# 11. OSCILLATIONS

## 11.1 Oscillations of a spring. Simple harmonic motion

We have been interesting in simple harmonic motion last term. We said that any vibrating system for which the restoring force is directly proportional to the negative of the displacement (as in oscillations of spring, simple pendulum, physical pendulum etc.) is said to exhibit simple harmonic motion describing by the equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx\ , \tag{11.1}$$

where $m$ is the mass which is oscillating and $k$ is the constant of proportionality. This constant is called spring constant. Rearranging this equation we obtain

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{m}{k}x = 0\ . \tag{11.2}$$

If we denote $\omega^2 = \frac{k}{m}$ then the equation gives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0\ . \tag{11.3}$$

This equation is called **equation of motion for a harmonic oscillator**. General solution of this second-differential equation is

$$x = A\cos(\omega t + \delta) + B\sin(\omega t + \delta), \tag{11.4}$$

where $A$ and $B$ are arbitrary constants and $\delta$ is called the phase constant. We can choose this value $\delta = 0$ . The speed of the harmonic motion is by definition

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)\ . \tag{11.5}$$

Applying the initial conditions $x(0) = A$ , $v(0) = 0$ at time $t = 0$ we have

$$x(0) = A\cos(0) + B\sin(0) = A$$

$$v(0) = -A\omega\sin(0) + B\omega\cos(0) = 0\ ,$$

where from the second equation we can see that constant $B$ has to equal zero, $B = 0$ . Then the motion is a cosine curve

$$x(t) = A\cos(\omega t + \delta)\ . \tag{11.6}$$

Situation is shown in Fig.11.1.

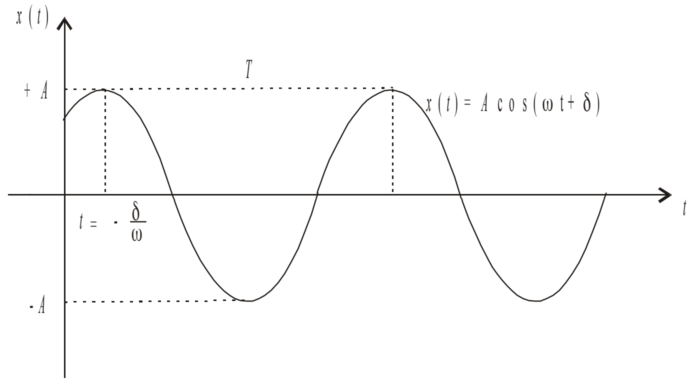


Fig.11. 1

Since the simple harmonic oscillator repeats its motion after a time equal to its period, *T*, and **sine** or **cosine** function repeats itself after $2\pi$ radians we have

$$\omega T = 2\pi$$

or

$$\omega = \frac{2\pi}{T} = 2\pi f ,$$

where *f* is **the frequency** of the motion and $\omega$ is called **angular frequency**. Since $\omega = \sqrt{\frac{k}{m}}$ then the period *T* equals

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{11.7}$$

and frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} . \tag{11.8}$$

From this equation we can see that **the period *T* is independent on the amplitude**. It is also clear that the greater mass *m* shifts the spring to the lower frequency. The frequency given by equation (11.8) is called **natural frequency**. The velocity *v* and acceleration *a* of simple harmonic oscillator is by the definition

$$v(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega A \sin(\omega t + \delta) \tag{11.9}$$

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 A \cos(\omega t + \delta) = -\omega^2 x(t) \tag{11.10}$$

From these equations we can see that the speed reaches its maximum of

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}} A \tag{11.11}$$

and acceleration of the oscillator has its maximum

$$a_{\max} = \omega^2 A = \frac{k}{m} A . \tag{11.12}$$

The displacement $x$, velocity $v$ and acceleration $a$ as a function of time $t$ are illustrated in Fig.11.2.

