10. MAGNETIC FIELD

The phenomenon of magnetism have been known to the Greeks around 800 B. C. They discovered that certain stones, now called magnetit (Fe_3O_4) attract pieces of iron. In 1269 Pierre de Maricourt found that the directions formed lines that encircle the sphere passing through two points diametrically opposite to each other, which he called poles of the magnet. The experiments showed that every magnet, regardless of its shape, has two poles, called north and south poles, which exhibits forces on each other in a manner analogous to electric charges. That is, like poles repel each other and unlike poles attract each other.

In 1600 Wiliam Gilbert extended these experiments to a variety of materials. Using the fact that a compass needle orients in preferred direction, the suggested that the earth itself is a large permanent magnet. In 1750 John Michell used a torsion balance to show that magnetic poles exert attractive or repulsive forces on each other and these forces vary as the inverse square of their separation. Although the forces between two magnetic poles are similar to the forces between two electric charges, there is an important difference. **Electric charges can be isolated** (electron or proton), whereas **magnetic poles cannot be isolated**. It means, that **magnetic poles are always in pairs**. No matter how many times a permanent magnet is cut, each pieces will always have a north and south pole.

The relationship between magnetism and electricity have discovered in 1819 by Danish scientist Hans Oersted. He found that electric current in a wire deflected a nearby compass needle. Thereafter, Andre Ampere obtained quantitative laws of magnetic force between current-carrying conductors. In 1820, further connections between electricity and magnetism have demonstrated by Faraday and independently by Joseph Henry. They showed that an electric current could be produced near the circuit or by changing the current in another, nearby circuit.

10.1 Properties of magnetic field. Magnetic force

Experiments on the motion of various charged particles moving in magnetic field give the following results:

1. The magnetic force is proportional to the charge q and speed v of the particle.
2. The magnitude and direction of the magnetic force depend on the velocity of the particle and on the magnitude and direction of the magnetic field.
3. When a charge particle moves in a direction parallel to magnetic field vector, the force on the charge is zero.
4. When the velocity vector makes an angle φ with magnetic field, the magnetic force acts in a direction perpendicular to both v and B.

5. The magnetic force on positive charge is in the direction opposite the direction of the force and on a negative charge moving in the same direction.
6. If the velocity vector makes an angle φ with the magnetic field, the magnitude of the magnetic force is proportional to $\sin\varphi$.

These observations can be summarized in the form

$$F = qv \times B, \qquad (10.1)$$

where the direction of the magnetic force is given by direction of $v \times B$, it means that its direction is given by right-hand rule as is shown in Fig.10.1 for a positive and negative charge.

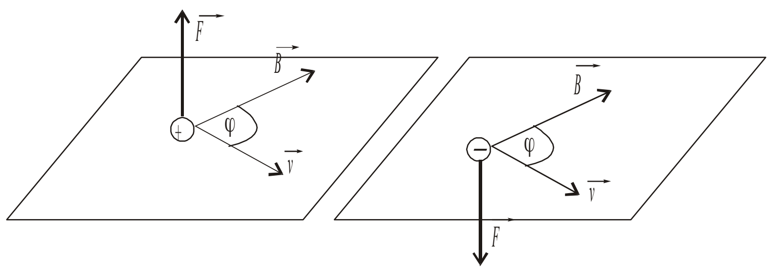

Fig.10. 1

The magnitude of the magnetic force has value

$$F = qvB\sin\varphi, \qquad (10.2)$$

where φ is the angle between v and B. Furthermore, the force has its maximum value

$$F = qvB, \qquad (10.3)$$

where v is perpendicular to B and minimum value

$$F = 0, \qquad (10.4)$$

if v is the same direction (or opposite direction) as B. In this case a particle with velocity v moves along the magnetic field B. The SI unit of magnetic field is **weber per square meter** (W/m^2) also called **the tesla** (T). For example, conventional laboratory magnets can produce magnetic field as large 2.5 T, supperconducting magnets can generate magnetic field as high as 25 T. The magnetic near the earth`s surface is about 0.5×10^{-4} T.

10.2 Magnetic force on a current-carrying conductor

Let us consider a straight segment of wire of length l and cross-sectional area A, carrying a current I in a uniform magnetic field B as in Fig.10.2.

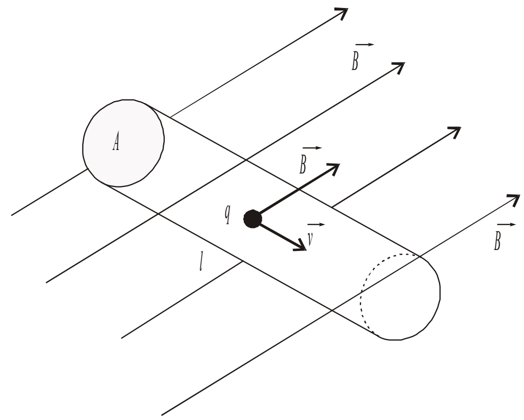

Fig.10. 2

The magnetic force on a charge q moving with the velocity v is $qv \times B$. To find the total force on a wire, we multiply this force by the number of charges in the segment

$$F = (qv \times B) nlA,$$

where n is the number of charges per unit volume and lA is the volume segment. This expression can be written in a more general form using the equation $I = nqvA$. Therefore F is expressed as

$$F = Il \times B, \qquad (10.5)$$

where l is a vector in the direction of the current I. The magnitude of l equals the length l of the segment.

Now consider an arbitrary shaped wire of a uniform cross section in external magnetic field, as is shown in Fig.10.3. It follows from eq(10.5) that the magnetic force on a very small segment $\mathrm{d}l$ in the magnetic field B is given by

$$\mathrm{d}F = I\mathrm{d}l \times B. \qquad (10.6)$$

To get a total force F on the wire, we integrate this equation over the length of the wire

$$\vec{F} = I \int_a^b \mathrm{d}\vec{l} \times \vec{B}, \qquad (10.7)$$

where a, b represent the end points of the wire. Note, this expression is valid for the steady I (I=constant).

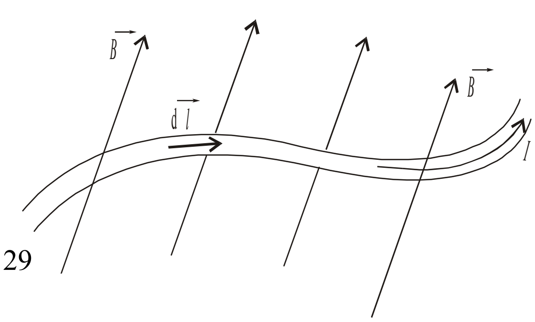

Fig.10. 4

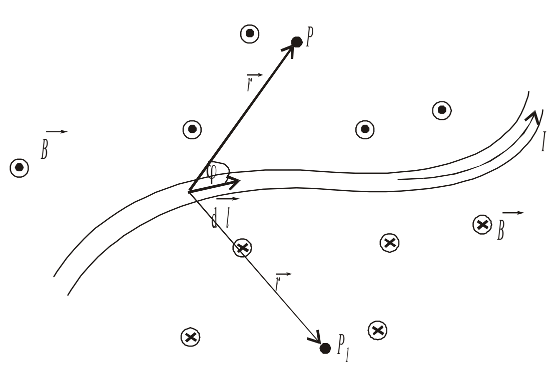

Fig.10. 3

10.3 Biot-Savart law

Jean Babtiste Biot and Felix Savart found that a conductor carrying a steady current produces a force on a magnet. From their experimental results, Biot and Savart were able to arrive at an expression that gives the magnetic field at some point in space in term of current that produces the field. **Biot-Savart law** says that if a wire carries a steady current, *I*, the magnetic field $\mathrm{d}B$ at a point *P* associated with an element $\mathrm{d}l$ equals (see Fig.10.4)

$$\mathrm{d}\vec{B} = k_m \frac{I\mathrm{d}l \times r}{r^3}, \tag{10.8}$$

where k_m is a constant, r is the vector pointing from the element $\mathrm{d}l$ to the given point. The constant k_m is usually written as

$$k_m = \frac{\mu_0}{4\pi}, \tag{10.9}$$

where μ_0 is constant, called **permeability of free space** and its value is $\mu_0 = 4\pi \times 10^{-7}$ Wb/A.m. Hence the Biot-Savart law can be written

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}l \times r}{r^3}. \tag{10.10}$$

To find the total magnetic field B at some point due to a conductor of finite size, **we must sum up contributions from all current elements** making up the conductor

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}l \times r}{r^3} = \frac{\mu_0 I}{4\pi}\int\frac{\mathrm{d}l \sin\angle(\mathrm{d}l, r)}{r^2}\vec{b}^{\circ}, \tag{10.11}$$

where the integral is taken over the conductor. The magnetic field B has the following properties:

1. The vector B is perpendicular both to $\mathrm{d}l$ (which is in the direction of current) and to the vector r directed from the element $\mathrm{d}l$ to the point *P*.
2. The magnitude of B is inversely proportional to r^2, where *r* is the distance from the element to the point *P*.
3. The magnitude of B is directly proportional to the current *I* and to the length $\mathrm{d}l$ of the element.
4. The magnitude of B is directly proportional to the $\sin\varphi$, where φ is the angle between the vectors $\mathrm{d}l$ and r.
5. The direction of B (or unit vector b°) is given by the right-hand rule.

Example

Calculate the magnetic field at the point *P* located at the distance *a* from the wire carrying a constant current *I* and placed along *x* axis, as is shown in Fig.10.5.

