + v_o}{\lambda} \tag{12.89}$$

Since $\lambda = \frac{v}{f}$ then the new frequency equals

$$f' = \left(1 + \frac{v_o}{v}\right) f \tag{12.90}$$

if **the observer moves toward stationary source**. When **the observer is moving away from the source** the wave velocity relative to the observer is now

$$v' = v - v_o \tag{12.91}$$

and

$$f' = \left(1 - \frac{v_o}{v}\right) f \tag{12.92}$$

Example

The siren emits sound at frequency of 500 Hz. This sound wave falls on an object moving at a speed 3.4 m/s toward the stationary source. What is the frequency of the reflected wave?

Solution:

First, we must calculate the frequency of the moving object (observer). This frequency is given by eq. (12.90) as

$$f' = \left(1 + \frac{v_o}{v}\right) f$$

Substituting the values of $v_o = 3.4\ \mathrm{m/s}$, frequency of the source $f = 500\ \mathrm{Hz}$ and speed of the sound on air $v = 340\ \mathrm{m/s}$ gives

$$f' = 505\ \mathrm{s}^{-1}. \tag{1}$$

At second we calculate the frequency f'' of the reflected wave. In this case the object (observer) acts like a moving source emitting the reflected wave of the frequency f'' given by eq.(12.83)

$$f'' = \frac{f'}{1 - \frac{v_s}{v}} = 510\ \mathrm{s}^{-1} \tag{2}$$

Comparing eqs.(1), (2) gives that the reflected wave is shifted in frequency relative to the incident wave by 5 s^{-1}.

Remarks:

Doppler effect is a phenomenon common to all harmonic wave. For example, there is a shift in frequencies of light waves (electromagnetic waves) produced by the relative motion of source and observer. Astronomers use this effect to determination the relative motion of stars, galaxies and other celestial objects. There is known **the red** (or low frequency end) **shift of the visible light** of receding galaxies toward the red end of spectrum, lending confirmation to the theory of an expanding universe.

Another example of applicability of Doppler effect is in police radar system to measure the speed of cars.

12.12 Electromagnetic waves. Maxwell`s equations

In mechanics we found set of equations that would define completely the object or particle. These equations are called Newton`s law of gravitation.

The Scottish physicist Maxwell showed that electric and magnetic phenomena could be described by the help four equations involving electric and magnetic field. In electric field we introduced idea that a magnetic field is producing by any current

$$\oint_l \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \left(I + I_d \right) = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}\phi_e}{\mathrm{d}t} \tag{12.93}$$

where the second term in this equation, I_d , called displacement current is defined as $I_d = \varepsilon_0 \frac{\mathrm{d}\phi_e}{\mathrm{d}t}$ and

$\phi_e = \int_A E \cdot \mathrm{d}A$. Farady`s law of induction states that

$$\varepsilon = \int \vec{E} \cdot \mathrm{d}\vec{l} = - \frac{\mathrm{d}\phi_m}{\mathrm{d}t} \tag{12.94}$$

Gauss` law of electricity relates the electric field to its source and electric charges in form

$$\oint_A \vec{E} \cdot \mathrm{d}\vec{A} = \frac{\sum q_i}{\varepsilon_0} \tag{12.95}$$

In opposite the Gauss` law of magnetism states that the lines of induction are continuous it means that they do not begin or end. This fact is expressed in form

$$\oint_A B \cdot \mathrm{d}A = 0 \tag{12.96}$$

These four equations are called **the Maxwell`s equations in integral form**. The properties of material is described by three equations in forms

$$D = \varepsilon E \tag{12.97}$$

$$B = \mu H \tag{12.98}$$

$$j = \sigma E \tag{12.99}$$

12.13 Hertz`s discoveries

Maxwell showed that the electromagnetic waves are a natural consequence of the fundamental Maxwell`s equations. These waves were first generated and detected in 1887 by Hertz using electrical source. His experimental apparatus is shown in Fig.12.19.