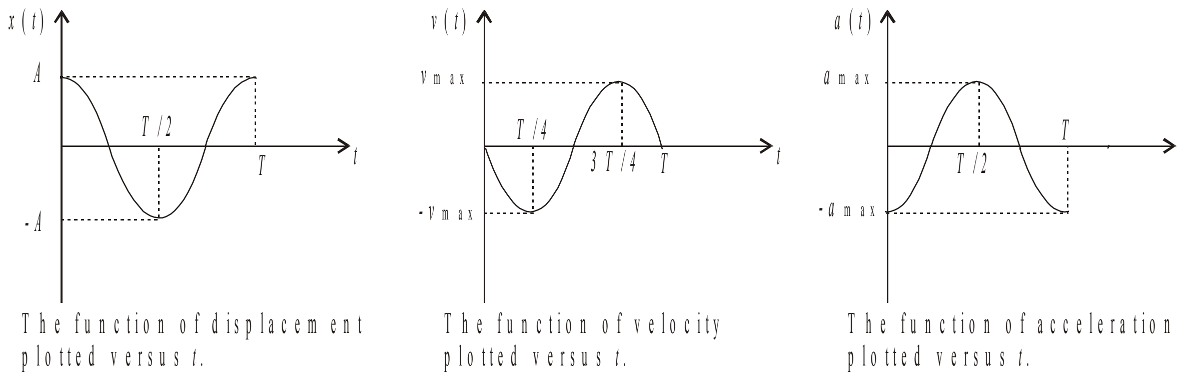


Fig.11. 2

## 11.2 Energy stored in a simple harmonic oscillator

Consider the simple harmonic oscillator such a mass $m$ oscillating on the end of massless spring. The motion of this oscillator caused by the restoring force is in the form

$$F = - kx \tag{11.13}$$

By the definition the potential energy of this oscillator equals

$$U = - \int_0^x \vec{F} \cdot \mathrm{d}\vec{x} = + \int_0^x kx\mathrm{d}x = k \int_0^x x\mathrm{d}x = \frac{1}{2} kx^2 \tag{11.14}$$

since the constant of integration equals zero for initial conditions $x = 0$, $U = 0$. The total mechanical energy equals $E = K + U$ . Therefore

$$E = \frac{1}{2} mv^2 + \frac{1}{2} kx^2 \tag{11.15}$$

$$E = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t) + \frac{1}{2} kA^2 \cos^2(\omega t) = \frac{1}{2} kA^2 \left(\sin^2(\omega t) + \cos^2(\omega t)\right) = \frac{1}{2} kA^2 \tag{11.16}$$

where $v$ is the velocity at a distance $x$ measured from the equilibrium position. There are two important situations:

1. For the extreme points $x = A$, $x = - A$ is the velocity equals zero. Then the total energy is the potential energy only, it means

   $$E = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 . \tag{11.17}$$

   From this equation we can see that **the total mechanical energy of a simple harmonic motion is proportional to the square of the amplitude**.

2. At a equilibrium $(x = 0)$ all energy is kinetic one

$$E = \frac{1}{2} m v_{\max}^2 = \frac{1}{2} k A^2 \tag{11.18}$$

where $v_{\max} = -\omega A$ is the maximum value of velocity during the harmonic motion.

3. For any point we can write eqs.(11.15), (11.16).

The graph of potential energy $U$ versus $x$ is shown in Fig.11.3.

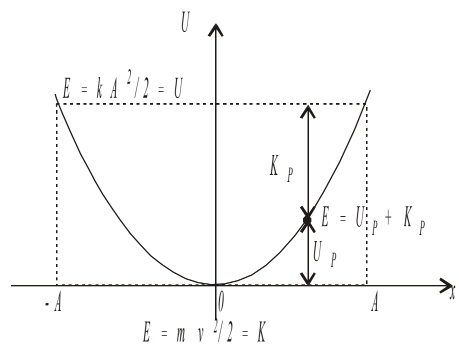


Fig.11. 3

Notes:

1. The phase constant corresponds to the choise of the zero of time.
2. If we start with $x$ positive and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, the return force gives an acceleration which induces a negative velocity. By the time $t$, $x$ returns zero and the negative velocity is maximum. Then the return force becomes positive. Finally the velocity is zero, but by that time the displacement is large and negative, and the process reverses.
3. The angular frequency of oscillation $\omega$ is related to physical properties of the system: $\omega^2$ equals the return force per unit displacement per unit mass.
4. The examples of the simple harmonic motion are simple pendulum and physical pendulum, if the angles through they pass are small.