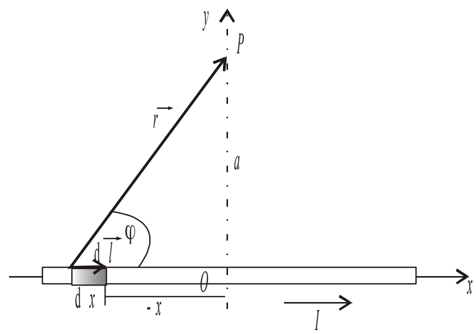

Fig.10. 5

<u>Solution:</u>

An element $\mathrm{d}l$ is at distance *r* from point *P*. Then

$$\mathrm{d}x \times r = |\mathrm{d}l||r| \sin\varphi . b^{\circ} = \mathrm{d}x r \sin\varphi . b^{\circ} .$$

Substituting this expression into Biot-Savart law (eq.10.10) gives

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \frac{I \mathrm{d}x \sin\varphi}{r^2} \vec{b}^{\circ} \qquad (1)$$

where b° is a unit vector pointing out of the paper. In order to integrate this expression, we must relate the variable φ, *x* and *r*. From geometry in Fig.10.5 we obtain

$$r = \frac{a}{\sin\varphi} \qquad (2)$$

$$\cot\varphi = -\frac{x}{a} \text{ or } x = -a\cot\varphi$$

and $$\mathrm{d}x = \frac{a}{\sin^2\varphi}\mathrm{d}\varphi \qquad (3)$$

Substituting (2) and (3) into (1) gives

$$\mathrm{d}B = \frac{\mu_0 I}{4\pi} \sin\varphi \, \mathrm{d}\varphi .$$

We can now obtain the total field at point *P* by integrating this equation over all elements subtanding angles ranging from φ_1 to φ_2. Then

$$B = \frac{\mu_0 I}{4\pi a} \int_{\varphi_1}^{\varphi_2} \sin\varphi \, \mathrm{d}\varphi = \frac{\mu_0 I}{4\pi a}\left(\cos\varphi_2 - \cos\varphi_1\right) \qquad (4)$$

Consider a special case of an infinitely long, straight wire. In this case $\varphi_1 = 0$ and $\varphi_2 = \pi$. Since $\left(\cos 0^{\circ} - \cos\pi\right) = 2$, equation (4) becomes

$$B = \frac{\mu_0 I}{2\pi a}. \qquad (10.12)$$

Example

Calculate the magnetic field at an axial point *P* at a distance *x* from the center of loop of wire of radius *R* located in the (*x*, *y*) plane and carrying a steady current *I* (Fig.10.6).

<u>Solution:</u>

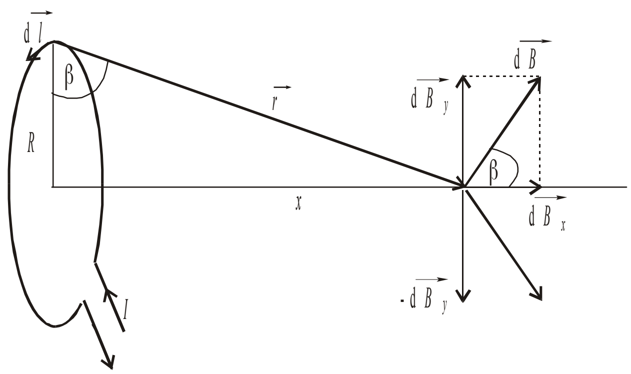

Fig.10. 6

In this case, any element $\mathrm{d}l$ is perpendicular to r° (unit vector oriented toward a point *P*). Furthermore, all elements around the loop are at the same distance *r* from point *P* given by

$$r^2 = x^2 + R^2 .$$

Hence, the magnitude of $\mathrm{d}B$ due to the element $\mathrm{d}l$ is given by

$$\left|\mathrm{d}\vec{B}\right| = \frac{\mu_0 I}{4\pi} \frac{\left|\mathrm{d}l \times r^{\circ}\right|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l \sin 90^{\circ}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{\mathrm{d}l}{r^2} .$$

The direction of $\mathrm{d}B$ is perpendicular to the plane formed by r° and $\mathrm{d}l$. This vector can be resolved into a component $\mathrm{d}B_x$, along the *x* axis and component $\mathrm{d}B_y$, which is perpendicular to *x* axis. When the components perpendicular to *x* axis are summed over the whole loop, the result is zero. It is given by the symmetry any element on one side of the loop. Therefore, the resultant field at *P* must be along *x* axis and can be found by integrating the components

$$\mathrm{d}B_x = \mathrm{d}B \cos\beta$$

as

$$B_x = \oint \mathrm{d}B \cos\beta = \frac{\mu_0 I}{4\pi} \oint \frac{\mathrm{d}l \cos\beta}{x^2 + R^2} .$$

Since β , *x* and *R* are constants for all elements of the loop and since

$$\cos\beta = \frac{R}{\sqrt{x^2 + R^2}}$$

we get

$$B_x = \frac{\mu_0 IR}{4\pi\left(x^2 + R^2\right)^{3/2}} \oint \mathrm{d}l = \frac{\mu_0 IR^2}{2\left(x^2 + R^2\right)^{3/2}}, \tag{1}$$

where we have used the fact that $\oint \mathrm{d}l = 2\pi R$. To find the magnetic field at the center of loop, we set $x = 0$ into (1) and then

$$B = \frac{\mu_0 I}{2R}. \tag{10.13}$$

10.4 The magnetic force between two parallel conductors. Definition of ampere.

We know that when the current carrying conductor is placed in an external magnetic field the magnetic force will act on it. It is easy to understand that two current-carrying conductors will exert magnetic forces upon each other. As we shall see, such forces can be used as the basis for defining the ampere and the coulomb.

Consider two long, straight, parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same directions, as is shown in Fig.10.7.

Wire (2), which carries a current I_2, set up a magnetic field B_2 at the position of wire (1). The direction of B_2 is perpendicular to the wire. According to eq. (10.5), the magnetic force on the length l of the wire is $F_1 = I_1 l \times B_2$. Since l is perpendicular to B_2, the magnitude of F_1 is given by $F_1 = I_1 l B_2$. Since the field due to wire (2) is given by equation

$$B_2 = \frac{\mu_0 I_2}{2\pi a}. \tag{10.14}$$

We see that

$$F_1 = I_1 l B_2 = I_1 l\left(\frac{\mu_0 I_2}{2\pi a}\right) = \frac{l\mu_0 I_1 I_2}{2\pi a}. \tag{10.15}$$

We can rewrite this in terms of the force per unit length as

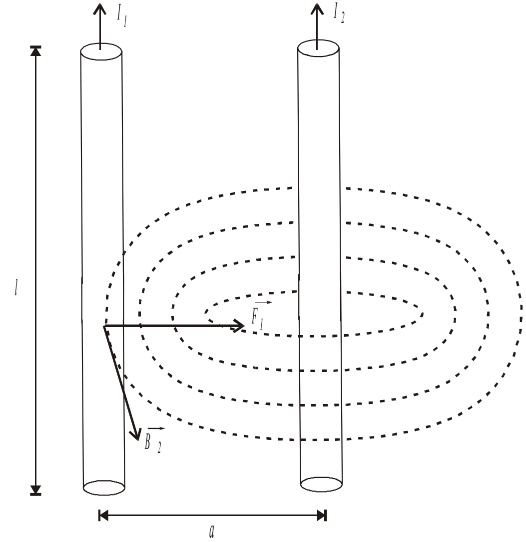

Fig.10. 7

$$\frac{F_1}{l} = \mu_0 \frac{I_1 I_2}{2\pi a}. \tag{10.16}$$

The direction of F_1 is perpendicular toward wire (2), since $l \times B_2$ is oriented toward the wire (2). If one considers the field set up at wire (2) due to wire (1), the force F_2 on wire (2) is found to be equal to and opposite F_1. This is what one would be expected, because Newton`s third law of action and reaction must be obeyed. When the currents are in opposite directions, the forces are reversed. We can say that **parallel conductors carrying currents in the same directions attract each other.** The force between two parallel wires each carrying a current is used to definition of the ampere as follows:

If two long, parallel wires 1 m apart carry the same current and force per unit length on each wire is 2×10^{-7} N/m, then the current is defined to be 1 A. This numerical value of 2×10^{-7} N/m is obtained from eq. (10.16), with $I_1 = I_2 = 1$ A and $a = 1$ m.

The SI unit of charge, the coulomb, can now be defined in terms of the ampere as follows:

If a conductor carries a steady current of 1 A, then the quantity of charge that flows through a cross section of the conductor in 1 s is 1 C.

10.5 Ampere`s law

A simple experiment carried out by Oersted (in 1820), demonstrate the fact that a current-carrying conductor produces a magnetic field. This observation showed that the direction of B is consistent with right-hand rule, as is shown in Fig.10.8.

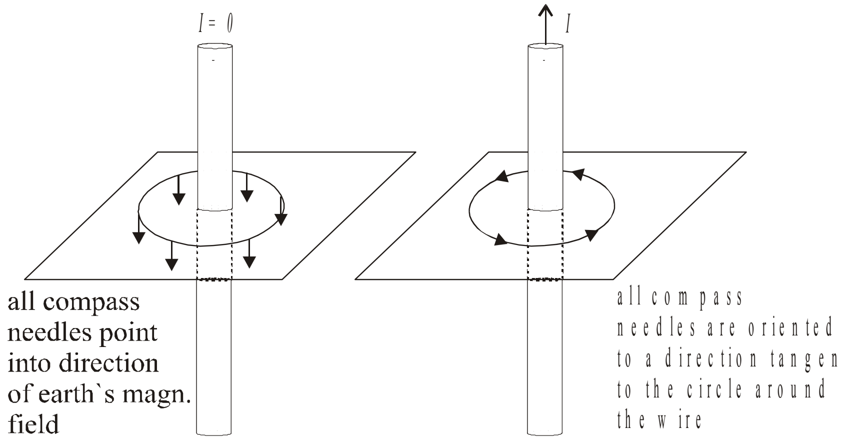

Fig.10. 8

When the wire carries a strong, steady current, the compass needles will all deflect in a direction tangent to the circle. When there is no current in the vertical wire, all compass needles point in the same direction given by the earth`s magnetic field.

