8. ELECTRIC FIELD

A number of simple experiments can be performed to demonstrate the existence of electric and forces. There are two kinds of electric charges, which were given the names positive and negative by Benjamin Franklin (1706 -1790). Electric forces between charged objects were measuring quantitatively by Coulomb using the torsion balance. Coulomb confirmed that the electric force between two small charged spheres is proportional to the inverse square of their separation that is $F \sim \frac{1}{r^2}$. From a number of experiments was concluded that electric charge has the following properties:

1. There are two kinds of charges in nature, with the property that unlike attract one another and like charges repel one another.
2. The force between charges varies as the inverse square of their separation.
3. Charge is conserved.
4. Charge is quantized, it means that any charge is multiple of the charge of electron.

8.1 Insulators, conductors and semiconductors

It is convenient to classify substances in terms of their ability to conduct electric charge:

1. **Conductors** are materials in which electric charges move quite freely (copper, silver, ...).
2. **Insulators** are materials that not readily transport charge (glass, rubber, ...).
3. **Semiconductor** are third class of materials, and their electrical properties are somewhere between these of insulators and conductors (silicon, germanium, ...).

8.2 Coulomb`s law

In 1785 established the fundamental law of electric force between two stationary, charged particles. Experiments show that an electric force has the following properties:

1. The force is universally proportional to the square of the separation, *r*, between the two particles.
2. The force is proportional to the product of the charges q_1 and q_2 on the particle.
3. The force is attractive if the charges are of opposite sign and repulsive if the charges have the same sign.

From these observations we can expressed the magnitude of the electric force between the two charges as

$$F = k\frac{|q_1||q_2|}{r^2}, \tag{8.1}$$

where k is the constant called Coulomb`s constant. This constant depends on the choise of units. In SI units it has the value $k = 8.9875 \times 10^{9}\ \mathrm{Nm^2/C^2}$. The unit of charge in SI units is the Coulomb (C). It is defined in terms of a unit current called ampere (A). The constant k can be also written

$$k = \frac{1}{4\pi\varepsilon_0}, \tag{8.2}$$

where the constant ε_0 is known as **the permitivity of free space** and has the value $\varepsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C^2/Nm^2}$.

The smallest unit of charge in nature is the charge on an electron or proton and has the value $|e| = 1.60219 \times 10^{-19}\ \mathrm{C}$. Therefore, 1C of charge is equal to the charge of 6.3×10^{18} electrons ($= 1/e$). This can be compared with the number of free electrons in 1cm^3 of copper, which is of the order of 10^{23}.

We must remember that force is vector quantity and must be treated accordingly. Note that Coulomb`s law applies exactly only to point charges or particles. The Coulomb`s law can be expressed in vector form as

$$\vec{F}_{21} = k\frac{|q_1||q_2|}{r^2}\cdot \vec{r}_{12}^{\,o}, \tag{8.3}$$

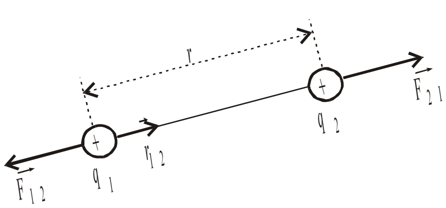

Fig.8. 1

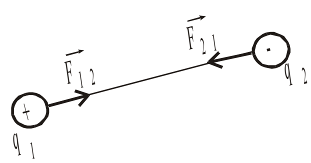

Fig.8. 2

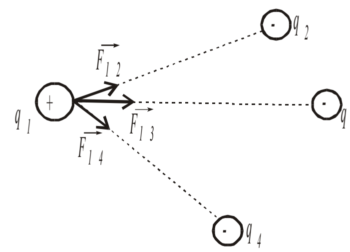

Fig.8. 3

where r_{12}^{o} is a unit vector directed from q_1 to q_2 as in Fig.8.1.

Since Coulomb`s law obeys Newton`s second law, the electric force on q_2 due to q_1 is equal in magnitude to the force on q_1 due to q_2 and in opposite direction

$$F_{12} = -F_{21}. \tag{8.4}$$

From eq.(8.3) we see that if q_1 and q_2 have the same sign, **the product** $q_1 q_2$ **is positive and force is repulsive**. On the other hand, if q_1 and q_2 are opposite sign, as is in Fig.8.2, **the product** $q_1 q_2$ **is negative and force is attractive**.

When more than two charges are present, the forces between any pair of charges is given by equation (8.3) (Fig.8.3). Therefore, the sum of the force on any one of them equals the vector sum of forces due to the various individual charges.

This principle of superposition for four particles is given by

$$F_1 = F_{12} + F_{13} + F_{14} . \qquad (8.5)$$

Example

Three charges lie along the *x* axis. The positive charge $q_1 = 15 \mu C$ is at $d = 2$ m, and the positive charge $q_2 = 6 \mu C$ is at the origin (Fig.8.4). Where must a negative charge q_3 be placed on the *x* axis such that the resultant force on it is zero?

<u>Solution:</u>

Since q_3 is negative and both q_1 and q_2 are positive, the forces F_{31} and F_{32} are both attractive, as is shown in Fig.8.4. If we let *x* be the coordinate of q_3, then the forces F_{31} and F_{32} have magnitudes given by

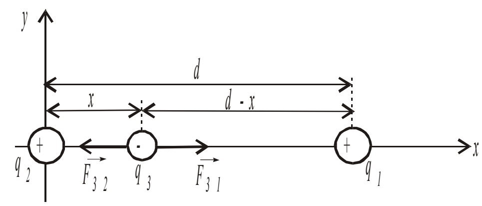

Fig.8. 4

$$F_{31} = k \frac{|q_3||q_1|}{(d - x)^2}$$

$$F_{32} = k \frac{|q_3||q_2|}{x^2} .$$

If the resultant force on q_3 is zero, then

$$k \frac{|q_3||q_1|}{(d - x)^2} = k \frac{|q_3||q_2|}{x^2}$$

or

$$(d - x)^2 |q_2| = x^2 |q_1|$$

$$(d^2 - 2dx + x^2)|q_2| = x^2 |q_1| .$$

Solving this quadratic equation for *x*, we find that $x = 0.775$ m . The second root of quadratic equation $x = - 3.44$ m is not conform to our example.

8.3 The electric field

The gravitational field at any point was defined to be $\vec{E} = \frac{F}{m_0}$, where m_0 is the test mass. In similar manner, an electric field at a point in space can be defined in terms of the electric force acting on

the test charge q_0 placed at that point

$$\vec{E} = \frac{F}{q_0} \quad \text{(N/C)}. \tag{8.6}$$

Note that the E is the field external to the test charge - not the field produced by the test charge. The direction of E is in direction of F since we have assumed that F acts on **the positive test charge**. When eq.(8.6) is applied, we must assume, that test charge q_0 is small enough such that it does not disturb the charge distribution responsible for the electric field.