## 11.3 Damped harmonic motion

**The amplitude of any real oscillating object decreases in time**. The damping is due to the resistance of the air and to the friction. Fig.11.4 shows a typical curve for such an oscillating motion.

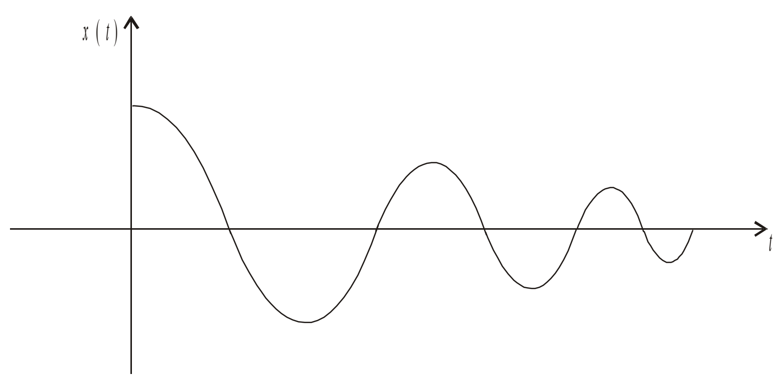




Fig.11. 4

The damping force opposes the motion and in many cases is directly proportional to the speed

$$F_{\text{dam}} = -bv\,, \tag{11.19}$$

where $b$ is a constant. For a mass oscillating on the end of spring the restoring force is $F = -kx$. Using Newton`s second law we have

$$ma = -kx - bv \tag{11.20}$$

or

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = 0 \tag{11.21}$$

This equation is **equation of motion for damped oscillator**. The solution of this equation can be written in form

$$x(t) = Ae^{-\alpha t}\cos(\omega' t)\,, \tag{11.22}$$

where $A$, $\alpha$ and $\omega'$ are assumed to be constant and at $t = 0$ is $x = A$. To determine the constants $\alpha$ and $\omega'$ we take the first and second derivate of equation (11.22) as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\alpha Ae^{-\alpha t}\cos(\omega' t) - \omega' Ae^{-\alpha t}\sin(\omega' t)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \alpha^2 Ae^{-\alpha t}\cos(\omega' t) + \alpha\omega' Ae^{-\alpha t}\sin(\omega' t) + \alpha\omega' Ae^{-\alpha t}\sin(\omega' t) - \omega'^2 Ae^{-\alpha t}\cos(\omega' t)\,.$$

If we substitute these expressions to the equation of an oscillatory motion then we obtain

$$Ae^{-\alpha t}\left[(m\alpha^2 + m\omega'^2 - b\alpha + k)\cos(\omega' t) + (2\alpha\omega' m - b\omega')\sin(\omega' t)\right] = 0 \tag{11.23}$$

**This equation must be zero for all time**.

1. We choose the time $t = 0$. For this time $\sin(\omega' t) = 0$ and the relation (11.23) is reduced to

$$m\alpha^2 + m\omega'^2 - b\alpha + k = 0 \tag{11.24}$$

2. For the time equal $t = \frac{\pi}{2\omega'}$ is $\cos(\omega' t) = 0$ and relation (11.23) reduces to

$$2\alpha m - b = 0 \text{ or } \alpha = \frac{b}{2m} \tag{11.25}$$

Inserting this value into eq.(11.24) gives

$$\omega' = \sqrt{\alpha^2 - \frac{b\alpha}{m} + \frac{k}{m}} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\,. \tag{11.26}$$

Then solution of the equation of oscillatory motion can be written as

$$x = Ae^{\frac{-b}{2m}t}\cos(\omega' t)\,, \tag{11.27}$$

where $\omega'$ is given by eq.(11.26).