Now let us evaluate the product $B \cdot \mathrm{d}l$ and the sum of these products **over the closed circular path** centered on the wire as

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = B\oint \mathrm{d}l = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I \tag{10.17}$$

because the $B \cdot \mathrm{d}l = B\mathrm{d}l$ (the vector $\mathrm{d}l$ and B are parallel at each point, and $\oint \mathrm{d}l = 2\pi r$ is the circumference of the circle. This result is known as **Ampere`s law**. It says that the line integral of $B \cdot \mathrm{d}l$ around any closed path equals $\mu_0 I$, where *I* is the total steady current passing through any surface bounded by the closed path

$$\oint B \cdot \mathrm{d}l = \mu_0 I\,. \tag{10.18}$$

Note that **Ampere`s law is valid only for steady current**.

Example

Calculate the magnetic field a long, straight wire of radius *R* carries a steady current *I* at a distance *r* from the center of wire in the regions $r \geq R$ and $r<R$.

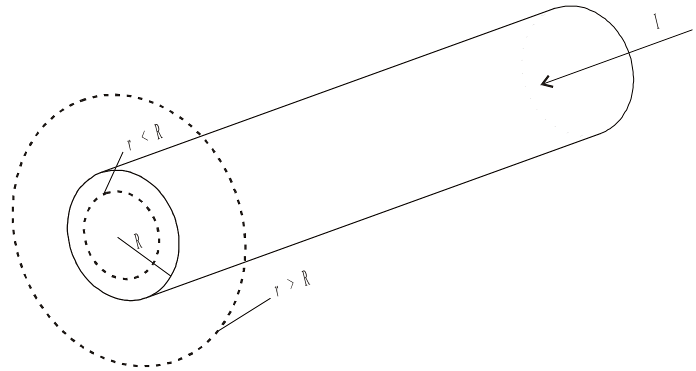

Fig.10. 9

Solution:

Situation is shown in Fig.10.9. In region, where $r \geq R$ we choose a circular path of radius *r* centered at the wire. From symmetry, we see that B must be constant in magnitude and parallel to $\mathrm{d}l$ at every point on the path. Ampere law applying to the path gives

$$\oint B \cdot \mathrm{d}l = B\oint \mathrm{d}l = B(2\pi r) = \mu_0 I$$

or

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi}\frac{1}{r}.$$

This result is identical to eq.(10.12). Now consider the interior of the wire, it means *r*<*R*. In this case the current I_1 enclosed by the path is less that the total current *I*. Since the current is assumed to be uniform over the cross section of the wire, we see that the fraction of the current enclosed the path of radius *r*<*R* must be equal the ratio of the area πr^2 enclosed the path *r*<*R* and the cross sectional area πR^2 of the wire. Then

$$\frac{I_1}{I} = \frac{\pi r^2}{\pi R^2} \text{ or } I_1 = \frac{r^2}{R^2} I .$$

Now we apply Ampere`s law to path *r*<*R*.. This gives

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = B(2\pi r) = \mu_0 I_1 = \mu_0 \left(\frac{r^2}{R^2}\right) I$$

or

$$B = \frac{\mu_0 I}{2\pi R^2} r \text{ for } r<R.$$

The magnetic field *B* versus *r* is drawn in Fig.10.10.

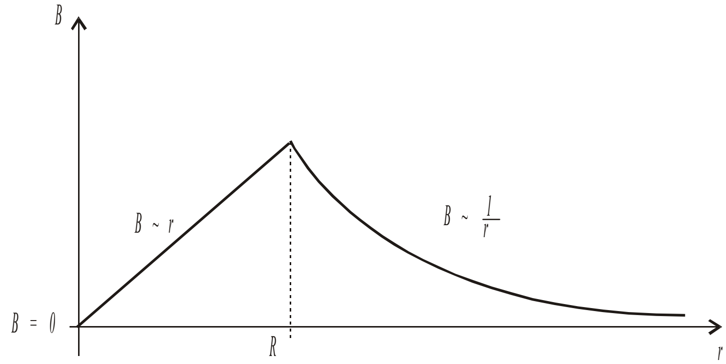

Fig.10. 10

Note that inside the wire $B \to 0$ as $r \to 0$. This result is similar to that of the electric field inside the uniformly charge rod.

10.6 A current loop in a uniform magnetic field. Torque. Torque on a current loop in uniform magnetic field

Consider rectangular loop carrying a current *I* in the presence of a uniform magnetic field in the plane of the loop as is shown in Fig.10.11. The forces on sides *a* are zero since these wires are parallel to the magnetic field B and hence

$$F = I \int \mathrm{d}l \times B = 0 \tag{10.19}$$

for these sides. The magnitude of the forces on the sides of length *b* is equal

$$\left|\vec{F}_1\right| = \left|\vec{F}_2\right| = \left|I \int_0^b \mathrm{d}\vec{l} \times \vec{B}\right| = IbB \tag{10.20}$$

Since $\left|\mathrm{d}l \times B\right| = \mathrm{d}lB\sin 90^\circ = \mathrm{d}lB$. The direction of F_1 (the force acting on the left side of the loop) is out of the paper and that of F_2 (the force on the right side of the loop) is into the paper. Remember that these directions are determined by the right-hand rule. Situation is shown in Fig.10.12.

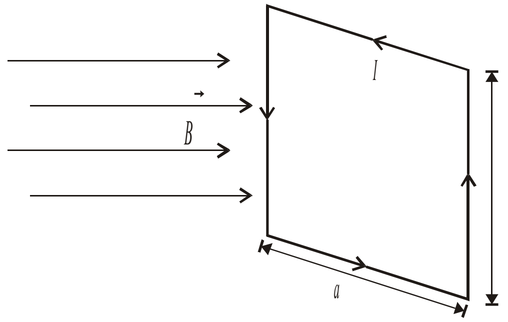

Fig.10. 11

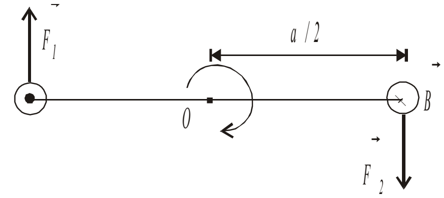

Fig.10. 12

If you can see from this figure, the forces F_1 and F_2 on the sides of length *b* create a torque that tends to twist the loop clockwise. The magnitude of this torque, $\tau_{\max}$, is given by

$$\tau_{\max} = F_1 \frac{a}{2} + F_2 \frac{a}{2} = IbB\frac{a}{2} + IbB\frac{a}{2} = IabB\ , \tag{10.21}$$

where the moment arm about *O* equals $\frac{a}{2}$ for each force. Since the area of the loop is $A = ab$, the torque can be expressed as

$$\tau = IAB\ . \tag{10.22}$$

Remember that this result is valid only when the field B is **perpendicular to the plane of loop**.

Now suppose the magnetic field makes an angle ϕ with respect to line perpendicular to the plane of loop, as is shown in Fig.10.13.

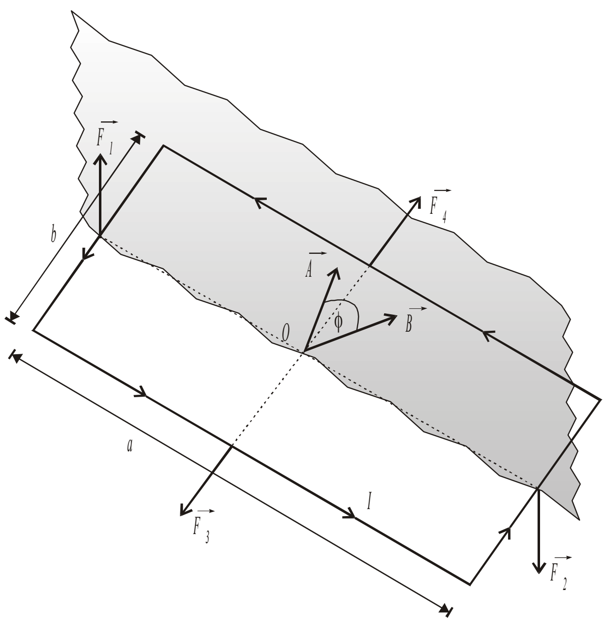

Fig.10. 13

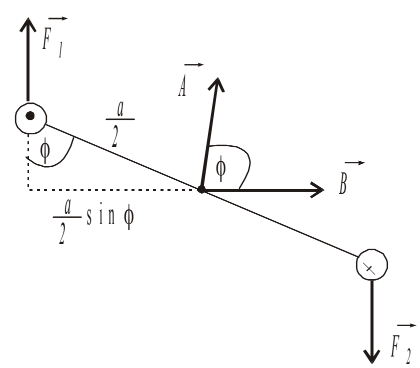

Fig.10. 14

We shall assume that the field B is perpendicular to the sides of length *b*. In this case, the magnetic forces F_3 and F_4 on the sides of length *a* cancel each other and produce no torque. However, the forces F_1 and F_2 acting on the sides of length *a* form a couple and hence produce a torque about any point, as is shown in Fig.10.14. We note that the moment arm of the force F_1 equals the moment arm of force F_2 and equals $\frac{a}{2}\sin\phi$. Since the forces F_1 and F_2 are equal in magnitude

$$F_1 = F_2 = IbB\ , \tag{10.23}$$

therefore the net torque about *O* has a magnitude given by

$$\tau = \tau_1 + \tau_2 = F_1 \frac{a}{2} \sin\phi + F_2 \frac{a}{2} \sin\phi = IbB \frac{a}{2} \sin\phi + IbB \frac{a}{2} \sin\phi = IabB \sin\phi = IAB \sin\phi ,$$

where $A = ab$ is the area of the loop.