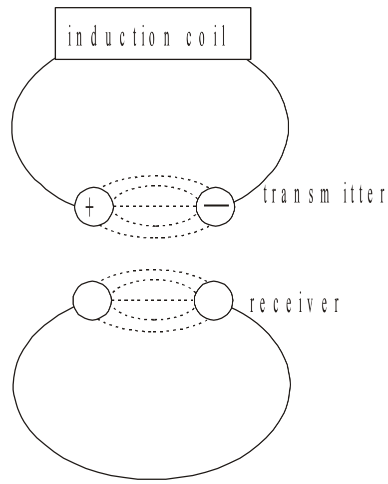

Fig.12. 19

An induction coil is connected to two spherical electrodes a narrow gap between them-transmitter. The coil provides short voltage surges to the spheres, making one positive, the other negative. A spark is generated between the spheres when the voltage between them reaches the breakdown voltage for air. As the air in the gap is ionized, it conducts more readily and the discharge between the spheres becomes oscillatory. This is equivalent to an CL curcuit, where the inductance L is that of the loop and the capacitance C is due to the spherical electrodes. Since L and C are quite small, the frequency of oscillation is very hight $\approx$ 100 MHz. Electromagnetic waves are radiated at this frequency as a result of the oscillation on free charges in the loop. Hertz was able to detect these waves using a single loop of wire with its own spark gap (receiver). This loop, placed several meters from the transmitter, has its own effective inductance, capacitance and natural frequency of oscillation. Sparks were induced across the gap of the receiving electrodes when the frequency of the receiver was adjusted to match that of the transmitter. Thus, Hertz demonstrates that the oscillating current induced in receiver was produced by electromagnetic waves radiated by the transmitter.

12.14 Plane electromagnetic waves

The properties of electromagnetic waves can be deduced from Maxwell`s equations. The following assumptions will be made

1. We shall assume that the electromagnetic wave is a plane wave it means that it travels in one direction. If the wave propagates in the *x* direction, the electric field E is in *y* direction and magnetic field B is in *z* direction as is shown in Fig12.20. Waves in which the electric and magnetic fields are restricted to being parallel to certain lines in *yz* plane are said to be **linearly polarized waves**.

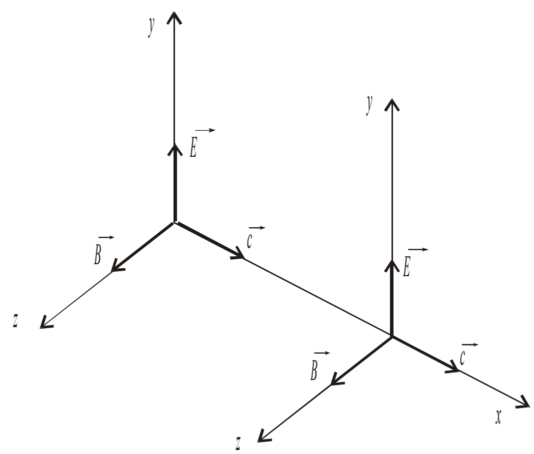

Fig.12. 20

2. We assume that E and B at any point depend upon *x* and *t* and not *y* and *z* coordinates of this point.

We shall use third and fourth Maxwell`s equations in form

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t} \qquad (12.100)$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\mathrm{d}\phi_e}{\mathrm{d}t} \qquad (12.101)$$

In empty space $Q = 0$, $I = 0$ and therefore we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t}$$

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \varepsilon_0 \frac{\mathrm{d}\phi_e}{\mathrm{d}t} \qquad (12.102)$$

Using these equations and plane wave assumption we obtain the differential equations relating E and B. From these equations follows that

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_y}{\partial t} \qquad (12.103)$$

$$\frac{\partial B_y}{\partial x} = - \varepsilon_0 \mu_0 \frac{\partial E_y}{\partial t} \tag{12.104}$$

(see derivation of these equations on the end of chapter)

Now we take the derivate of eq.(12.103) with respect to *x* and combine this with eq.(12.104). We drop the subscripts E_x and B_y and so

$$\frac{\partial^2 E}{\partial x^2} = - \frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) = - \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right) = - \frac{\partial}{\partial t}\left(- \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}\right)$$

or

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} \tag{12.105}$$

This equation is called **the wave equation for electromagnetic wave in free space**.

In the same manner taking a derivate of eq.(12.104) with respect to *x* and combining this with eq. (12.103) we get

$$\frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu \frac{\partial^2 B}{\partial t^2} \tag{12.106}$$

Eqs(12.104),(12.105) both have the form of the general equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{12.107}$$

with the speed

$$v = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \tag{12.108}$$

Taking $\mu_0 = 4\pi \times 10^{-7}$ Wb/A and $\varepsilon_0 = 8.85 \times 10^{-12}$ C/Nm^2 we find the value of velocity of the electromagnetic wave as $c = 2.99792 \times 10^8$ m/s . Since the speed is the same as the speed of light in empty space one is led to believe that **the light is an electromagnetic wave**.

Simple plane wave solution is sinusoidal wave for which the field amplitudes *E* and *B* vary with *x* and *t* according to the expression

$$E = E_{\max} \cos(kx - \omega t) \tag{12.109}$$

$$B = B_{\max} \cos(kx - \omega t) \tag{12.110}$$

where $E_{\max}$, $B_{\max}$ are the maximum values of the fields. Constant $k = 2\pi/\lambda$ is the wave number and ω is the angular frequency equals $\omega = 2\pi f$. Remember that frequency *f* is the number of cycles per second. The ratio

$$\frac{\omega}{k}=\frac{2\pi f}{\frac{2\pi}{\lambda}}=\lambda f=c \qquad (12.111)$$

If you can see this ratio equals the velocity of the light.