Consider the Coulomb`s law we find that the electric field at the position of q_0 due to the charge q is given by

$$\vec{E} = k\frac{q}{r^2}\vec{r}^{\,o}. \tag{8.7}$$

In order to calculate the electric field due to a group of point charges, we first calculate the electric field vectors at the point *P* individually using eq.(8.7) and then add them vectorially. We can say that **the total electric field** due to a group of charges equals the vector sum of the electric fields of all charges

$$\vec{E} = k\sum_{\mathrm{i}} \frac{q_i}{r_i^2}\vec{r}_i^{\,o}, \tag{8.8}$$

where r_i is the distance from *i*-th charge, q_i, to the test charge, r_i^o is a unit vector directed from q_i toward q_0.

Example

An electric dipole consists of a positive charge $+q$ and a negative charge $-q$ separated by a distance *2a*. Find the electric field E due to these charges along the *y* axis at point *P*, which is a distance *y* from the origin (Fig.8.5). Assume that $y \gg a$.

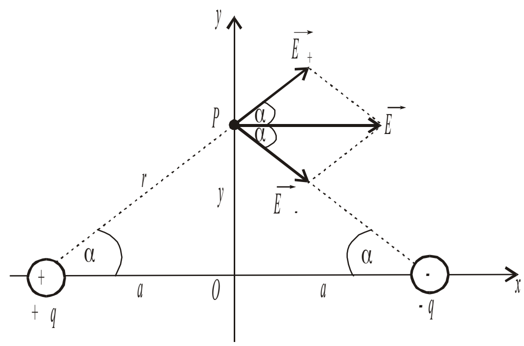

Fig.8. 5

Solution:

At point *P*, the fields E_+ and E_- due to the two charges are equal in magnitude, since point *P* is equidistant from the two equal and opposite charges as is shown in Fig.8.5. The total field by the principle

of superposition equals

$$E = E_+ + E_- \, ,$$

where the magnitudes of E_+ and E_- are given by

$$E_+ = E_- = k\frac{q}{r^2} = k\frac{q}{y^2 + a^2} \, .$$

The *x* components are equal since they are both along the *x* axis. The y components of E_+ and E_- cancel each other. Therefore, E lies along the *x* axis and has a magnitude

$$E = 2E_+ \cos\alpha = 2k\frac{q}{\left(y^2 + a^2\right)}\frac{a}{\left(y^2 + a^2\right)^{1/2}} = k\frac{2qa}{\left(y^2 + a^2\right)^{3/2}} \, ,$$

where we used

$$\cos\alpha = \frac{a}{r} = \frac{a}{\left(y^2 + a^2\right)^{1/2}} \, .$$

Using the approximation $y \gg a$, we can neglect a^2 in the denominator and write

$$E \approx k\frac{2qa}{y^3} \, . \qquad (8.9)$$

We see that along the y axis the field of a dipole at distant point varies as $\frac{1}{y^3}$, whereas the more slowly varying field of a point charge goes as $\frac{1}{y^2}$. The $\frac{1}{y^3}$ variation in E for the dipole is also obtained for a distant point along the x axis and for a general distant point. The dipole is a good model of many molecules, such as HCl. These molecules are permanent dipoles (HCl is an H^+ ion combined with a Cl^- ion).

8.4 Electric field of a continuous charge distribution. Electric field lines

To evaluate the electric field of a continuous charge distribution, we divide the charge distribution into small elements each of which contains a small charge Δq, as is shown in Fig.8.6.

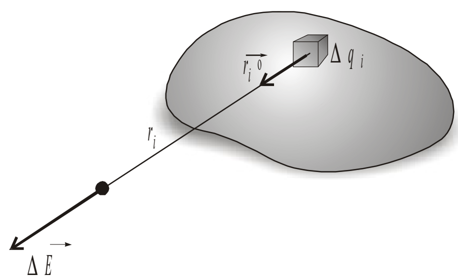

Fig.8. 6

Next we use the Coulomb`s law to calculate the electric field due to one of these elements at a point P and finally, we evaluate the total field at P due to charge distribution by summing the contribution of all the charge elements. The electric field at P due to one element given by

$$\Delta\vec{E} = k\frac{\Delta q_i}{r_i^2}\vec{r_i^{\,o}}. \tag{8.10}$$

The total electric field at P due to all elements in the charge distribution is given by

$$\vec{E} = k\sum_i \frac{\Delta q}{r_i^2}\vec{r_i^{\,o}}, \tag{8.11}$$

where r_i is the distance from charge element to point P and $r_i^{\,o}$ is a unit vector directed from the charge element toward the P. If the separation between elements in the charge distribution is small compared with the distance to P, the charge distribution can be approximated to be continuous. Therefore, the total field at P in the limit $\Delta q_i \to 0$ becomes

$$\vec{E} = k\lim_{\Delta q_i \to 0}\sum_i \frac{\Delta q_i}{r_i^2}\vec{r_i^{\,o}} = k\int\frac{dq}{r^2}\vec{r^{\,o}}. \tag{8.12}$$

If a charge Q is distributed through out a volume V, **the charge per unit volume,** ρ, is defined as

$$\rho = \frac{\mathrm{d}Q}{\mathrm{d}V} \quad (\mathrm{C/m^3})\,. \tag{8.13}$$

If a charge Q is distributed on a surface of area A, **the surface charge density**, σ , is defined by

$$\sigma = \frac{\mathrm{d}Q}{\mathrm{d}A} \quad (\mathrm{C/m^2})\,. \tag{8.14}$$

Finally, if a charge Q is distributed along a line of length l, **the linear charge density** is defined by

$$\lambda = \frac{\mathrm{d}Q}{\mathrm{d}l} \quad (\mathrm{C/m})\,. \tag{8.15}$$

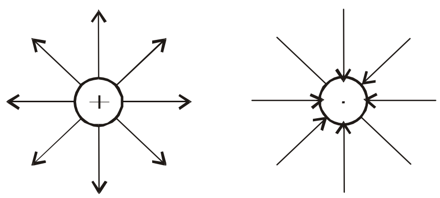

Fig.8. 7

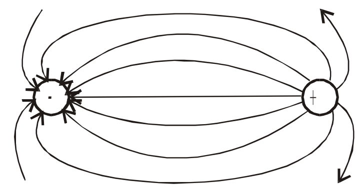

Fig.8. 8

A convenient aid for visual electric field patterns is to draw lines pointing in the same direction as the electric field vector at any point. These lines, called **electric field lines**, are related to the electric field in any region of space in the following manner:

1. The electric field vector E is the tangent to the electric field line at each point.
2. The number of lines per unit area through a surface perpendicular to the lines is proportional to the strength of the electric field in that region.

   Some representative electric field lines for a single positive point charge are shown in Fig.8.7.
3. The lines start only on positive charges and only on negative charges Fig.8.8.