This results shows:

1. **the torque has the maximum value** IAB, when the field is parallel to the plane of loop ⇒ the angle $\phi = 90^\circ$
2. **the torque is zero** when the field is perpendicular to the plane of loop ⇒ $\phi = 0$.

A convenient vector expression for the torque is the following cross-product relationship

$$\tau = IA \times B , \tag{10.24}$$

where A is the vector perpendicular to the plane of loop and having the magnitude equal to the area of the loop. The sense of A is given by the right-hand rule: by rotating the fingers of the right hand in the direction of the current in the loop, the thumb points in the direction of A.

We shall defined **the magnetic moment μ** of the loop as

$$\mu = IA . \tag{10.25}$$

The SI unit of magnetic moment μ is $A.m^2$. Using this equation, the torque can be expressed as

$$\tau = \mu \times B . \tag{10.26}$$

Notes:

1. This result is analogous to the torque acting on an electric dipole moment p in the presence of an external electric field E, where $\tau = p \times E$.
2. Although the torque was obtained for a particular orientation of B with respect to the loop, eq. (10.26) is valid for any orientation.
3. Although the torque expression was divided for a rectangular loop, the result is valid for a loop af any shape.

10.7 Motion of the charged particle in a magnetic field

We found that the magnetic force (given by eq.(10.1)) acting on a charged particle in a magnetic field is always perpendicular to the velocity.

Consider the special case of a positively charged particle moving in a uniform magnetic field with its initial velocity vector perpendicular to the field, as is shown in Fig.10.15.

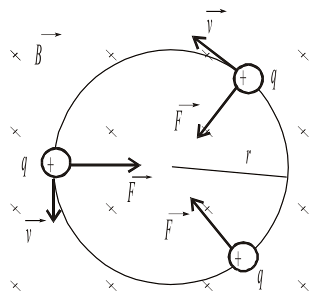

Fig.10. 15

Figure shows that the particle moves in a circle whose plane is perpendicular to the magnetic field. This is because the magnetic force F is at right angles to v and B and has a constant value $F = qvB$. If the particle is deflected by the force the directions of v and F are continuously changing. Therefore **the force F is centripetal force**. The sense of the rotation in this case is counter-clockwise for a positive charge.

From Newton`s second law follows

$$qvB = \frac{mv^2}{r} \tag{10.27}$$

or

$$r = \frac{mv}{qB} . \tag{10.28}$$

We see, that the radius of the circle is proportional to the momentum mv of the particle and is inversely proportional to the magnetic field. The angular frequency of the rotating particle is given by eqs.(2.27-2.30)

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{10.29}$$

and the period

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} . \tag{10.30}$$

These results show that the angular frequency and period of the circular motion do not depend on the speed of particle or the radius of the orbit.

10.8 Motion of charged particle in both an electric field and the magnetic field. Lorentz`s force

For many situations, the charge under consideration will be moving with a velocity v in the presence of both an electric field E and a magnetic field B. Therefore, the charge particle will experience both electric force qE and a magnetic force $qv \times B$, and so **the total force on charge** will be given by

$$F = qE + qv \times B \tag{10.31}$$

This force is known as **Lorentz`s force**.

10.9 The magnetic field of solenoid

A solenoid is a long wire wound in the form of helix. With this configuration, one can produce a reasonably uniform magnetic field within a small volume of the solenoid`s interior region if the consecutive turns are closely spaced. The net magnetic field is the vector sum of the fields due to all the turns.

If the turns are closely spaced and the solenoid is of finite length, the field lines are shown in Fig.10.16. The field inside the solenoid is nearly uniform and strong. An inspection of this field distribution exterior to the solenoid shows a similarity with the field of a bar magnet. It means, one end of the solenoid behaves like the north pole of a magnet while the opposite end behaves like the south pole. As the length of the solenoid increases, the field within it becomes more and more uniform. One approaches the case of **an ideal solenoid** when the turns are closely spaced and length is long compared with the radius. In this case, the field outside the solenoid is weak compared with the field inside the solenoid and the field inside is uniform over a large volume.

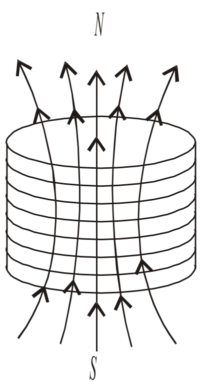

Fig.10. 17

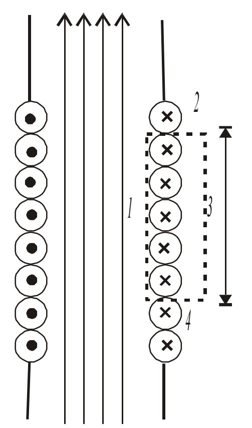

Fig.10. 16

We can use Ampere`s law to obtain an expression for the magnetic field inside an ideal solenoid. A longitudinal cross section of part of our ideal solenoid, as is shown in Fig.10.17, carries a current *I*. For an ideal solenoid, B inside the solenoid is uniform and parallel to the axis and B outside is zero. Consider a rectangular path of length *l* and width W. We can apply Ampere`s law to this path as

$$\oint B \cdot dl = \int_{(1)} B \cdot dl + \int_{(2)} B \cdot dl + \int_{(3)} B \cdot dl + \int_{(4)} B \cdot dl = \int_{(1)} B \cdot dl = B\int dl = Bl \tag{10.32}$$

where contribution along side (3) is clearly zero, since $B = 0$ in this region and contributions from sides (2) and (4) are both zero since B is perpendicular to $\mathrm{d}l$ along this path.

The right side of Ampere`s law involves the total current that passes through the area bound by the path of integration. In our case, the total current through the rectangular path equals the current through each turn multiplied by the number of turns. If N is the number of turns in the length l, then the total current through the rectangle equals NI. Therefore, Ampere`s law applied to this path gives

$$\oint B \cdot \mathrm{d}l = Bl = \mu_0 NI \qquad (10.33)$$

or

$$B = \mu_0 \frac{N}{l} I = \mu_0 nI \ , \qquad (10.34)$$

where $n = N/l$ is the number of turns per unit length. This equation is valid only for points near center of a very long solenoid.

10.10 Lines of induction. Magnetic flux

Just as we represented the electric field by electric lines we can represent magnetic field by **lines of induction** as follows:

1. The tangent to a line of induction B at any point gives the direction of B at that point.
2. The lines of induction are drawn so that the number of lines per unit cross-sectional area $\mathrm{d}A$ is proportional to the magnitude of B.

There is, however, a great difference between lines of E which represent the electric field and lines of induction. The lines of induction they can never start and they never stop. They will close back on themselves, making closed loops.

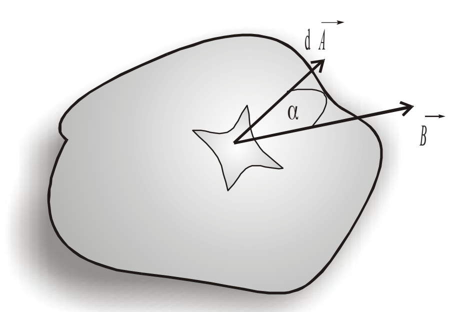

Fig.10. 18

Consider an element of area $\mathrm{d}A$ on an arbitrary shaped surface, as is shown in Fig.10.18. If the magnetic field at this element is $B \cdot \mathrm{d}A$, where $\mathrm{d}A$ is the vector perpendicular to the area $\mathrm{d}A$. Hence, the magnetic flux ϕ_m through the surface is given by

$$\phi_m = \int B \cdot \mathrm{d}A . \qquad (10.35)$$

Consider the special case of a plane of area A and a uniform field B, which makes an angle α with the vector $\mathrm{d}A$. The magnetic flux through the plane in this case is given by

$$\phi_m = BA\cos\alpha . \qquad (10.36)$$

If the magnetic filed lies in the plane (see Fig.10.19.a) then $\alpha = 90^\circ$ and flux is zero. If the field is perpendicular to the plane (see Fig.10.19.b), then $\alpha = 0^\circ$ and the flux equal BA.

Since B has units of Wb/m^2, or T, the unit of flux is Wb.

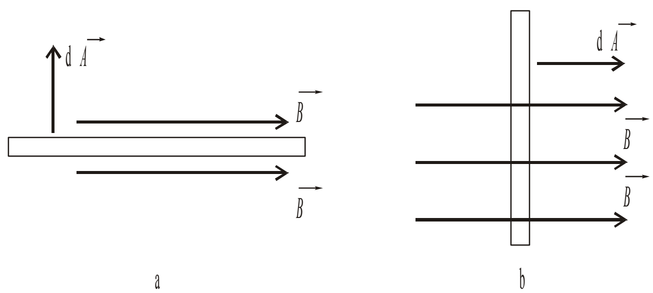

Fig.10. 19

10.11 Gauss` law of magnetism

We found that the flux of the electric field through the close surface surrounding a net charge is proportional to that charge (see eq.8.44). In order words, the number of electric filed lines leaving the surface depends only on the net charge within in. This property is based on the fact that electric field lines originate on electric lines.