Remarks:

1. The solution of third and fourth equation are wavelike, where E and B satisfy the same wave equation
2. Electromagnetic waves travel through empty space with the speed of light (12.108)
3. The electric and magnetic field components are perpendicular to each other and also perpendicular to the speed of wave propagation
4. Electromagnetic waves obey the principle of superposition

12.15 Spectrum of electromagnetic waves

Since all electromagnetic waves travel through the vacuum with a speed c their frequency f and wavelength λ are related by expression

$$c=f\lambda \qquad (12.112)$$

The various types of electromagnetic waves are shown in table (Fig.12.21)

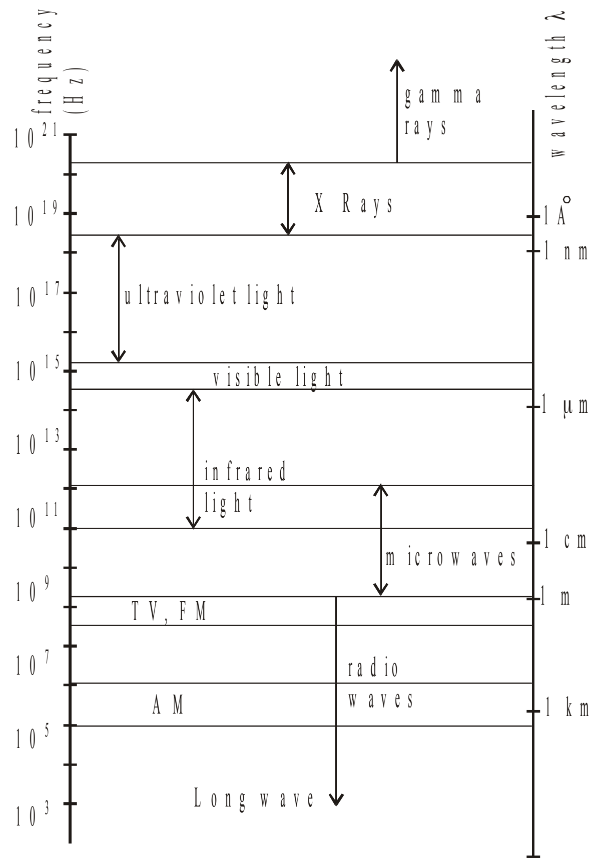

Fig.12. 21

APPENDIX

We start with Faraday`s law

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t}$$

We assume that the electromagnetic wave travel in the *x* direction with the electric field E in positive *y* direction and the magnetic field B in the positive *z* direction.

Consider a thin rectangle lying in the *xy* plane as is shown in Fig12.22. Then the integral

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = E(x + \mathrm{d}x, t)l - E(x,t)l = \frac{\partial E}{\partial x}\mathrm{d}xl \qquad (12.113)$$

We know that the magnetic flux through the rectangle of area $\mathrm{d}A = l\mathrm{d}x$ is approximately equals

$$\phi_m = Bl\mathrm{d}x$$

since $\mathrm{d}x$ is small compared with λ. Taking the derivate of ϕ_m with respect to *t* gives

$$\frac{\mathrm{d}\phi_m}{\mathrm{d}t} = -\, l\mathrm{d}x \frac{\mathrm{d}B}{\mathrm{d}t} \tag{12.114}$$

Substituting eq.(12.113) into (12.114) gives

$$\frac{\partial E}{\partial x}\mathrm{d}xl = -\,\mathrm{d}xl\frac{\partial B}{\partial t}$$

or

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{12.115}$$

In similar way we can derivate the eq.(12.104).

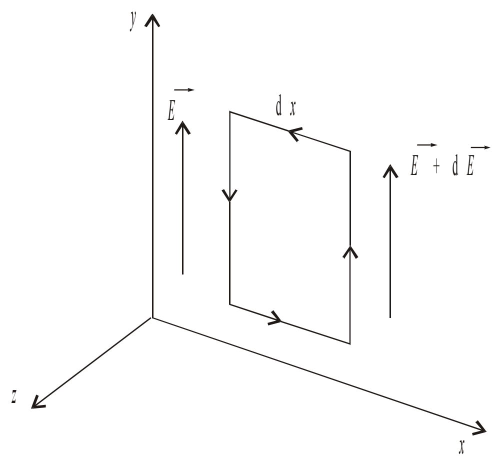

Fig.12. 22