8.5 Motion of charged particle in a uniform electric field

When a particle of charge q is placed in an electric field E , the electric field on the charge is qE . If this is the only force exerted on the charge, then Newton`s second law applied to the charge gives

$$F = qE = ma\,. \tag{8.16}$$

The acceleration of the particle is therefore given by

$$\vec{a} = \frac{q}{m}\vec{E}\,. \tag{8.17}$$

We know that

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\,, \tag{8.18}$$

then the equation (8.17) is in the form

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{q}{m}\vec{E} \tag{8.19}$$

or

$$\int_{v_0}^{v} d\vec{v} = \frac{q}{m}\int_0^t \vec{E}dt \,. \tag{8.20}$$

Integrating eq. (8.20) we obtain

$$\vec{v} = \vec{v}_0 + \frac{q}{m}\vec{E}t \,. \tag{8.21}$$

Suppose a particle q entering the uniform electric field E with velocity v_0. The vector v_0 makes an angle α with the horizontal, as is shown in Fig.8.9.

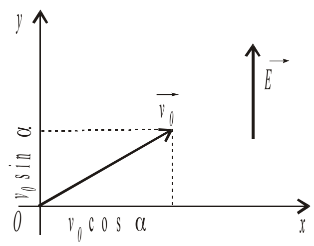

Fig.8. 9

We can write the initial conditions as $t = 0$, $x_0 = y_0 = z_0 = 0$, $v_0(v_{0x}, v_{0y}, 0)$, where $v_{0x} = v_0 \cos\alpha, v_{0y} = v_0 \sin\alpha$. Electric field is oriented along the y axis, i.e. it has the nonzero y component only ($E_x = 0$, $E_y = E$).

Let us write the vector v given by eq. (8.21) as the two scalar equations

$$v_x = v_{0x} + \frac{q}{m}E_x t = v_{0x}$$

$$v_y = v_{0y} + \frac{q}{m}E_y t \,. \tag{8.22}$$

With respect to initial conditions we obtain

$$v_{0x} = v_0 \cos\alpha$$

$$v_{0y} = v_0 \sin\alpha \,. \tag{8.23}$$

By the definition of the velocity is

$$v_x = \frac{dx}{dt} \text{ and } v_y = \frac{dy}{dt} \,.$$

Then for x component of the velocity we have

$$dx = v_x dt \,. \tag{8.24}$$

Inserting the value of v_x from eq.(8.23) and integrating we obtain

$$\int_0^x \mathrm{d}x = \int_0^t v_0 \cos\alpha \mathrm{d}t$$

or

$$x = v_0 t \cos\alpha \,. \tag{8.25}$$

Similar procedure for y component of velocity gives

$$y = v_0 t \sin\alpha + \frac{q}{m}\frac{E_y t^2}{2} \,. \tag{8.26}$$

Eliminating t from (8.25) and inserting this value into (8.26) we obtain

$$y = \frac{qE}{2mv_0^2 \cos^2\alpha} x^2 + x\mathrm{tg}\alpha \,, \tag{8.27}$$

which is the equation of parabola. The influence of electric field on the motion of charged particle is used, for example, in cathode-ray oscilloscope for detection of electron beam.

8.6 A dipole in an electric field

In the proceeding section 8.4 we have seen that the electric field due to the dipole (see eq.(8.9)) is given by

$$E \approx k\frac{2aq}{y^3} \,,$$

where $2a$ is the distance between charges $-q$ and $+q$ of the dipole and y is the distance between the center of the dipole and the point P lying on the axis of symmetry of the dipole, as is shown in Fig.8.10.

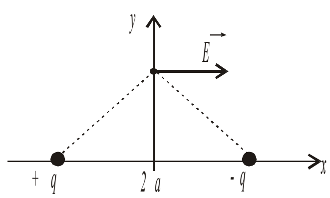

Fig.8. 10

Fig.8. 11

The product $2aq$ is called **electric dipole moment** and is represented by the symbol p. The dipole moment can be considered to be vector, that points from the negative to positive charge as is shown in Fig.8.11. Then the vector form of the dipole moment is

$$p = 2qa \,. \tag{8.28}$$

Then the equation (8.9) can be rewrite for distant points along the perpendicular bisector as

$$E = k\frac{p}{y^3} \tag{8.29}$$

Let us consider a dipole placed in uniform external field E as is shown in Fig.8.12. The dipole

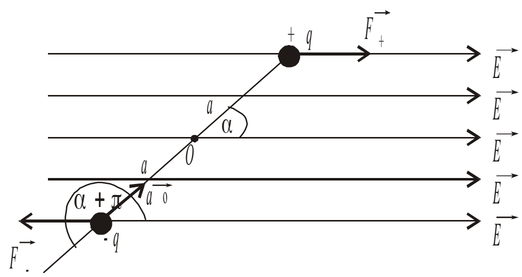

Fig.8. 12

moment makes an oriented angle α with the external electric field E. The forces on two charges are equal and opposite, as is shown in Fig.8.12. They have the magnitudes

$$F = qE. \tag{8.30}$$

We see that the net force on the dipole is zero.

However, the two forces produce a net torque on the dipole, so that the net torque about O is given by

$$\tau = qE\sin\alpha \cdot a + (-qE)\sin(\pi + \alpha) \cdot a = 2qaE\sin\alpha \tag{8.31}$$

or

$$\tau = pE\sin\alpha, \tag{8.32}$$

where $p = 2qa$ is the moment of the dipole. It is convenient to express the torque in vector form as the product of the vector p and E. Then the equation (8.31) can be written in a vector notation as

$$\tau = p \times E. \tag{8.33}$$

Thus electric dipole placed in a external electric field E experiences a torque. The effect of the torque is to try to turn the dipole so p is parallel to E.

8.7 Electric flux and Gauss` law

The concept of electric field lines we described qualitatively in section 8.5. We shall now use the concept of electric flux to put this idea on a quantitative basis. **Electric flux** is a measure of the number of electric field lines penetrating some surface. The number of lines that go through the surface is proportional to the net charge within the surface.

Consider a general surface divided into a large number of small elements, each of area ΔA as is shown in Fig.8.13. The variation in the electric field over the element can be neglect if the element is small. It is convenient to define a vector ΔA_i whose magnitude represents the area of the *i*-th element and whose direction is defined to be perpendicular to the surface. The electric flux $\Delta\phi_i$ through this small element is defined by

$$\Delta\phi_i = E_i \Delta A_i \cos\varphi = E_i \cdot \Delta A_i, \tag{8.34}$$

where we have used the definition of the scalar product of two vectors. By summing the contributions of all elements, we obtain the total flux

$$\phi = \lim_{\Delta A_i \to 0} \sum E_i \cdot \Delta A_i = \int_{\text{surface}} E \cdot \mathrm{d}A. \tag{8.35}$$

The integral is surface one, which must be evaluated over the hypothetical surface in question. In general, the value of ϕ depends both on the field pattern and on the specified surface. From the SI units of E and A, we see that electric flux has the unit of $\mathrm{Nm^2/C}$.