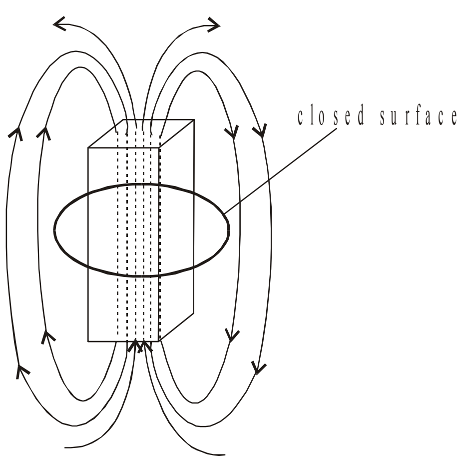

Fig.10. 20

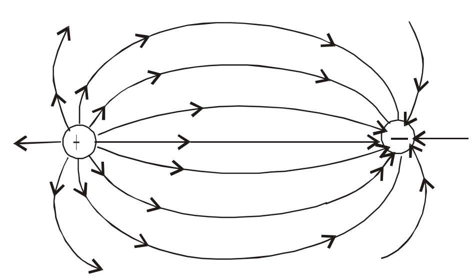

Fig.10. 21

The situation is different for magnetic field lines, which are continuous and form closed loops. Magnetic field due to currents do not begin or end at any point. The magnetic field lines of the bar magnet is shown in Fig.10.20. Note that for any close surface, the number of line entering that surface equals the number leaving that surface, and so **the net magnetic flux is zero**. This is in contrast to the case of a surface surrounding one charge of an electric dipole as is shown in Fig.10.21. In this case **the net electric flux is not zero**.

Gauss` law in magnetic field states that the net magnetic flux through any close surface is always zero

$$\oint B \cdot \mathrm{d}A = 0 . \qquad (10.37)$$

This statement is based on the experimental fact that isolated magnetic poles (or monopoles) have not been detected. The only known sources of magnetic fields are magnetic dipoles-current loops, even in magnetic materials. In facts, all magnetic effects in matter can be explain in terms of **magnetic dipole moments** associated with electrons and nuclei.

10.12 Electric and magnetic field in matter

10.12.1 Electric field in matter

The simplest way of introducing the problem of the effects of matter in electrostatic is to consider the capacitor. Now we use the capacitor, instead of having the space between plates filled with the some n**on-conducting material or dielectrics**.

First of all we shall try to understand in atomic terms, what happens when we place a dielectric in an electric field. There are two possibilities:

1. The molecules of some dielectric have the asymmetric arrangements of their atoms. For instance, the molecule of water (H_2O) has asymmetric arrangement (as is shown in Fig.10.22) of hydrogen and oxygen atoms. There is an average positive charge on a hydrogen and negative charge on the oxygen. Since the effective center of the negative charge and the effective center of the positive charge do not coincide the total charge distribution of the molecule has a dipole moment p . Such a molecule is called **a polar molecule**. When materials, called polar, are placed in an external electric field, the electric dipole moments p tend to align themselves with an external electric field as is shown in Fig.10.23. Because the molecules are in constant thermal agitation, the degree of alignment will not be complete but will increase as the electric field is increased or as the temperature decreases.

2. The molecules of some gases, like oxygen, will have a symmetric pair of atoms in each molecule. They have no inherent dipole moment because the effective center of the positive and negative charge is the same. These molecules are called **non-polar molecules**.

We shall discuss the simplest case, monoatomic gas, for instant helium. When this atom is an

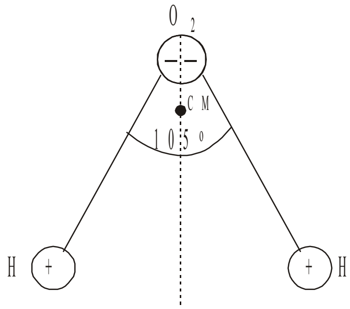

Fig.10. 22

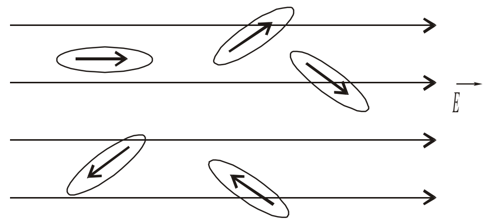

Fig.10. 23

electric field, the electrons are pulled one way by the field while the nucleus is pulled the other way. There is **a slight net displacement** of the effective center of charge and **a dipole moment is induced**.

We shall now study from the macroscopic view what happens if a dielectric is placed in the electric field. Faraday in 1837 first investigated the effect of filling the space between the plates of a parallel-plate capacitor with a dielectric. This experiment showed that the capacitance of such a capacitor is increased when a dielectric material is put between the plates.

Let us imagine that we have two identical capacitors in one of which we placed a dielectric. Let the capacitance of capacitor with dielectric be C and the capacitance of the second capacitor be C_0. Let us place the same charge on both of them. From definition (eq.8.75)

$$Q = CV = C_0 V_0$$

or

$$\frac{C}{C_0} = \frac{V_0}{V}. \tag{10.38}$$

From this equation we have for V

$$V = V_0 \frac{C_0}{C}. \tag{10.39}$$

Let us denote $\varepsilon_r = \frac{C}{C_0}$. If you can see ε_r is the dimensionless constant. Then the eq.(10.39) is in form

$$V = \frac{V_0}{\varepsilon_r}. \tag{10.40}$$

From this equation follows the potential difference V between two plates of capacitor with dielectric is

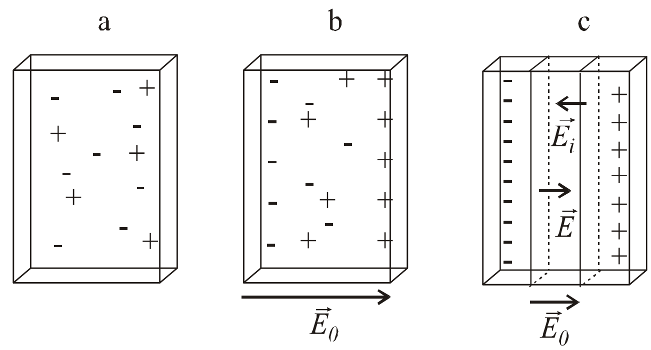

Fig.10. 24

smaller that for a capacitor without a dielectric by a factor $1/\varepsilon_r$.

Let us go explain this fact. We consider **a dielectric slab**, showing the random distribution of positive and negative charges, as is shown in Fig.10.24a. An external field E_0 separates the center of positive charge in the slab slightly from the center of negative charge, resulting in the appearance of surface charges as is shown in Fig10.24b. After any time no net charge exists in any volume element located in the interior, as is shown in Fig.10.45c. The surface charges set up a field E_i , which oppose the external field. The positive induced surface charge must be equal in magnitude to the negative induced surface charge. Note that in this process electrons in dielectric are displaced from their equilibrium positions **by distances that are considerably less that an atomic diameter** ($\sim 10^{-11}$ m). The resultant field E in the dielectric is the sum of E_0 and E_i , that is

$$E = E_0 + E_i . \tag{10.41}$$

Its direction is in the same direction as E_0 but smaller. From this we can see that if we place a dielectric in an electric field, induced surface charges appear and tend to weaken the original field within the dielectric.

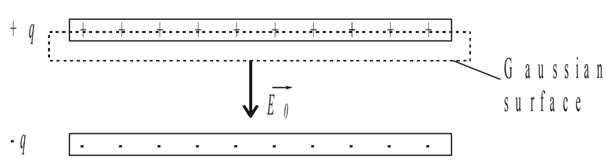

Fig.10. 25

Let us apply Gauss` law to the parallel-plate capacitor without dielectric (see Fig.10.25)
Thus we obtain

$$\varepsilon_0 \oint E_0 \cdot dA = \varepsilon_0 E_0 A = q$$

or

$$E_0 = \frac{q}{\varepsilon_0 A} \tag{10.42}$$

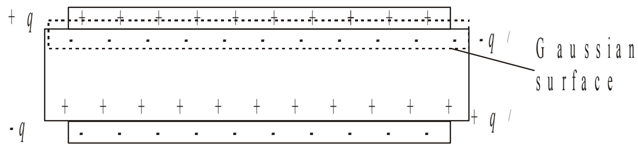

Fig.10.26

If the dielectric is present (see Fig.10.26), Gauss` law gives

$$\varepsilon_0 \oint E \cdot dA = \varepsilon_0 EA = q - q' \tag{10.43}$$

or

$$E = \frac{q}{\varepsilon_0 A} - \frac{q'}{\varepsilon_0 A}, \tag{10.44}$$

where q' is the induced charge. It must be distinguished from q. These charges, both of which lie within a gaussian surface, are opposite sign. $(q - q')$ is the net charge within a gaussian surface.