The frequency has the value

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \qquad (11.28)$$

From this expression we can see that **the frequency is less than for an undamped simple harmonic motion.**

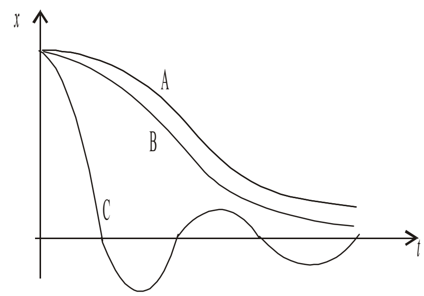


Fig.11. 5

From this we can make the following conclusions:

1. If $b^2 > 4mk$ , the damping is so large and the system is for a long time in the equilibrium. This system behaves as **overdamped** (Fig.11.5 curve A).
2. If $b^2 = 4mk$ the equilibrium is reached in the short time. This system is called **critical damping** (Fig.11.5 curve B).
3. If $b^2 < 4mk$ the system makes several swings before coming to rest. This system is **underdamped** (Fig.11.5 curve C).

### 11.4 Forced vibrations. Resonance

Let us suppose that an object may be vibrated at the frequency $f$ due to the external force acting on it. In this case the vibrations are called **forced vibrations**.

We shall take an interest in the important case when the external force is represented by the expression

$$F_{\text{ext}} = F_0 \cos(\omega t), \qquad (11.29)$$

where $\omega = 2\pi f$ is an angular frequency and $F_0$ is the amplitude of the applied force. Then the equation of motion will be in form

$$ma = -kx - bv + F_0 \cos(\omega t) \qquad (11.30)$$

or

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0 \cos(\omega t). \qquad (11.31)$$

Solution of this second-order differential equation is

$$x(t) = A_0 \sin(\omega t + \varphi_0) \qquad (11.32)$$

where

$$A_0 = \frac{F_0}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(b\frac{\omega}{m}\right)^2}} \tag{11.33}$$

$$\varphi_0 = \arctan\left(\frac{\omega_0^2 - \omega^2}{\omega\frac{b}{m}}\right) \tag{11.34}$$

**If you can see the amplitude of motion, $A_0$, depends strongly on the difference between the applied frequency $\omega$ and the natural frequency $\omega_0$**. The graph of $A_0$ versus $\omega$ for three various values of the damping constant *b* is shown in Fig.11.6a.

From eq.(11.33) follows:

1. If $\omega \approx \omega_0$ (see curve A) the damping is not too large. This case is known as **resonance** and the natural frequency $\omega_0$ of the system is called **resonant frequency** of this system.
2. If $b = 0$, resonance occurs at $\omega = \omega_0$ and the resonant peak of $A_0$ becomes infinity (see Fig.11.6b)
3. For real system, *b* is never zero and the resonant peak is finite and it does not occur precisely at $\omega = \omega_0$.
4. If the damping is large, there is little or no peak (see curve C)

