6. MEASUREMENT OF THE MOMENT OF INERTIA BY THE TORSIONAL PENDULUM

ASSIGNMENT

1. Measure the period of the torsional pendulum
2. Calculate the moment of inertia
3. Determine the shear modulus of the steel wire
4. Analyse the source of errors

THEORETICAL PART

Every substance is elastic to some degree. A material is to be elastic after a deformation of any kind, it returns readily to its original shape. There is a limit to the force, which may to applied to a body and still it return to its original, distorted state. This is called **the elastic limit**. If the stretching force is increased beyond the elastic limit, the material will eventually pull apart or break.

We shall discuss the elastic properties of solids in terms of the concept of **stress** and **strain**. We assume that a body is subjected to a force F tangential to one of it faces while the opposite face is held in fixed position (see Fig.6.1).

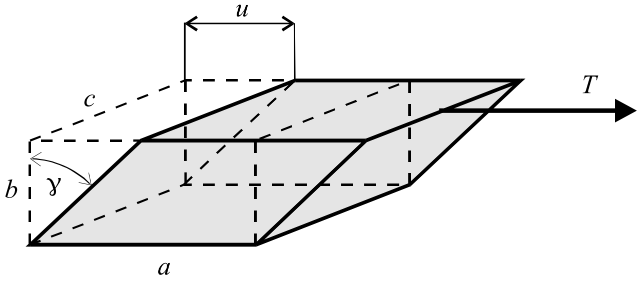


Fig.6.1

The object is originally a rectangular block, and the stress result in a shape, whose cross-section is a parallelogram. For this situation the stress called a **shear stress**.

We define **the shear modulus S** as the ratio of the shear stress F / A and shear strain dx / h i.e.

$$S = \frac{Fh}{Adx} \quad (6.1)$$

where: dx is the horizontal distance the shear face moves

h is the height of the object

The shear strain can be calculated from the Hook's law as

$$dx = \frac{1}{S}\frac{F}{A}l \qquad (6.2)$$

where dx is perpendicular to h . The shear modulus is constant for the materials and is generally one-third to one-half the value of the elastic modulus called Young's modulus.

Values of the shear modulus, for example, for steel are $8.4 \cdot 10^{10}$ Nm^{-2} and for cooper are $4.2 \cdot 10^{10}$ Nm^{-2} in contradistinction to the elastic modulus that are for steel $2x10^{11}\,Nm^{-2}$ and for the cooper $1x10^{11}\,Nm^{-2}$.

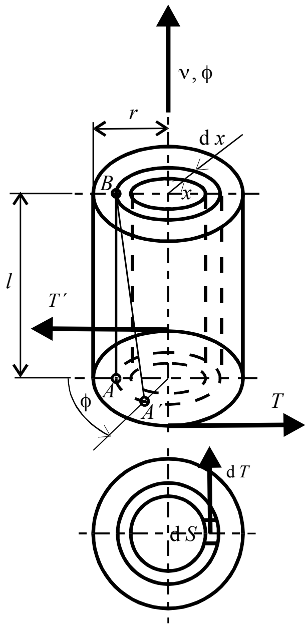
Now we calculate the torque of torsion's force acting on the uniform solid cylinder. The cylinder of radius *r* and length *l* is clamped at one end and the second end we will action with torque τ . The shear stress is defined as

$$S = \frac{dF}{dA} \tag{6.3}$$

From this equation the tangential force acting on the infinitesimal element *dA* equals

$$dF = SdA$$

From the geometry in Fig.6.2 follows that the elementary area dA in which acts the tangential force is



$$dA = 2\pi x dx \tag{6.4}$$

So we can obtain an expression for the force acting on the shall of radius x of the cylinder

$$dF = SdA\frac{x}{l} = S\frac{2\pi}{l}x^2 dx \tag{6.5}$$

Then the torque $d\tau$ due to the external tangential force acting on the elementary shell of the cylinder is by the definition

$$d\tau = x\phi dF = S\frac{2\pi}{l}\phi x^3 dx \tag{6.6}$$

Integration of eq.6.6 gives the resultant torque on the cylinder as

$$\tau = \int d\tau = \int_0^r \frac{S\varphi x^3}{l} 2\pi dx = \frac{\pi S r^4}{2l}\phi \tag{6.7}$$

Fig.6.2

Comparing this result with Hook's law in form

$$\tau = \tau_0 \phi \tag{6.8}$$

we have

$$\tau_0 = \frac{\pi G r^4}{2l} \tag{6.9}$$

The constant τ_0 is called **the torsional constant.**

The torsional pendulum offers a standard method of determine the moment of inertia of an irregular object. A torsional pendulum consists of a rigid body suspended by a wire attached to a rigid support (see Fig.6.3).The body (disc) oscillate about line *AB* with the amplitude ϕ.

The equation of motion for a oscillating body is in form $\tau = I\varepsilon$. If we insert the value of the torque given by eq 6.7 into this equation we have.