We know that $V = Ed$ for a parallel-plate capacitor we can write with respect to eq.(10.40)

$$\frac{V_0}{V} = \frac{E_0}{E} = \varepsilon_r \tag{10.45}$$

or

$$E = \frac{E_0}{\varepsilon_r}. \tag{10.46}$$

Inserting the value of $E_0 = \frac{q}{\varepsilon_0 A}$ gives

$$E = \frac{q}{\varepsilon_0 \varepsilon_r A}. \tag{10.47}$$

Then from eq.(10.44) we have

$$\frac{q}{\varepsilon_0 \varepsilon_r A} = \frac{q}{\varepsilon_0 A} - \frac{q'}{\varepsilon_0 A} \tag{10.48}$$

or

$$q' = q\left(1 - \frac{1}{\varepsilon_r}\right). \tag{10.49}$$

This expression shows that **the induced surface charge q' is always less in magnitude than the free charge q** and is equal to zero if no dielectric is present (it means if $\varepsilon_r = 1$).

Let us substitute eq.(10.49) into eq.(10.43). We give

$$\varepsilon_0 \oint \vec{E} \cdot \mathrm{d}\vec{A} = \varepsilon_0 EA = q - q\left(1 - \frac{1}{\varepsilon_r}\right) \qquad (10\ 50)$$

and after rearrangement we obtain

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{q}{\varepsilon_0 \varepsilon_r}. \qquad (10.51)$$

This equation is valid generally. From this equation follows

a) The electric flux now contains a dielectric constant ε_r

b) The charge q contained within gaussian surface is taken to be free charge only. Induced surface charge is taken into account by the introduction of ε_r .

Now we rewrite eq.(10.48) as

$$\frac{q}{A} = \varepsilon_0 \left(\frac{q}{\varepsilon_0 \varepsilon_r A} \right) + \frac{q'}{A}. \qquad (10.52)$$

The quantity in parentheses is the electric field E in the dielectric. The last term is induced surface charge per unit area. We introduce **the electric polarization** P as

$$P = \frac{q'}{A} \quad (\mathrm{C/m^2}). \qquad (10.53)$$

Then the equation (10.48) is in form

$$\frac{q}{A} = \varepsilon_0 E + P. \qquad (10.54)$$

The quantity on the right side named **electric displacement** *D*, or

$$D = \varepsilon_0 E + P. \qquad (10.55)$$

Since P and E are vectors, D must be vector too, so we have

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}. \qquad (10.56)$$

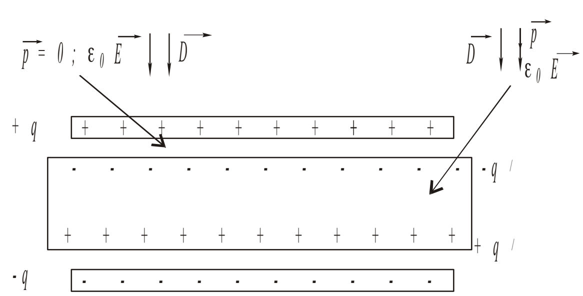

Fig.10.27

Fig.10.27 shows vectors D, $\varepsilon_0 E$ and P in the dielectric (upper right) and in the gap (upper left) for a parallel late capacitor with dielectric.

From the eq.(10.56) follows:

1. vector D is connected with the free charge only
2. vector P is connected with the polarization charge only
3. vector E is connected with all charges there are actually present
4. vector P vanishes outside the dielectric. D has the same value in the dielectric and in the gap and E has different values in the dielectric and in the gap.
5. vectors D and P can both be expressed in the terms of E alone as

$$D = \varepsilon_0 \varepsilon_0 E \tag{10.57}$$

$$P = \varepsilon_0 (\varepsilon_r - 1) E . \tag{10.58}$$

These equations can be obtained using eqs.(10.46), (10.42) and $D = \frac{q}{A}$.

10.12.2 The magnetic moments of atom. Magnetic field in matter

We shall use a classical model of the atom in which electrons move in circular orbits about the much more massive nucleus. In this model, an orbiting electron is viewed as a tiny current loop, and atomic magnetic moment is associated with the orbital motion.

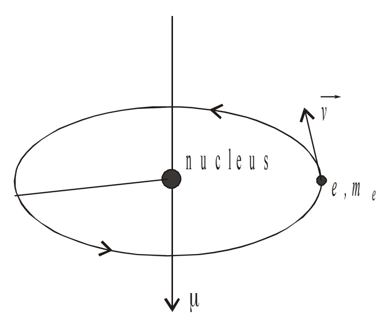

Fig.10. 28

Consider an electron with constant speed *v* in a circular orbit of radius *r* about the nucleus as is shown in Fig.10.28. Since the electron travels a distance $2\pi r$ in a time *T* (the time for one revolution), the orbital speed of the electron is $v = 2\pi r / T$. The current associated with the orbiting electron equals

$$I = \frac{e}{T} = \frac{e\omega}{2\pi} = \frac{ev}{2\pi r} , \tag{10.59}$$

where we use expressions $T = \frac{2\pi}{\omega}$ and $\omega = \frac{v}{r}$. The magnetic moment associated with this current loop is given by $\mu = IA = I\pi r^2$ (see eq.(10.25)). Therefore,

$$\mu = \left(\frac{ev}{2\pi r}\right)\pi r^2 = \frac{1}{2}evr\,. \tag{10.60}$$

Since the orbital angular momentum

$$\left|\vec{L}\right| = \left|\vec{r} \times m_e \vec{v}\right| = m_e r v \tag{10.61}$$

the magnetic moment can be written as

$$\vec{\mu} = \left(\frac{e}{2m_e}\right)\vec{L}\,, \tag{10.62}$$

where m_e is the mass of an electron. Its value is 9.01×10^{-31} kg. From this result we can see, that **the magnetic moment of electron is proportional to its orbital angular momentum.** Note that since the electron is negatively charged, the vectors $\vec{\mu}$ and $\vec{L}$ point in opposite directions. Both vectors are perpendicular to the plane of orbit.

A fundamental outcome of quantum physics is that **the orbital angular momentum must be quantized** as

$$L = 0, \hbar, 2\hbar, 3\hbar, \ldots$$

where the integer $\hbar = \frac{h}{2\pi} = 1.06 \times 10^{-34}$ J.s, h is called as **Planck`s constant**. Hence the smallest nonzero value of the magnetic moment of atom is

$$\mu = \frac{e}{2m_e}\hbar\,. \tag{10.63}$$

The magnetic moments of atoms or ions are in order to 10^{-24} J/T.

Since all substances contain electrons, you may wonder why all are not magnetic. The main reason is that in most substances, the magnetic moment of one electron in an atom is canceled by the moment of another electron in the atom orbiting in the opposite direction. The net result is that the magnetic effect produced by orbital motion of electrons is either zero or very small.

Note that the nucleus of an atom has also a magnetic moment associated with its constituent protons and neutrons. However, the magnetic moment of proton or neutron is small compared to the magnetic moment of electron and can be usually neglected.

Last time we saw that if a dielectric is placed in a electric field, polarization charges will appear on its surface. These surface charges, which find their origin in the elementary electric dipoles (permanent or induced) that make up the dielectric, set up a field that modifies the origin field. In magnetism is similar situation. If magnetic materials are placed in an external magnetic field, the elementary magnetic dipoles

$\mu = IA$ will act to set up a field of their own that will modify the origin field. To describe this situation we find it useful to introduce two other magnetic vectors, the magnetization M and the magnetic strength H .

The magnetic state of a substance is distributed by a quantity called **the magnetization vector,** M **.** The magnitude of this vector is defined as the magnetic moment per unit volume

$$M = \frac{d\mu}{d\tau}. \qquad (10.64)$$

Consider a region where there exists a magnetic field B_0 produced by the current-carrying conductor, such as the interior of solenoid. If we now fill that region with a magnetic substance, the total field B in that region will be given by

$$B = B_0 + B_m, \qquad (10.65)$$

where B_m is the field produced by the magnetic substance. This contribution can be expressed in terms of magnetization vector as

$$B_m = \mu_0 M . \qquad (10.66)$$

Hence the total field in the substance becomes

$$B = B_0 + \mu_0 M \qquad (10.67)$$

The SI unit of both terms in this equation is T.

It is convenient to introduce another field quantity H , called **the magnetic field strength**. This vector quantity is defined by the relation

$$\vec{H} = \frac{B}{\mu_0} - \vec{M} \quad \text{(A/m)} \qquad (10.68)$$

or

$$B = \mu_0 (H + M) \qquad (10.69)$$

To better understanding these expressions, consider the region inside a solenoid with current *I*. If the interior region is vacuum, then $M = 0$, $B = B_0 = \mu_0 H$. Since $B_0 = \mu_0 nI$ inside a solenoid, where *n* is the number of turns per unit length in its windings, then $H = \frac{B_0}{\mu_0} = \frac{\mu_0 nI}{\mu_0}$ or

$$H = nI . \qquad (10.70)$$

That is, the magnetic field strength inside the solenoid is due to the current in their windings. If the solenoid is filled with some substance, and the current is kept constant, then H inside the substance will remained unchanged, with a magnitude *nI*. However, the total field strength H changes. From eq.

(10.69) we see that the part of B arises from $\mu_0 H$ associated with the current across the solenoid. The second contribution to B is the term $\mu_0 M$ due to the magnetization of the substance.

For a charge class of substances, specially paramagnetic and diamagnetic substances, the magnetization is proportional to H as

$$M = \kappa H \,, \tag{10.71}$$

where κ is a dimensionless factor called **the magnetic susceptibility**.

If the substance is **diamagnetic**, κ is negative, and M is opposite to H. For example, κ for copper has value of -9.8×10^{-6}.

It is important to note that this **linear relationship does not apply to ferromagnetic substances**.

If the substance is **paramagnetic**, κ is positive, and M is in the same direction as H. For example, κ for platinum is 2.9×10^{-4}.