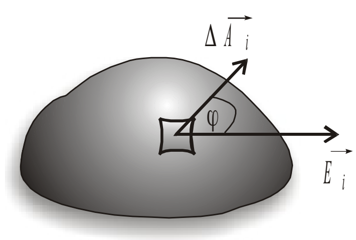

Fig.8. 13

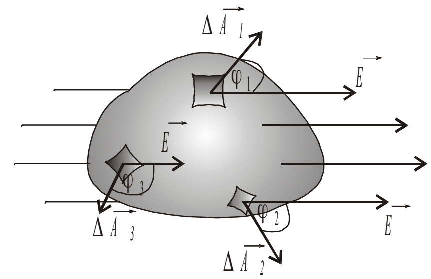

Fig.8. 14

We shall be interested in the flux through a closed surface, as is shown in Fig.8.14. At each point the vectors ΔA_1 are normal to the surface and, by convention, always point outward. At the elements *1* and *2* E is outward and $\varphi < 90^\circ$ then the flux through these elements is positive $\Delta\phi = E \cdot \Delta A > 0$. For element such as *3*, where the field lines are directed into the surface, $\varphi > 90^\circ$ and the flux is negative with $\cos\varphi$. Then the net, or total, **flux through the surface is proportional to the net number of field lines leaving the volume surroundings the surface mines the number of field lines entering the surface**. Using the symbol $\oint$ to represent an integral over a closed surface, we can write the net flux, ϕ_C, through the closed path as

$$\phi_C = \oint E \cdot \mathrm{d}A. \tag{8.36}$$

Now we find the relation between the net electric flux through a closed surface and the charge enclosed by the surface. This surface is often called **a gaussian surface**. This relation, knows as Gauss` law, is of fundamental importance in the study of electric fields.

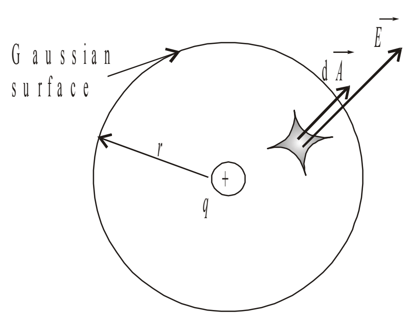

Fig.8. 15

Let us consider a positive point charge q located at the center of the sphere of radius r as is shown in Fig.8.15. From Coulomb`s law we know that the magnitude of the electric field everywhere on the surface of the sphere is

$$E = k\frac{q}{r^2}. \tag{8.37}$$

The electric lines are radial outward, and hence are perpendicular to the surface at each point. It means at each point E is parallel to the vector $\mathrm{d}A$. Therefore

$$E \cdot \mathrm{d}A = E\mathrm{d}A \tag{8.38}$$

and from eq.(8.36) we find that the net flux through the gaussian surface is given by

$$\phi_C = \oint E\mathrm{d}A = E\oint \mathrm{d}A. \tag{8.39}$$

Since E is constant over the surface and given by eq.(8.37). For a spherical gaussian surface is

$$\oint \mathrm{d}A = A = 4\pi r^2. \tag{8.40}$$

Hence the net flux through the gaussian surface is

$$\phi_C = k\frac{q}{r^2}4\pi r^2 = 4\pi qk \tag{8.41}$$

Recalling that $k = \dfrac{1}{4\pi\varepsilon_0}$, we can write this in the form

$$\phi_C = \frac{q}{\varepsilon_0}. \tag{8.42}$$

Note that this result, which is independent of r, says that the net flux through a spherical gaussian surface is proportional to the charge inside the surface.

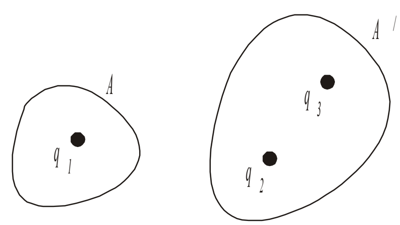

Fig.8. 16

We can extend this equation to the generalized case of many point charges or a continuous distribution of charge. We shall make use of the superposition principle, which says that the electric field due to many charges is the vector sum of the electric field produce by the individual charges. Consider the system of charges shown in Fig.8.16. The surface A surround only one charge q_1. Hence the net flux through A is $\dfrac{q_1}{\varepsilon_0} = \phi_{C_1}$. The flux through A due to the charges outside it is zero since each electric field line that enters at one point leaves it at another. The surface A' surrounds charges q_2 and q_3. Hence the net flux through A' is $\frac{q_1 + q_2}{\varepsilon_0} = \phi_{C_{A'}}$. Generally

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint \left(\vec{E}_1 + \vec{E}_2 + \ldots\right) \cdot \mathrm{d}\vec{A} = \frac{\sum q_{in}}{\varepsilon_0}. \tag{8.43}$$

Gauss` law states that the net flux through any closed surface is given by

$$\phi_C = \oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{\sum q_{in}}{\varepsilon_0}, \tag{8.44}$$

where $\sum q_{in}$ represents the net charge inside the gaussian surface and E represents the electric field at any point on the gaussian surface. In words,

Gauss` law states that the net electric flux through any closed gaussian surface is equal to the net charge inside the surface divided by ε_0.

Example

Electric charge Q_+ is distributed uniformly throughout a nonconducting sphere of radius *R*. Determine the electric field

a) outside the sphere (*r*>*R*)

b) inside the sphere (*r*<*R*).

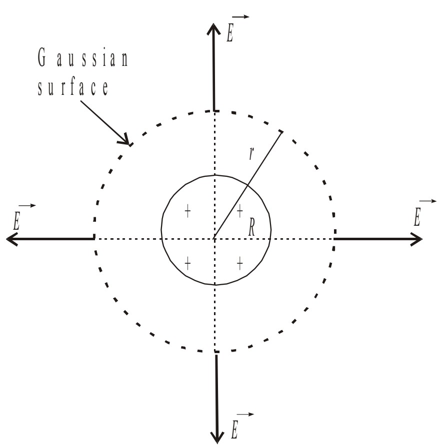

Fig.8. 17a

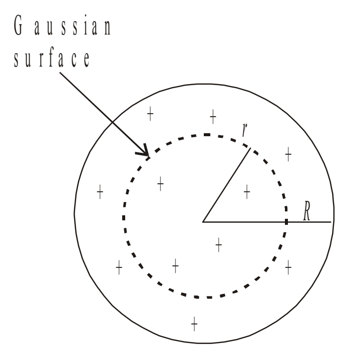

Fig.8. 17b

<u>Solution:</u>

a) The volume density ρ has a constant value. Since the charge is distributed symmetrically the electric field at all points of the sphere must be also symmetric that E is directed radially outward. The vector E is perpendicular to the surface so that the angle between E and $\mathrm{d}A$ is zero. Therefore

$$E \cdot \mathrm{d}A = E \cdot \mathrm{d}A\,.$$

Thus we have

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint E\mathrm{d}A = E4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\,.$$

b) Inside the sphere we choose for gaussian surface a concentric sphere of radius $r<R$. Then from Gauss` law we have

$$\oint \vec{E} \cdot \mathrm{d}A = \frac{Q'}{\varepsilon_0}\,, \tag{8.45}$$

where Q' is the part of charge Q which is contained within the sphere of radius r. For a uniform charge distribution we can write

$$Q' = \rho\frac{4}{3}\pi r^3$$

$$Q = \rho\frac{4}{3}\pi R^3\,.$$

From these equations may be calculated the charge Q' inside the gaussian sphere of radius r as

$$Q' = Q\frac{r^3}{R^3}\,.$$

Inserting this value into eq.(8.45) gives

$$\oint \vec{E} \cdot d\vec{A} = E4\pi r^2 = \frac{Q}{\varepsilon_0}\frac{r^3}{R^3}$$

$$E = \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\,.$$

Magnitude of the electric field as a function of the distance r from the center of a uniformly charged non-conducting sphere is shown in Fig.8.18.