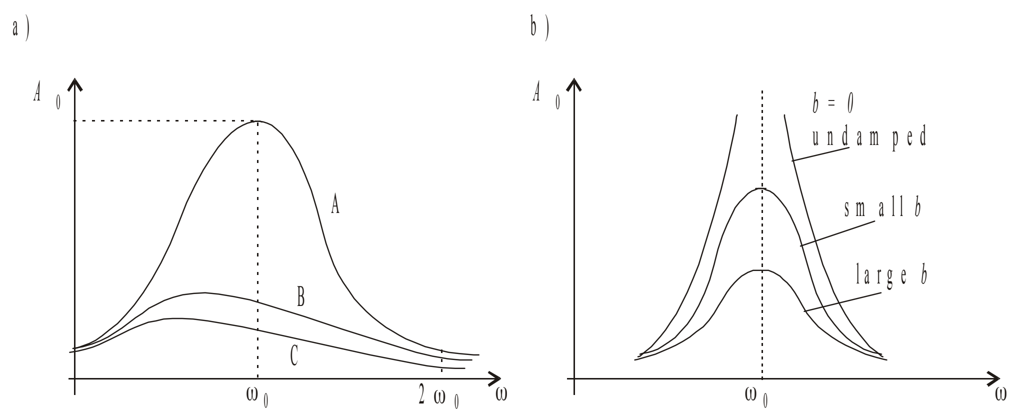


Fig.11. 6

Notes:

Every system has a natural fundamental vibration frequency. If the forces are exerted on that system at the right frequency and phase, sympathetic vibrations can be excited. Oscillating forces at the right frequency can cause sympathetic vibrations of catastrophic proportions. For example, if the forces applied to bridge by the wind are applied in pulses at a natural frequency of the bridge, the amplitude of the bridge`s oscillations increases and the bridge could no larger stand the stress.

Another example is vibrations of mechanical machines. These are often broken apart if one vibrating part is at resonance with some other moving part.

Finally, soldiers marching in cadence across bridges have been known to set up resonance vibrations in the structure, causing the bridges to collapse.

### 11.5 Superposition of two harmonic motions

Consider the two simple harmonic motions of the same directions. We shall assume that both motions have the same frequencies $\omega_1 = \omega_2 = \omega$, different amplitudes $A_1 \neq A_2$ and different phase constants $\varphi_1 \neq \varphi_2$. Hence the first motion is described by the equation

$$x_1(t) = A_1 \cos(\omega t + \varphi_1) \tag{11.35}$$

and second one by

$$x_2(t) = A_2 \cos(\omega t + \varphi_2)\,. \tag{11.36}$$

To find the resultant motion we shall use the principle of superposition, which says that the displacement of the resultant motion equals to the sum of displacements of both motions. Thus,

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) = A_1 \cos(\omega t + \varphi_1) + A_2 \cos(\omega t + \varphi_2) = \\ &= (A_1 \cos\varphi_1 + A_2 \cos\varphi_2)\cos(\omega t) - (A_1 \sin\varphi_1 + A_2 \sin\varphi_2)\sin(\omega t) \end{aligned} \tag{11.37}$$

Note that the terms in parenthesis are constant. If you can see from this expression the resultant motion is also **harmonic**, with the same frequency $\omega$ but with different amplitude and phase. To find these unknown quantities we express the displacement of the resultant motion in standard form

$$x(t) = A\cos(\omega t + \varphi) = A\cos\varphi\cos(\omega t) - A\sin\varphi\sin(\omega t) \tag{11.38}$$

where $A$ and $\varphi$ represents the amplitude and phase of the resultant motion. The equations (11.37) and (11.38) must be identical at every instant of time and so the coefficients at $\cos(\omega t)$ and $\sin(\omega t)$ must be the same. Hence

$$A\cos\varphi = A_1 \cos\varphi_1 + A_2 \cos\varphi_2 \tag{11.39}$$

$$A\sin\varphi = A_1 \sin\varphi_1 + A_2 \sin\varphi_2 \tag{11.40}$$

Dividing of these equations gives

$$\tan\varphi = \frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_2 \cos\alpha\varphi_2} \tag{11.41}$$

To find the amplitude of the resultant motion we square equations (11.39) and (11.40) and after their adding we have

$$A^2 = A_1^2 + 2A_1A_2\cos(\varphi_2 - \varphi_1) + A_2^2 \qquad (11.42)$$

From this equation follows:

1. If $\varphi_2 - \varphi_1 = 2k\pi$, where $k = 0, 1, 2, \ldots$ then $\cos(\varphi_2 - \varphi_1) = 1$ and the amplitude of the resulting motion reaches its maximum

   $$A_{\max} = A_1 + A_2 \qquad (11.43)$$

2. If $\varphi_2 - \varphi_1 = (2k+1)\pi$, where $k = 0, 1, 2, \ldots$ then $\cos(\varphi_2 - \varphi_1) = -1$ and the amplitude of resultant motion reaches its minimum

   $$A_{\min} = A_1 - A_2 \text{ if } A_1 > A_2 \qquad (11.44)$$

   $$A_{\min} = A_2 - A_1 \text{ if } A_1 < A_2 \qquad (11.45)$$

**Example**

Consider a particle which undergoes **simple harmonic motion along two perpendicular directions *x* and *y*** describing by the equations

$$x = A_x\cos(\omega t + \varphi_x)$$

$$y = A_y\cos(\omega t + \varphi_y) \qquad (1)$$

Determine the resultant motion for these conditions:

1. If phases are the same, it means $\varphi_x = \varphi_y = \varphi$
2. The amplitudes are equal $A_x = A_y = A$ and phase difference $\varphi_y - \varphi_x = \pm 90^\circ$
3. Phase difference $\varphi_y - \varphi_x = \pm 90^\circ$ and $A_x \neq A_y$.

<u>Solution:</u>

1. Since the phases are equal then

   $$x = A_x\cos(\omega t + \varphi)$$

   $$y = A_y\cos(\omega t + \varphi) = \frac{A_y}{A_x}x$$

The result is equation of a straight line of slope $A_y / A_x$. From this follows that the resultant motion will be straight line motion in (*xy*) plane as is shown in Fig.11.7.

2. $\varphi_y - \varphi_x = \pm\frac{\pi}{2}$, $A_x = A_y = A$. We rewrite these equations into form

$$x = A\cos(\omega t + \varphi)$$

$$y = A\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A\sin(\omega t + \varphi)$$

From these equations we have

$$x^2 + y^2 = A^2$$

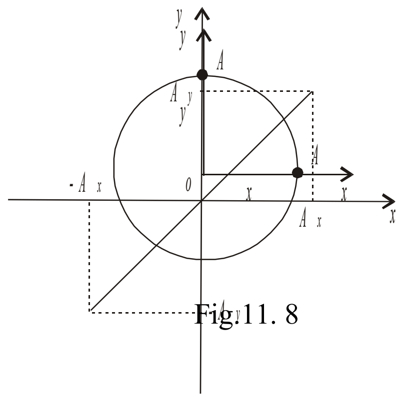

Fig.11. 8

Fig.11. 7

which is the equation of a circle in the (*xy*) plane of radius *A* as is shown in Fig.11.8.

3. $\varphi_y - \varphi_x = \pm\frac{\pi}{2}$, $A_x \neq A_y$. Inserting these expressions into eq.(1) gives

$$x = A_x\cos(\omega t + \varphi)$$

$$y = A_y\cos\left(\omega t + \varphi - \frac{\pi}{2}\right) = A\sin(\omega t + \varphi)$$

or

$$\frac{x}{A_x} = \cos(\omega t + \varphi)$$

$$\frac{y}{A_y} = \sin(\omega t + \varphi)$$

We shall calculate the sum of square of these equations. Hence

$$\frac{x^2}{A_x^2} + \frac{y^2}{A_y^2} = \cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi) = 1$$

which is the equation of the elipse with **major axis equals** $2A_x$ and **minor axis equals** $2A_y$ as is shown in Fig.11.9.

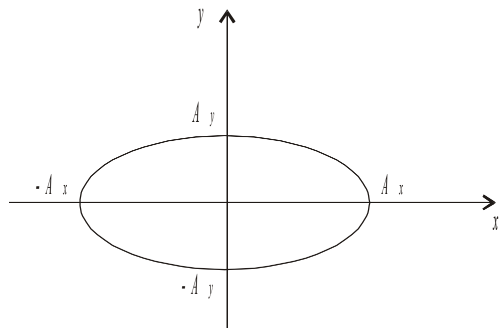


Fig.11. 9