Inserting eq.(10.70) into eq.(10.71) gives

$$B = \mu_0 \left(H + M\right) = \mu_0 \left(H + \kappa H\right) = \mu_0 (1 + \kappa) H \tag{10.72}$$

or

$$B = \kappa_m H \,, \tag{10.73}$$

where the constant κ_m is called **permeability of the substance** and has the value

$$\kappa_m = \mu_0 (1 + \kappa) \,. \tag{10.74}$$

Substances may also be classified in their permeability κ_m compared to μ_0 as follows

paramagnetic $\kappa_m > \mu_0$

diamagnetic $\kappa_m < \mu_0$

ferromagnetic $\kappa_m >> \mu_0$.

For ferromagnetic substances κ_m is several thousand times larger than μ_0.

The magnetization of a ferromagnetic substance depends on the history of the substance as well as the strength of the applied field. All ferromagnetics materials contain microscopic regions called **domains** within which all magnetic moments are aligned. These domains have volumes af about 10^{-12} to 10^{-8} cm^3 and contains 10^{17} to 10^{21} atoms. In an unmagnetized sample, the domains are randomly oriented such that the net magnetic moment is zero. When the sample is placed in an external magnetic field, the domains tend to align with the field by rotating slightly, which result is magnetized sample. When the external field is removed, the sample may retain a net magnetization in the direction of the original field. This effect is called **magnetic hysteresis** for a ferromagnetic material. Its shape and size depend on the properties of the electromagnetic substance and on the strength of the maximum applied field. This curve is shown in Fig.10.29. Note that at point *b*, the field, B, is not zero, although the external field is $B_0 = 0$. This is explained by the fact that the iron is now magnetized due to the alignment of a large numbers of domains ($B = B_m$). At this point, the iron is said to have **a permanent magnetization**. If the external field is reversed in direction and increased in strength by reversing the current, the domains reorient until the sample is again unmagnetized at point *c*, where $B = 0$. A further increase in the reverse current causes the iron to be magnitized in the opposite direction at point *d*. Then the magnetization curve follows the path *def*. If the current is increased sufficiently, the magnetization curve returns to point *a*. **The area enclosed by the magnetization curve represents the work required to take the material through the hysteresis cycle**.

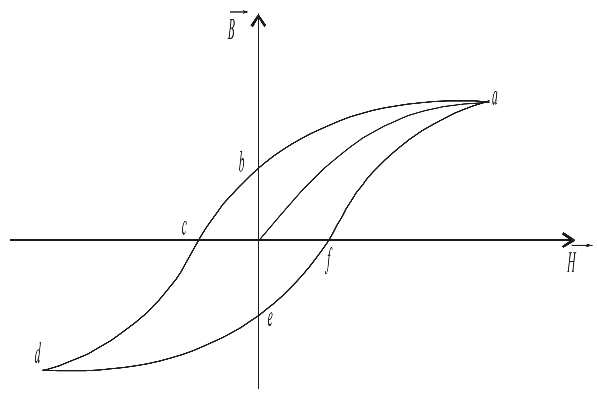

Fig.10. 29

10.13 Faraday`s law of induction

We shall take an interest in electric fields that originate **from changing magnetic field**.

Experiments provided by Michael Faraday in England in 1831 and independently by Joseph Henry in United States showed that an electric current could be induced in a circuit by a changing magnetic field. As we shall see, an induced *emf* can be produced in many ways. For instance, an induced *emf* and an induced current can be produced in a closed loop of wire when the wire moves into magnetic filed.

Let us describe an experiment, first conducted by Faraday that is shown in Fig.10.30. In this experiment when the switch in the primary circuit at the left is close, the galvanometer in the secondary circuit at the right deflects. The *emf* induced in the secondary circuit is caused by the changing magnetic field through the coil in this circuit. At the instant the switch in the primary circuit is closed, the galvanometer in the secondary circuit deflects in one direction and then returns to zero. When the switch is open, the galvanometer deflects in the opposite direction and again returns to zero. Finally, the galvanometer reads zero when there is steady current in the primary circuit. As result of these observations, Faraday concluded that an electric current can be produced by the changing magnetic field.

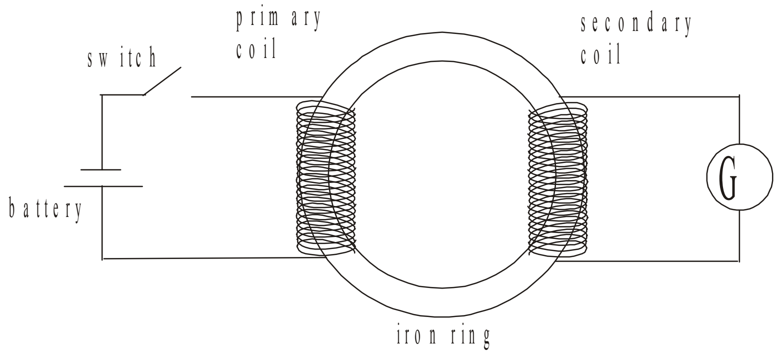

Fig.10. 30

This statement, known as Faraday`s law, can be written

$$\varepsilon = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t}, \tag{10.75}$$

where ϕ_m is the magnetic flux threading the circuit, which can be expressed as

$$\phi_m = \int B \cdot \mathrm{d}A, \tag{10.76}$$

where the integral is taken over the area bounded by the circuit. We can derive the Faray`s law by the help of the moving conductor slides along a stationary U-shape conductor as is shown in Fig.10.31. From the previous analysis it is clear that the current *I* will be established within the U-shape conductor. Because of the existence of this current, the field exerts a force toward the left on the moving conductor, and therefore an external force provided by some working agent is needed to maintain the motion. The work done by this agent is the work done on the circulating charge.

The force exerted by the magnetic field on the moving conductor is by the definition

$$F_m = \int I\mathrm{d}l \times B = IlBf^{o}, \tag{10.77}$$

where *I* is the current in the circuit, *B* is the external magnetic field, f^{o} is the unit vector oriented toward left.

The force exerted by an external agent has an equal magnitude but opposite direction. So

$$F_{ext} = -IBl. \tag{10.78}$$

We know that the electromotive force (*emf*) is defined as the ratio of the work done on the circulating charge to the quantity of this charge as

$$\varepsilon = \frac{\mathrm{d}W}{\mathrm{d}q}, \tag{10.79}$$

where $\mathrm{d}W = F_{ext} \cdot \mathrm{d}s$. Because that F_{ext} is in the same direction as $\mathrm{d}s$ then the work equals

$$\mathrm{d}W = F_{ext}\mathrm{d}s. \tag{10.80}$$

Inserting eq.(10.78) into eq.(10.80) gives

$$\mathrm{d}W = -IBl\mathrm{d}s \tag{10.81}$$

where $I = \frac{\mathrm{d}q}{\mathrm{d}t}$ and $\mathrm{d}s = v\mathrm{d}t$. Then we have for the work expression

$$\mathrm{d}W = -Blv\mathrm{d}q\,. \tag{10.82}$$

Therefore, the electromotive force given by eq.(10.79) equals

$$\varepsilon = -Blv = -Bl\frac{\mathrm{d}s}{\mathrm{d}t}\,. \tag{10.83}$$

The magnetic flux through the area $\mathrm{d}A = l\mathrm{d}s$ bounded by the circuit is by the definition

$$\mathrm{d}\phi_m = B\cdot\mathrm{d}A = -Bl\mathrm{d}s\,. \tag{10.84}$$

Combining eqs. (10.83) and (10.84) gives the Faraday`s law

$$\varepsilon = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t}\,.$$

The direction of the induced *emf* in eq.(10.75) can be found from Lenz`s law (1804-1865), which

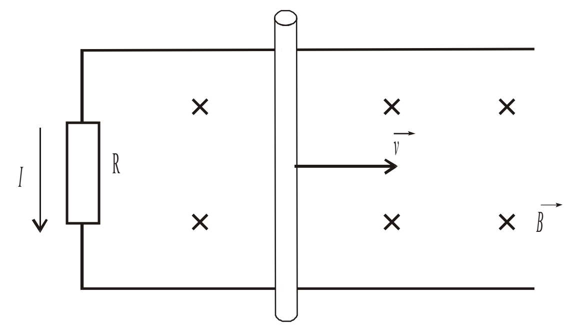

Fig.10. 31

can be stated as follows:

The polarity of the induced *emf* is such that it tends to produce a current that will create a magnetic flux to oppose the change in magnetic flux through the loop. In order to obtain a better understanding of Lenz`s law we use the example of a bar moving to the right in presence of a uniform magnetic filed, as is shown in Fig.10.31. As the bar moves to the right, the magnetic flux through the circuit increases with time since the area of the loop increases. Lenz`s law says that the induced current must be in a direction such that the flux it produces oppose the change in the external magnetic flux. Since the flux due to the external field is increasing into the paper, the induced current must produce a flux out of the paper. Hence, the induced current must be counterclockwise when the bar moves to the right (use the right-hand rule to verify this direction).

10.14 Inductance. Self inductance

We know that currents and *emf* are induced in a circuit when the magnetic flux through the circuit changes with time. From Faraday`s law the induced *emf* is given by the negative time rate of change of the magnetic flux (see eq.(10.75)). The magnetic flux is defined as (10.76). From this definition we can see that the flux is proportional to the magnetic field, which in turn is proportional to the current of the circuit. Therefore, the self-induced *emf* is always proportional to the time rate of change of the current. To determination the value of L we consider a closed loop carrying a current I as is shown in Fig.10.32.