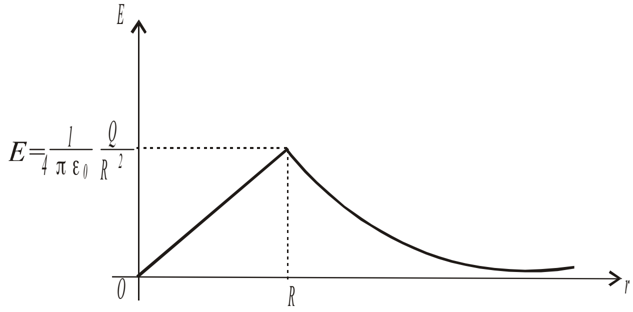

Fig.8. 18

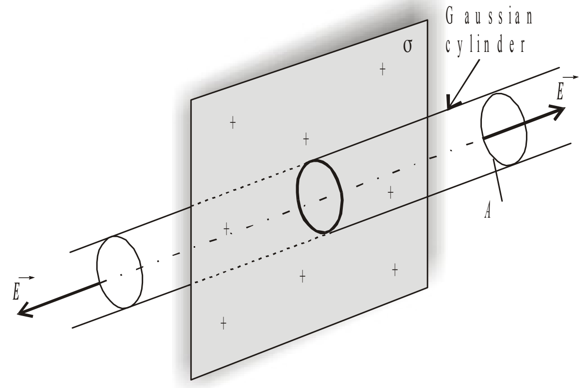

Fig.8. 19

Example

Find the electric field due to a non-conducting, infinite plane with uniform charge per unit area σ (Fig.8.19).

Solution:

It is convenient to choose for our gaussian surface a small cylinder whose axis is perpendicular to the plane whose ends each have an area A. Using Gauss` law we give

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint E\mathrm{d}A = 2EA = \frac{Q}{\varepsilon_0},$$

where $Q = \sigma A$. Then

$$E = \frac{\sigma}{2\varepsilon_0}.$$

Notes:

a) because E is perpendicular to the cylindrical surface, there is no flux through this surface

b) since the distance of the surfaces from the plane does not appear we conclude that $E = \frac{\sigma}{2\varepsilon_0}$ at any distance from the plane. It means that the field is uniform everywhere.

8.8 Electric field and conductors. Coulomb`s equation

A good electrical conductor contains electrons that are not bound to any atom and are free to move about within material. Where there is no net motion of charge within conductor, the conductor is in **electrostatic equilibrium**. A conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside the conductor.
2. Any excess charge on a isolated conductor must reside entirely on its surface.

3. The electric field just outside a charge conductor is perpendicular to the conductor`s surface and has magnitude σ/ε_0 .

We shall return to first property. The free charges would accelerate under the action of electric field. Before the electric field is applied, the electrons are uniformly distributed throughout the conductor. When the external field is applied, the free electrons accelerate to the left and of positive charge on the right, as is shown in Fig.8.20. These charges create their own electric field, which opposes the external field. The surface charge density increases until the magnitude of the electric field set up by these charges equals that of external electric field, giving net field of zero inside the conductor. In a good conductor, the time it takes the conductor to reach equilibrium is of the order 10^{-16} s .

Note to second and third property. We can use Gauss` law to verify these properties of a conductor in electrostatic equilibrium.

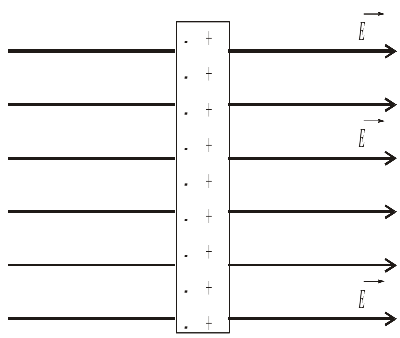

Fig.8. 20

The fact that the electric field inside the conductor equals zero has one interesting consequence that is any net charge on conductor distributes itself on the outer surface. This can be easily shown using Gauss` law.

Consider charge conductor of a arbitrary shape, A, as is shown in Fig.8.21, which carries a net charge Q. Let us choose the gaussian surface A_1 (dashed line) inside the conductor. The electric field is zero at all points on this gaussian surface when it is in electrostatic equilibrium. Since the electric field is also zero at every point on the gaussian surface the net flux through surface is

$$\phi = \oint_{A_1} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\varepsilon_0} = 0 . \qquad (8.46)$$

We can see that the net charge inside this gaussian surface must be zero too.

Now we choose the gaussian surface A_2 , outside the conductor in equilibrium (see Fig.8.21). Because this surface encloses the charge Q we can write the Gauss` law as

$$\oint_{A_2} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\varepsilon_0} \tag{8.47}$$

If we calculate the limit case for $A_2 \to A$ we have

$$\lim_{A_2 \to A} \oint_{(A)} \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q}{\varepsilon_0} \tag{8.48}$$

and

$$\lim_{A_1 \to A} \oint_{(A)} E \cdot \mathrm{d}A = 0 . \tag{8.49}$$

From these results we can see that the charge cannot be inside the conductor. It is only on the surface of the conductor.

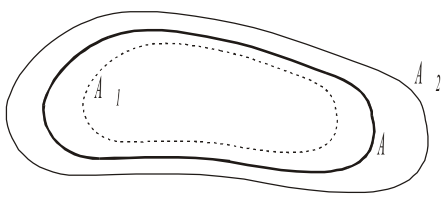

Fig.8. 21

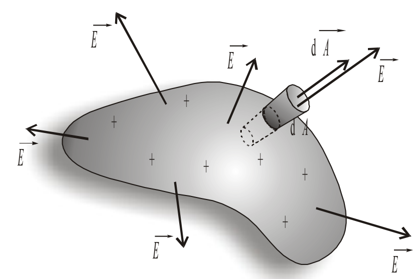

Fig.8. 22

Gauss` law allows us to determine the magnitude of electric field E just outside the surface of any conductor of arbitrary shape as is shown in Fig.8.22. We choose the gaussian surface as a small cylindrical surface, $\mathrm{d}A$, which is very small in hight with end faces parallel to the surface. Part of the cylinder is just outside the conductor, and part is inside. There is no flux through the face on the inside of the cylinder since $E = 0$ inside the conductor. There is no flux through the cylindrical face of the gaussian surface since E is tangent to this surface. Hence, the net flux through the gaussian surface is only $E \cdot \mathrm{d}A$. Applying Gauss` law to this surface gives

$$\vec{E} \cdot d\vec{A} = E\mathrm{d}A = \frac{\mathrm{d}Q}{\varepsilon_0} = \frac{\sigma \mathrm{d}A}{\varepsilon_0} , \tag{8.50}$$

where $\sigma = \frac{\mathrm{d}Q}{\mathrm{d}A}$ is the surface charge density. Thus the magnitude of electric field at surface of conductor we obtain

$$E = \frac{\sigma}{\varepsilon_0} . \tag{8.51}$$

This is very useful result, which is valid for any shape conductor. This expression is known as **Coulomb`s equation**.