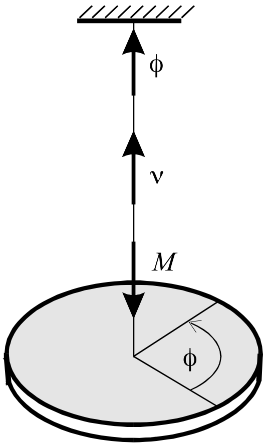


Fig. 6.3

$$I\frac{d^2\phi}{dt^2} = -\frac{\pi G r^4}{2l}\phi \qquad (6.7)$$

The mines sign indicates that the force acts to the equilibrium position. Using the notation

$$\tau_0 = \frac{\pi G r^4}{2l} \qquad (6.8)$$

we obtain

$$\frac{d^2\phi}{dt^2} = -\frac{\tau_0}{I}\phi \qquad (6.9)$$

We denote the ratio $\frac{\tau_0}{I}$ by the symbol ω^2 , i.e.

$$\frac{\tau_0}{I} = \omega^2 \qquad (6.10)$$

then the equation 6.9 can be written in the form

$$\frac{d^2\phi}{dt^2} = -\omega^2\phi \qquad (6.11)$$

What we now require is a solution to eq. (6.11). This second-order differential equation is often referred to as the harmonic equation. Therefore, the solution of is

$$\phi = \phi_0 \sin(\omega t + \alpha) \qquad (6.12)$$

where ϕ_0 is the maximum angular displacement

$$\omega = \sqrt{\frac{\tau_0}{I}} \quad \text{is angular frequency} \qquad (6.13)$$

α is called the phase constant

T is the period of the harmonic motion given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\tau_0}} = 2\pi\sqrt{\frac{2lI}{\pi S r^4}} \qquad (6.14)$$

where *I* the moment of inertia to the axis passing through the centre of mass of the cylinder, *l* is the length of the string , *S* is the shear modulus of the string and *r* is the radius of the string.

THE METHOD - PRACTICAL PART

Eq. (6.14) represents a possible experimental technique for measuring the moment of inertia of unknown body. The method is based on the measuring of the period of the torsional pendulum made of this body. Note that we must have at known body disposition. For the disc of radius *R* and mass *M* is the moment of inertia about the axis through the centre of mass:

$$J = \frac{1}{2} MR^2 \qquad (6.15)$$

Now we add to this disk the unknown body with the same pendulum axis. Moment of inertia of this compound system is given by

$$I = I + I' \qquad (6.16)$$

where I' is the moment of inertia of the unknown body. The period such a torsional pendulum is

$$T' = 2\pi \sqrt{\frac{I + I'}{\tau_0}} \qquad (6.17)$$

By eliminating between eqs. (6.14) and (6.17) we have

$$I' = I \frac{T'^2 - T^2}{T^2} \qquad (6.18)$$

From this equation we see that the two-part experiment is conducted with the measuring of the both the period T of known body and the period T' of the compound system .

From eq. (6.14) we can calculate the shear modulus as

$$S = \frac{64\pi l MR^2}{d^4 T^2} \qquad (6.19)$$

MEASUREMENT

APPARATUS: known disc, measured body (unknown disc), steel hinge, stopwatch, balance, meter stick, micrometer

Using a laboratory balance determine the mass M of the disc. Measure the diameter of the disc. Determine the diameter of the wire with the micrometer (take several measurements and average the results). Measure the periods T' and T a few times and record them into Tab.1. There is no small angle restriction in this situation, as long as, the elastic limit of the wire is not exceeded.

CALCULATION

Using eq. 6.18 calculate the value of I' . Eq. (6.18) may be rewriting into form

$$I = \frac{1}{8} MD^2 \frac{T^2 - T^2}{T^2} \qquad (6.20)$$

since $I = \frac{1}{8} MD^2$ is the moment of inertia of the known disc.

Using eq.6.19 calculate the shear modulus. Calculate the percentage error of moment of inertia as

$$\frac{u_I}{I}\cdot 100\% = \sqrt{\left(\frac{u_M}{M}\right)^2 + 4\left(\frac{T'^2}{T'^2 - T^2}\right)^2 + \left(\frac{u_{T'}}{T'}\right)^2 + \left(\frac{u_T}{T}\right)^2 + 4\left(\frac{u_D}{D}\right)^2}\cdot 100\%$$

Calculate the percentage error of the measurement of shear modulus given by

$$\frac{u_S}{S}\cdot 100\% = \sqrt{\left(\frac{u_l}{l}\right)^2 + \left(\frac{u_M}{M}\right)^2 + 4\left(\frac{u_R}{R}\right)^2 + 16\left(\frac{u_d}{d}\right)^2 + 4\left(\frac{u_T}{T}\right)^2}\cdot 100\%$$

where: u_m is accuracy of the balance

u_D is accuracy of the meter- stick

u_d is accuracy of the micrometer

u_T, $u_{T'}$ are accuracies of the stopwatch

Analyse the source of error in your experiment