GRAVITATIONAL FIELD

The law of universal gravitation

Prior to 1686, a great number of data had been collected on the motions of the moon and the planets but a clear understanding of the forces that caused these bodies to move the way they did was not available. In the year 1686 Isaac Newton provided the key that unlocked the secrets of the heaven. He knew, from first law of motion that a net force had to be acting on the moon. If not, it would move in a straight-line motion. Newton reasoned that this force arose as a result of a gravitational attraction that the earth exerts on the moon. He also concluded that there could be nothing special about the earth-moon system or the sun and its planets that would cause gravitational forces act on them alone.

In 1687 Newton published his work on **the universal law of gravity**. This law states, that every particle in the universe attracts every particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This law is called **the universal law of gravity and it is the form**

$$F = G\frac{m_1 m_2}{r_2}, \tag{7.1}$$

Where m_1 and m_2 are masses, G is a universal constant called **gravitational constant**, which has been measured at the first by Sir Henry Cavendish in 1798. Its value in SI is $G = 6.6672 \times 10^{-11}\ \mathrm{Nm}^2/\mathrm{kg}^2$.

We can express this force in vector form by introducing the unit vector r_{12}^{o}, as is shown in Fig.7.1. Because the unit vector is in direction of the displacement vector r_{12} directed from m_1 to m_2, the force acting on m_2 due to m_1 is given by

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r_{12}^2}\vec{r}_{12}^{\,o}. \tag{7.2}$$

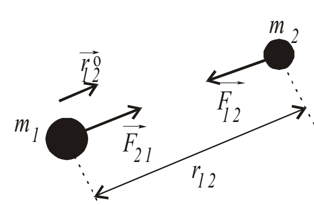

Fig.7. 1

The mines sign in this equation indicates that m_2 is attracted to m_1, and so the gravitational force must be directed toward on m_1. By Newton`s third law the force acting on m_1 due to m_2 , F_{12} , is equal in magnitude to F_{21} and in opposite direction

$$F_{12} = - F_{21} . \tag{7.3}$$

Example

Two stars of masses *M* and *4M* are separated by a distance *d*. Determine the location of a mass point measured from *M* at which the net force on a third mass would be zero.

Solution:

We assume $F_{Mm} = F_{4Mm}$. Inserting the Newton`s law of gravity into this identity gives

$$G\frac{mM}{x^2} = G\frac{m4M}{(d-x)^2}$$

Using this identity we can calculate the ditance *x* as follows:

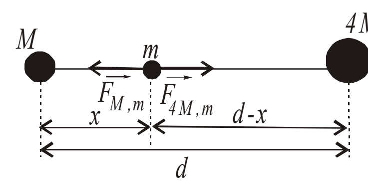

Fig.7. 2

$$4x^2 = d^2 - 2dx + x^2$$

$$3x^2 + 2dx - d^2 = 0$$

This quadratic equation has two roots:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad x_1 = \frac{1}{3}d \quad \text{and} \quad x_2 = -d .$$

The distance of a point measured from *M* at which the net gravitational force equals zero is x_1 . Note that second root is unreal for our case, because the mass point has to lie between masses *M* and *4M*.

The gravitational force between an extended body and particle

We have note that law of universal gravitation is valid only if interacting objects are considered as particles. We now determine the gravitational force associated with a system consisting of a point mass *m* and system of *N* particle with masses m_1, m_2, ..., m_N. **Resultant force acting on the point mass *m*** is given by

$$\vec{F} = \sum_{i-1}^{N} \vec{F}_i = -G\sum_{i=1}^{N} \frac{m_i m}{r_i^2} \vec{r}_i^{\,o} = -Gm\sum_{i=1}^{N} \frac{m_i}{r_i^2} \vec{r}_i^{\,o} , \tag{7.4}$$

Where m_i is the *i*th particle of the system, r_i is the distance between m_i and *m*, r_i^{o} is the unit vector directed toward the point mass *m*.

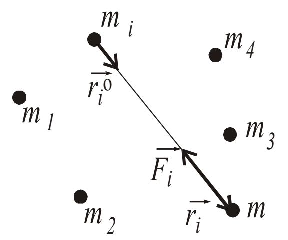

Fig.7. 4

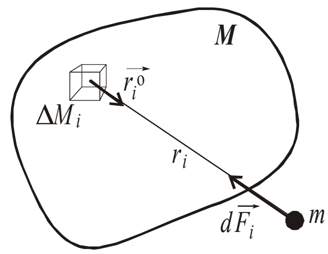

Fig.7. 3

Now we calculate the force acting between extended body of mass M and mass point m. We divide the body into very small segments each of mass ΔM_i as is shown in Fig.7.4. Using the Newton's law of gravity the force, $\mathrm{d}F_i$, associated with the element ΔM_i and particle m equals

$$\mathrm{d}\vec{F}_i = -G\frac{\Delta M_i m}{r_i^2}\vec{r}_i^{\,\mathrm{o}} . \qquad (7.5)$$

The total force on the particle of mass *m* due to the body of mass *M* is obtained by taking the sum over all elements as $\Delta M_i \to 0$. In this limit, the total force on the particle is given by

$$\vec{F} = -Gm\int_M \frac{dM}{r^2}\vec{r}^{\,0} , \qquad (7.6)$$

Where r^{o} is the unit vector directed from the element of the body $\mathrm{d}M$ toward the particle of mass m.