8.9 Electric potential. Equipotential surfaces

Last time we showed that the gravitational force is conservative one. Since the electric force, given by Coulomb law, is of the same form as the universal law of gravity, it follows that electric force is also conservative. Therefore, it is possible to define a potential energy function associated with this force.

When a charge q_0 is placed in a electric field E, the electrostatic force on the test charge is $Eq_0 = F$. This force is the vector sum of the individual forces exerted on q_0 by the various charges producing the field E. It follows that this force is conservative one.

Work done by it is equal

$$\mathrm{d}W = F \cdot \mathrm{d}r = q_0 E \cdot \mathrm{d}r \qquad (8.52)$$

By the definition, the work done by a conservative force equals the negative of the change in potential energy $\mathrm{d}U$. Then

$$\mathrm{d}W = -\mathrm{d}U$$

or

$$\mathrm{d}U = -\mathrm{d}W = -q_0 E \cdot \mathrm{d}r\ . \qquad (8.53)$$

For finite displacement of the test charge between points *A* and *B*, the change in potential energy is given by

$$\Delta U = U_B - U_A = -q_0 \int_A^B \vec{E} \cdot \mathrm{d}\vec{r} \qquad (8.54)$$

The integral is performed along the path by which q_0 moves from point *A* to point *B* and is called **a path integral** or line integral. Since force $F = q_0 E$ is conservative, this integral does not depend on the path taken between *A* and *B*.

The potential difference, $V_B - V_A$, between points *A* and *B* is defined as the change in potential energy divided by the test charge q_0

$$V_B - V_A = \frac{U_B - U_A}{q_0} = -\int_A^B \vec{E} \cdot \mathrm{d}\vec{r} \qquad (8.55)$$

or

$$\mathrm{d}V = -E \cdot \mathrm{d}r\ . \qquad (8.56)$$

The unit of potential in SI units is J/C. We shall usually choose the potential to be zero for a point at infinity. Which this choise, we can say that the electric potential at an arbitrary point equals **the work required per unit charge to bring a positive test charge from infinity to that point**

$$V_P = -\int_\infty^P \vec{E} \cdot \mathrm{d}\vec{r} \qquad (8.57)$$

Since potential difference is a measure of energy per unit charge, the SI unit of potential is joule per Coulomb, defined to be equal to a unit called the volt (V); 1V=1 J/C. From this follows that the SI unit of electric field (N/C) can also be expressed as volt per meter; 1N/C=1 V/m. Note that a unit of energy commonly used in atomic and nuclear physics is **the electron volt**, which is defined as the energy that an electron (or proton) gains when moving through the potential difference of magnitude 1V. The electron volt (eV) is related to the joule through the definition $1\ \mathrm{eV} = 1.602 \times 10^{-19}\ \mathrm{C.V} = 1.602 \times 10^{-19}\ \mathrm{J}$.

The electric potential can be graphically represented in three dimensions by equipotential lines, which are named equipotential surfaces. **An equipotential surface** is one on which all points are at the same potential. That is, the potential difference between any two points on the surface is zero. An equipotential surface must be perpendicular to the electric field at any point.

8.10 Electric potential due to point charges

Consider an isolated positive charge *q* as is shown in Fig.8.23. Note that such a charge produces an electric field that is radially outward from the charge. The general expression for the potential difference is given by

$$V_B - V_A = -\int_A^B \vec{E} \cdot \mathrm{d}\vec{r} \tag{8.58}$$

We know that the electric field due to the point charge *q* is given by

$$\vec{E} = k\frac{q}{r^2}\vec{r}^{\,\mathrm{o}}, \tag{8.59}$$

where r^{o} is the unit vector directed from the charge to the field point. Then

$$\vec{E} \cdot \mathrm{d}\vec{r} = k\frac{q}{r^2}\vec{r}^{\,\mathrm{o}} \cdot \mathrm{d}\vec{r} = k\frac{q}{r^2}\mathrm{d}r. \tag{8.60}$$

The dot product $r^{\mathrm{o}} \cdot \mathrm{d}r = |r^{\mathrm{o}}||\mathrm{d}r|\cos\alpha = \mathrm{d}r\cos\alpha$, where α is the angle between r^{o} and $\mathrm{d}r$. We note that $\mathrm{d}r\cos\alpha$ is the projection of $\mathrm{d}r$ onto r, so that $\mathrm{d}r\cos\alpha = \mathrm{d}r$. Inserting eq.(8.60) gives

$$V_B - V_A = -\int E\mathrm{d}r = -kq\int_{r_A}^{r_B}\frac{\mathrm{d}r}{r^2} = kq\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{8.61}$$

We can see that the integral is independent on the path between *A* and *B*.

If we choose the reference of potential to be zero at $r_A \to \infty$ then the electric potential due to a point charge at any distance *r* from the charge is given by

$$V = k\frac{q}{r}. \tag{8.62}$$

From this equation we see that V is constant on a spherical surface of radius r. We can conclude, that **the equipotential surface for an isolated charge consists of number of spheres concentric with the charge**, as is shown in Fig.8.23.

The electric potential of two or more point charges at point P is obtained by applying principle of superposition

$$V = k\sum_i \frac{q_i}{r_i}, \tag{8.63}$$

where the potential is again taken to be zero at infinity and r_i is the distance from the point P to the charge q_i.