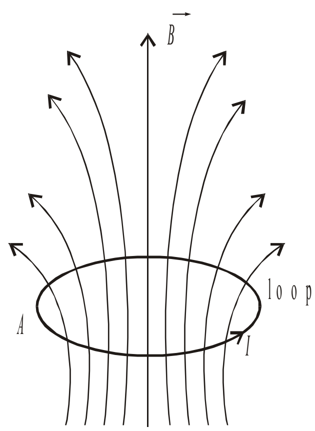

Fig.10. 32

The area of the circuit loop is A. magnetic flux through a surface A, which is bounded by this loop is

$$\phi_m = \int_A B \cdot \mathrm{d}A \,. \tag{10.85}$$

To obtain magnetic induction B in the point of the surface we can use Biot-Savart law

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{\text{loop}} \frac{\mathrm{d}l \times r^{\circ}}{\mathrm{r}^2} \,.$$

Inserting this expression into expression for magnetic flux given by eq.(10.85) gives

$$\phi_m = \int \frac{\mu_0 I}{4\pi r^2} \oint \left(\mathrm{d}\vec{l} \times \vec{r}^{\circ}\right) \cdot \mathrm{d}\vec{A} \,. \tag{10.86}$$

For a steady current (I= const) this equation can be rearanged into form

$$\phi_m = I \int_A \frac{\mu_0}{4\pi r^2} \oint_{\text{loop}} \left(\mathrm{d}\vec{l} \times \vec{r}^{\circ}\right) \cdot \mathrm{d}\vec{A} \tag{10.87}$$

Denoting

$$L = \int_A \frac{\mu_0}{4\pi r^2} \oint_{\text{loop}} \left(\mathrm{d}\vec{l} \times \vec{r}^{\circ}\right) \cdot \mathrm{d}\vec{A} \tag{10.88}$$

we can rewrite eq.(10.87) as

$$\phi_m = LI \; . \tag{10.89}$$

Let us now imagine a time varying current *I* passes through the loop. In this case a changing magnetic flux is produced. In this case in loop induced *emf* oppose the change in flux. The value of it is given by Faraday` law as

$$\varepsilon = -\frac{\mathrm{d}\phi_m}{\mathrm{d}t} \; . \tag{10.90}$$

We know that $\phi_m = LI$. Inserting this value into equation (10.90) gives

$$\varepsilon = -L\frac{\mathrm{d}I}{\mathrm{d}t} \; . \tag{10.91}$$

For a closely spaced coil of *N* turns of fixed geometry such as solenoid, we find that

$$\varepsilon = -N\frac{\mathrm{d}\phi_m}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \; , \tag{10.92}$$

where the constant of proportionality *L*, called **the self-inductance** of the device, depends on the geometry of the circuit and other physical quantities. From eq(10.92) the inductance of coil containing *N* turns is given by

$$L = N\frac{\phi_m}{I} \; , \tag{10.93}$$

where it is assumed that the same flux passes through each turn. The SI unit of inductance is Vs/A = H (henry).

Example

Find the inductance of a uniformly wound solenoid with *N* turns and length *l*. Assume that *l* is long compared with the radius and that the core of the solenoid is air.

<u>Solution:</u>

In this case, we can take the interior field to be uniform and given by eq.(10.34)

$$B = \mu_0 nI = \mu_0 \frac{N}{l} I \; ,$$

where *n* is the number of turns per unit length. The flux through each turn is given by

$$\phi_m = \int \vec{B} \cdot \mathrm{d}\vec{A} = BA = \mu_0 \frac{NA}{l} I \; ,$$

where area *A* is the cross-sectional area of the solenoid. Using expression (10.93) we find that

$$L = \frac{N\phi_m}{I} = \frac{\mu_0 N^2 A}{l} \; .$$

This expression shows that L depends on geometry and is proportional to the square of the numbers of turns. Since $N = nl$, the result can be also expressed in the form

$$L = \mu_0 \frac{(nl)^2}{l} A = \mu_0 n^2 Al = \mu_0 n^2 V \tag{10.94}$$

where $Al = V$ is the volume of solenoid.

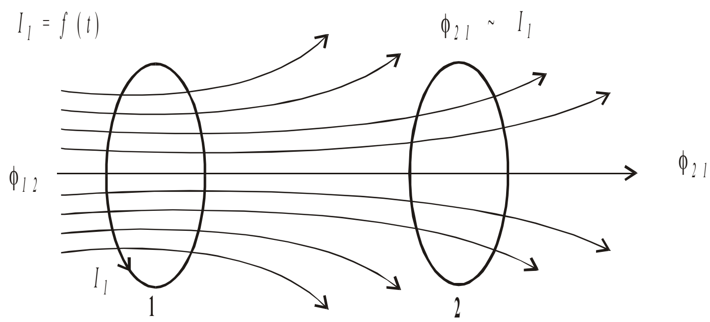

Fig.10. 33

10.15 Mutual inductance

Consider two closely wound coils as is shown in Fig.10.33. The current I_1 in loop (1),which creates magnetic field lines, some of which pass through loop (2). The corresponding flux through loop (2) produced by loop (1) is represented by ϕ_{21} . We defined **the mutual inductance** M_{21} of loop (2) with respect to loop (1) as

$$M_{21} = \frac{\phi_{21}}{I_1} \tag{10.95}$$

or

$$\phi_{21} = M_{21} I_1 . \tag{10.96}$$

The mutual inductance depends on the geometry of both circuits and on their orientation with respect to one other. As the circuit separation increases, the mutual inductance decreases because the flux linking circuits decreases.

If the current I_1 varies with time, the induced *emf* in loop (1) due to loop (2) is given by Faraday`s law

$$\varepsilon_2 = - \frac{d\phi_{21}}{dt} = - M_{21} \frac{dI_1}{dt} . \tag{10.97}$$

Similarly, if the current I_2 varies with time, the induced *emf* in loop (1) due to loop (2) is given by

$$\varepsilon_1 = - M_{12} \frac{dI_2}{dt} . \tag{10.98}$$

These results are similar to expression for the self-induced *emf* $\varepsilon = -L\frac{dI}{dt}$.

If the rates at which the currents change with time are equal, it means

$$\frac{dI_1}{dt} = \frac{dI_2}{dt} \quad (10.99)$$

then from equations (10.97), (10.98) follows

$$\varepsilon_1 = \varepsilon_2 . \quad (10.100)$$

Although the constant of proportionality M_{12} and M_{21} appear to be different, one can show that they equal $M_{12} = M_{21} = M$. Then these equations become

$$\varepsilon_2 = -M\frac{dI_1}{dt}$$

$$\varepsilon_1 = -M\frac{dI_2}{dt} .$$

The unit of mutual inductance is **Henry**.

10.16 Energy in a magnetic field

We found that the induced *emf* set up by a battery had to do work against an inductor to create a current. Part of energy supplied by the battery goes into joule heat dissipated in the resistor, while the remaining energy is stored in the conductor.

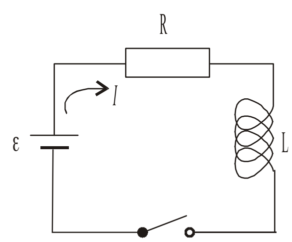

Fig.10. 34

Consider the circuit consisting of a resistor, inductor and battery shown in Fig.10.34. The internal resistance of battery will be neglect. If the switch is close at time $t = 0$, the current I will begin to increase and the inductor will produce an *emf* that is opposed the increasing current. The inductor acts like a battery whose polarity is opposite that of the real battery in the circuit. The back *emf* produced by the inductor is given by

$$\varepsilon_L = -L\frac{dI}{dt} . \quad (10.101)$$

Since the current is increasing, $\frac{\mathrm{d}I}{\mathrm{d}t}$ is positive and ε_L is negative. We can apply Kirchoff`s rule to this circuit as

$$\varepsilon - L\frac{\mathrm{d}I}{\mathrm{d}t} = RI\ , \tag{10.102}$$

or

$$\varepsilon = RI + L\frac{\mathrm{d}I}{\mathrm{d}t}\ , \tag{10.103}$$

where *IR* is the voltage drop across conductor. If we multiply each term of this equation by the current *I*, we have

$$\varepsilon I = I^2 R + LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{10.104}$$

This equation tell us, that the rate at which energy is supplied by the battery, εI, equal the sum of the rate, at which joule heat is dissipated in the resistor, $I^2 R$, and the rate at which energy is stored in the inductor. **This is an expression of energy conservation**. If we denote the energy stored in the inductor as U_m at any time, then the rate $\mathrm{d}U_m/\mathrm{d}t$ can be written

$$\frac{\mathrm{d}U_m}{\mathrm{d}t} = LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{10.105}$$

To find the total energy stored in the inductor we must integrate this equation

$$U_m = \int_0^{U_m}\mathrm{d}U_m = \int_0^I LI\mathrm{d}I = \frac{1}{2}LI^2\ . \tag{10.106}$$

We can also determine **energy density** u_m. For simplicity, consider a solenoid whose inductance is given by eq.(10.94). We know that the magnetic field of a solenoid is given by $B = \mu_0 nI$. Substituting the expression for *L* (eq.(10.94)) and $I = \frac{B}{\mu_0 n}$ into eq.(10.106) gives

$$U_m \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 Al\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}(Al)\ , \tag{10.107}$$

where $Al = V$ is the volume of the solenoid. Then the energy stored per unit volume in a magnetic field is

$$u_m = \frac{U_m}{V} = \frac{B^2}{2\mu_0}\ . \tag{10.109}$$

Although the eq.(10.109) was derived for the solenoid, it is valid for **any region of space in which a magnetic field exists**. The SI unit of energy density is J/m^3.