The gravitational field

The two masses interact even thought there are not contact with each other**. The space associated** with **some configuration of masses in which gravitational forces are detected is called gravitational field**. An alternative approach in describing the gravitational interaction is to introduce the concept of a **gravitational field vector (gravitational field) at every point in space.** It is defined as

$$\vec{E} = \frac{F}{m} \qquad \text{(N/kg).} \qquad (7.7)$$

That is, **the gravitational field at any point equals the gravitational force that a test mass, *m*, experience divided by the test mass**. From Newton`s second law the gravitational field equals to acceleration due to gravitational force acting on the free particle at every point in space:

$$\vec{E} = \frac{mg}{m} = \vec{g}$$

The gravitational field has the same direction as the gravitational force as in eq.7.7.

.When more then two masses are present, the force between any pair of masses is given by law of gravity. Therefore, the resultant gravitational field at any point in the space equals to vector sum of the gravitational fields due to various masses. This principle is called **the principle of superposition**. It is express as

$$\vec{E} = -G\sum_{i=1}^{N}\frac{m_i}{r_i^2}\vec{r}_i^{\,o} \qquad (7.9)$$

Where N is the number of particles of the system.

An alternative approach to evaluating **the gravitational force between a particle and extended body** is to perform a vector sum over all segments of the body. Using the same procedure in evaluating F we can obtain the expression for determination the total gravitational field between a particle and an extended body of mass M

$$\vec{E} = -G\int_M \frac{\mathrm{d}M}{r^2}\vec{r}^{\,o}\ , \qquad (7.10)$$

Where $\mathrm{d}M$ is the mass element of the body of mass M, r is the distance between $\mathrm{d}M$ and test mass and r^{o} is the unit vector oriented from $\mathrm{d}M$ toward the test mass.

We have to note that the electric field is independent on the mass point particle m, as follows from all expressions for determination of E.

Gravitational potential

The procedure above is not always recommended since working with a vector function is more difficult then working with the scalar function. This function is called **the gravitational potential** and it is defined by the equation

$$V = \frac{U}{m} \qquad \text{(J/kg)} \qquad (7.11)$$

It is defined as the potential energy U of the test mass divided by this mass. From dynamics we know that the potential energy of mass m at point r equals the negative of the work done by the force:

$$U = -\int_{r_i}^{r_f}\vec{F}\cdot \mathrm{d}\vec{r}\ . \qquad (7.12)$$

As always, the choice of a referent point for the potential energy is completely arbitrary. It is convenient to choose the reference point where the force is zero. Taking $r_i = \infty$ we obtained the important result

$$V = -\frac{1}{m}\int_{\infty}^{r}\vec{F}\cdot \mathrm{d}\vec{r} = -\int_{\infty}^{r}\vec{E}\cdot \mathrm{d}\vec{r}\ . \qquad (7.13)$$

This equation imagines the important relationship between the potential V and gravitational field E.

Now we express the equations for determination the gravitational potential for the basic types of gravitational field:

1. **The potential due to the mass point *m* .** By the eq.(7.13) is

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E}\cdot \mathrm{d}\vec{r} = Gm\int_{\infty}^{r}\frac{\vec{r}^{\,o}\cdot \mathrm{d}\vec{r}}{r^2} = Gm\int_{\infty}^{r}\frac{\mathrm{d}r}{r^2} = -Gm\frac{1}{r}\,, \tag{7.14}$$

since $\vec{r}^{\,o}\cdot \mathrm{d}\vec{r} = r^o \mathrm{d}r\cos\alpha = \mathrm{d}r$ is increment of $\mathrm{d}\vec{r}$ along $\vec{r}$ as is shown in Fig.7.5.

2. **The potential due to a system of *N* particles of masses *m₁*, *m₂*, ..., *m_N*.** From the superposition principle the potential of at any point due to this system is given by

$$V = \sum_{i=1}^{N} V_i = -G\sum_{i=1}^{N}\frac{m_i}{r_i}\,, \tag{7.15}$$

Where r_i is the distance between *i*th mass m_i and at any point in the space, in which we calculate the potential.