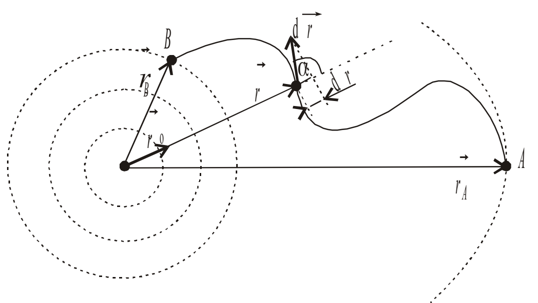

Fig.8. 23

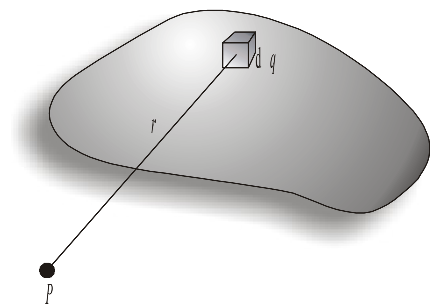

Fig.8. 24

8.11 Electric potential due to continuous charge distribution

The electric potential due to a continuous charge distribution can be calculated using eq.(8.62). We can consider the potential due to a small charge element $\mathrm{d}q$, treating this element as a point charge, as is shown in Fig.8.24. Then the potential $\mathrm{d}V$ at some point P is given by

$$\mathrm{d}V = k\frac{\mathrm{d}q}{r} \tag{8.64}$$

and the total potential at point P is then

$$V = k\int\frac{\mathrm{d}q}{r}. \tag{8.65}$$

8.12 Obtaining *E* from the electric potential

We know that the potential difference $\mathrm{d}V$ between two points at distance $\mathrm{d}r$ apart as

$$\mathrm{d}V = -E\cdot\mathrm{d}r. \tag{8.66}$$

If the electric field has only one component, E_x, then $E\cdot\mathrm{d}r = E_x\mathrm{d}x$. Therefore eq.(8.66) becomes

$$\mathrm{d}V = -E_x\cdot\mathrm{d}x$$

or

$$E_x = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{8.67}$$

That is the electric field is equal to the negative of the derivate of the potential with respect to some coordinate. If the charge distribution has spherical symmetry, where the charge density depends only on the radial distance *r*, then electric field is radial. In this case

$$E \cdot \mathrm{d}r = E\mathrm{d}r = -\mathrm{d}V. \tag{8.68}$$

Therefore

$$E_r = -\frac{\mathrm{d}V}{\mathrm{d}r}. \tag{8.69}$$

Note that the potential changes only in the radial direction, not in a direction perpendicular to r. Thus V is a function r. This is consistent with the idea that the equipotential surfaces are perpendicular to field lines.

When a test charge is displaced by a vector $\mathrm{d}r$ that lies within any equipotential surface then by the definition $\mathrm{d}V = -E \cdot \mathrm{d}r = 0$. From this equation we see that **the equipotential surfaces must be always perpendicular to the electric field lines**.

In general, the electric potential $V(r)$ is a function of all three spatial coordinates *x*, *y*, and *z* and then the electric field components E_x, E_y and E_z can readily be found from $V(x, y, z)$. We shall consider the value of potential at two near by points (x, y, z) and $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$. The change in V going from the first point to the second one is

$$\mathrm{d}V = \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z. \tag{8.70}$$

On the other hand from the definition of potential we have

$$\mathrm{d}V = -E \cdot \mathrm{d}r, \tag{8.71}$$

where the infinitesimal vector displacement $\mathrm{d}r$ is $\mathrm{d}r = \mathrm{d}xi + \mathrm{d}yj + \mathrm{d}zk$ and $E = E_x i + E_y j + E_z k$. Using eq.(8.70), (8.71) we have

$$\frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -E_x\mathrm{d}x - E_y\mathrm{d}y - E_z\mathrm{d}z \tag{8.72}$$

or

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}. \tag{8.73}$$

in these expressions the derivates are called **partial derivates**. This means that in the operation $\frac{\partial V}{\partial x}$ one takes a derivate with respect to *x* while *y* and *z* are held constant. In vector notation E is written

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k}\right) = -\nabla V = -\mathrm{grad}V, \tag{8.74}$$

where ∇ is called **the gradient operator (grad)**. The mines sign in this equation came in because the electric field points from a region of positive potential toward a region negative, whereas vector $\mathrm{grad} V$ is defined so that it points in the direction of increasing *V*.

Example

Find the electric potential, *V*, and electric field, E, along axis of uniformly charged disk of radius *R* and the charge surface density σ (Fig.8.25).

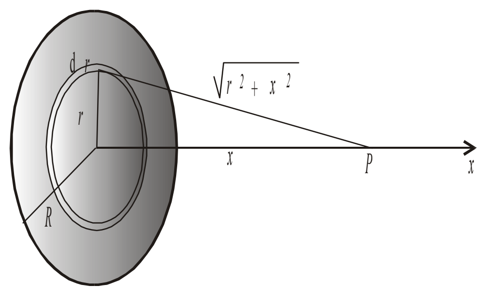

Fig.8. 25

<u>Solution:</u>

We divide the disk into a series if infinitesimal rings of radius *r* and width $\mathrm{d}r$. The potential of each ring is given by the definition

$$\mathrm{d}V = k\frac{\mathrm{d}q}{\sqrt{r^2 + x^2}},$$

where *x* is the distance between point *P* and center of the disk. The area of the ring is $\mathrm{d}A = 2\pi r \mathrm{d}r$ and hence the charge on the ring is $\mathrm{d}q = \sigma \mathrm{d}A = \sigma 2\pi r \mathrm{d}r$. To find the total potential at point *P* we sum over all rings making up the disk. That is, we integrate $\mathrm{d}r$ form $r = 0$ to $r = R$

$$V = k\pi\sigma\int_0^R \frac{2r\mathrm{d}r}{\sqrt{r^2 + x^2}} = 2\pi\sigma k\left[\left(x^2 + R^2\right)^{1/2} - x\right].$$

Using the relation between *V* and E in form $E_x = -\frac{\mathrm{d}V}{\mathrm{d}x}$. We can find the electric field at any axial point by taking the negative of the derivate of *V* with respect to *x* (*y* and *z* components are zero)

$$E_x = -\frac{\mathrm{d}}{\mathrm{d}x}\left(2\pi\sigma k\left[\left(x^2 + R^2\right)^{1/2} - x\right]\right) = 2\pi\sigma k\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right).$$

8.13 Capacitance

Consider two conductors having a potential difference *V* between them. Let us assume that the conductors have equal and opposite charges as is shown in Fig.8.26. Such a combination of two conductors is called **capacitor**. The potential difference is found to be proportional to the magnitude of the charge Q on the capacitor. **The capacitance**, C, of a capacitor is defined as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them

$$C = \frac{Q}{V} \quad (\mathrm{C/V} = 1\,\mathrm{F})\,. \tag{8.75}$$

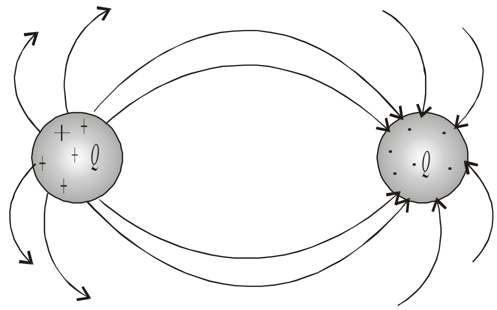

Fig.8. 26

Note that by the definition capacitance is always **a positive quantity** and it is measured in unit of farad (F). The farad is a very large unit of capacitance. Typical devices have capacitances ranging from microfarads (1μF=10^{-6}F) to picofarads (1pF=10^{-12}F).

The capacitance of a device depends on the geometrical arrangement of the conductors. Let us consider two large conducting plates which are parallel to each other and separated by a distance d small compared with the plate dimension, as is shown in Fig.8.27. The plates will have surface charge density $+\sigma$ and $-\sigma$ respectively. The electric field E for all points on each side of the plate is given by equation

$$E = \frac{\sigma}{2\varepsilon_0}\,. \tag{8.76}$$

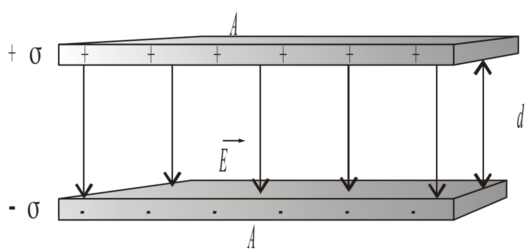

Fig.8. 27

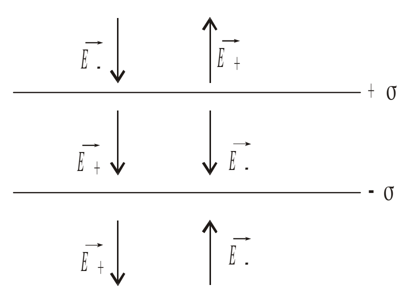

Fig.8. 28

To determine the electric field E between the plates it is possible to use the principle of superposition

$$E = E_+ + E_-\,, \tag{8.77}$$

where E_+ and E_- are the electric fields caused by positively and negatively charged plates, respectively. Then we obtain

$$E = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}. \tag{8.78}$$

Note that the electric field outside the plate equals to zero.

The plates will have different potential U_1 and U_2. Potential difference $V = U_2 - U_1$ is called **voltage.** This potential difference can be expressed as the work per unit charge

$$V = Ed = \frac{\sigma}{\varepsilon_0} d = \frac{d}{\varepsilon_0}\frac{\sigma A}{A} = \frac{d}{\varepsilon_0 A} Q, \tag{8.79}$$

where we used the expression $\sigma = \frac{Q}{A}$. Inserting eq.(8.79) into definition of capacitance given by eq. (8.75)we have

$$C = \frac{Q}{V} = \frac{Q}{\frac{d}{\varepsilon_0 A} Q} = \varepsilon_0 \frac{A}{d}. \tag{8.80}$$

The capacitance is constant for a given capacitor. Its value depends on the geometry of capacitor and the relative position of the two conductors and on the material that separates them (dielectric constant).

A single isolated conductor can also be said to have a capacitance. In this case *C* is defined as **the ratio of the charge to absolute potential *U*** on the conductor that is relative to $U_1 = 0$ at $r \to \infty$. If we denote $U_2 = U$, therefore, for isolated conductor we have

$$C = \frac{Q}{U}, \tag{8.81}$$

where *Q* is the charge on the isolated conductor. For example, the capacitance of conducting sphere of radius *R* carrying charge *Q* is

$$C = \frac{Q}{V} = \frac{Q}{\frac{1}{4\pi\varepsilon_0}\frac{Q}{R}} = 4\pi\varepsilon_0 R. \tag{8.82}$$

8.14 Energy stored in a charged capacitor

Consider a parallel-plate capacitor that is initially uncharged, so that the initial potential difference across the plates is zero. Imagine that the capacitor is connected to a battery and develops a maximum charge *Q*. We assume that the capacitor is charged slowly. The final potential difference of the capacitor is $V = Q/C$.

Suppose that q is the charge on the capacitor at some instant during the charging process. At the same instant, the potential difference across the capacitor is $V = q/C$. The work necessary to transfer of charge $\mathrm{d}q$ from the plate of charge -q to the plate of charge +q (which is at higher potential) is given by

$$\mathrm{d}W = V\mathrm{d}q = \frac{q}{C}\mathrm{d}q \tag{8.83}$$

and the total work done in charging the capacitor from $q = 0$ to some final charge Q is given by

$$W = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{Q^2}{2C}. \tag{8.84}$$

But the work done in charging the capacitor can be considered as potential energy U stored in capacitor. Using $Q = CV$, we can express the electrostatic energy stored in charged capacitor in this form

$$U = \frac{Q^2}{2C} = \frac{1}{2}QV = \frac{1}{2}CV^2. \tag{8.85}$$

This result can be applied to any capacitor, regardless of its geometry. From this equation we can see that the stored energy increases as C increases and as the potential difference increases.

Example

A cylindrical conductor of radius a and charge +Q is concentric with a larger cylindrical shell of radius b and charge -Q (see Fig.8.29). Find the capacitance of this cylindrical capacitor if its length is l.

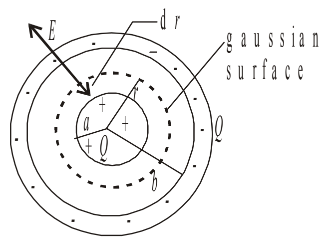

Fig.8. 29

Solution:

If we assume that l is long compared with a and b, we can neglect and effects. In this case the field is perpendicular to the axis of the cylinder. The electric field in the region $a < r < b$ can be calculated using Gauss` law as

$$\phi_c = \oint \vec{E} \cdot \mathrm{d}\vec{A} = E\oint \mathrm{d}\vec{A} = \frac{q_{\text{inside}}}{\varepsilon_0} = \frac{Q}{\varepsilon_0}.$$

Note, that E is parallel to $\mathrm{d}A$ everywhere on the cylindrical surface. But the area of the curved surface is $A = 2\pi rl$, therefore

$$E\,2\pi r l = \frac{Q}{\varepsilon_0}$$

or

$$E = 2k\frac{Q}{lr}, \tag{8.86}$$

where $k = \dfrac{1}{4\pi\varepsilon_0}$. Then the potential difference between the two cylinders is by the definition (see eq. (8.58))

$$V_b - V_a = -\int_a^b \vec{E}\cdot \mathrm{d}\vec{r} = -\int E\mathrm{d}r\,. \tag{8.87}$$

Inserting eq.(8.86) into eq.(8.87) gives

$$V_b - V_a = -2k\lambda\int_a^b \frac{\mathrm{d}r}{r} = -2k\frac{Q}{l}\ln\frac{b}{a}\,.$$

Substituting this into definition of capacitance given by the eq.(8.75), we get

$$C = \frac{Q}{V} = \frac{Q}{\frac{2kQ}{l}\ln\frac{b}{a}} = \frac{l}{2k\ln\frac{b}{a}}\,. \tag{8.88}$$

<u>Remarks</u>

1. The potential $V = V_b - V_a$ is the magnitude of the potential difference given by $2k\frac{Q}{l}\ln\frac{b}{a}$, a positive quantity. That is, $V = V_b - V_a$ is positive since the inner cylinder is at higher potential.
2. From result we see that the capacitance is proportional to the length of cylinder and also depends on the radii of the two cylindrical conductors.