3. **The potential of gravitational field due to the body of mass *M***. By the same procedure describing in precious chapter is

$$V = \int \mathrm{d}V = -G\int_{(M)}\frac{\mathrm{d}M}{r}\,. \tag{7.16}$$

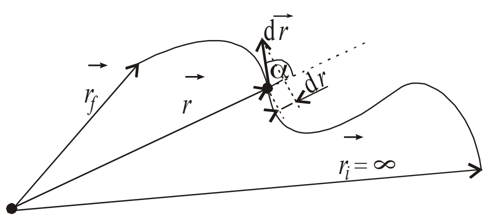

Fig.7. 6

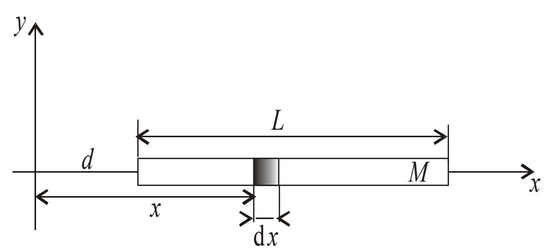

Fig.7. 5

Example

Calculate the potential of the gravitational field of a homogenous bar of length L and mass M at a point distance of d from the end of the bar.

<u>Solution:</u>

We divide the bar on the segments of length $\mathrm{d}x$ at distance *x* from the end of the bar. The mass element (see Fig.7.6) equals $\mathrm{d}M = \rho\mathrm{d}x = \frac{M}{L}\mathrm{d}x$, where the density of the bar is ρ. Inserting the value of $\mathrm{d}M$ into eq.(7.16) gives

$$\mathrm{d}V = \frac{dU}{m} = -G\frac{\mathrm{d}M}{r} \qquad \text{or} \qquad V = -G\frac{M}{L}\int_{d}^{d+L}\frac{\mathrm{d}x}{x} = -G\frac{M}{L}\ln\frac{d+L}{d}\,.$$

Gravitational field of the earth. Weight and gravitational force

Last time we defined the weight of a body of mass *m* as simply *mg*, where *g* is the magnitude of the acceleration due to gravity. Now we obtain a more fundamental description of *g*. Since the force on a freely falling body of mass near the surface of the earth is given by Newton`s law of gravity, we can equate *mg* to this equation to give

$$mg = G\frac{mM_e}{R_e^2}. \tag{7.17}$$

where M_e is the mass of the earth and R_e is the radius of the earth.

From this equation the magnitude of *g* near the surface of the earth is given by

$$g = G\frac{M_e}{R_e^2} \tag{7.18}$$

Using the fact that $g = 9.80\ \mathrm{m/s^2}$ at the earth surface and the radius of the earth $R_e \approx 6.38\times10^6\ \mathrm{m}$, we find from this equation that $M_e = 5.98\times10^{24}\ \mathrm{kg}$. Therefore, the density of the earth equals $\rho = \dfrac{M_e}{\frac{4}{3}\pi R_e^2} = 5.50\times10^3\ \mathrm{kg/m^3}$. Since this value is about twice the density of the most rocks at the earth`s surface, we conclude that the inner core of the earth has a much higher density.

Now consider a body of mass *m* a distance *h* above the earth`s surface. Since the distance from the earth's centre is $r = R_e + h$ eq,7.17 may be rewrite as

$$mg_h = G\frac{M_e m}{(R_e + h)^2}$$

or

$$g_h = G\frac{M_e}{(R_e + h)^2}. \tag{7.19}$$

From this equation we see that g_h decreases with increasing altitude. As $h \to \infty$ the true weight approaches zero.

Energy consideration in planetary and satellite motion

Consider a body of mass *m* moving with a speed *v* in the vicinity of the massive body *M*, where $M \gg m$ (Fig.7.7). The system may be a planet moving around the sun or satellite in orbit around the earth.

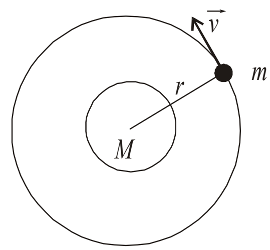

Fig.7. 7

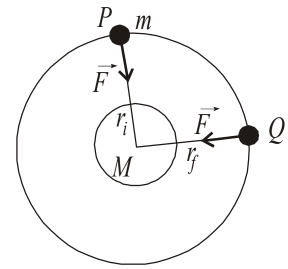

Fig.7. 8

If **we assume that *M* is at rest in an inertial system, then total energy is conserved**

$$E = K + U$$

Inserting the expressions for the kinetic and potential energy into this expression gives

$$E = \frac{1}{2} m v^2 - G\frac{Mm}{r^2}, \tag{7.20}$$

where r is the distance between a mass m and earth`s centre. Therefore, as the mass m moves from point P to Q (see Fig.7.8) then the total energy remains constant:

$$\frac{1}{2} m v_i^2 - G\frac{Mm}{r_i} = \frac{1}{2} m v_f^2 - G\frac{Mm}{r_f}. \tag{7.21}$$

This result shows that *E* m a b **ay be positive, negative or zero**. The sign of the energy depends on the velocity of mass *m*. **For bound system, the energy *E* is less then zero**. We can easily establish that $E < 0$ for a system consisting of a mass *m* moving in circular orbit about body of mass *M*, where $M >> m$. Newton`s second law applied on the body of mass *m* moving around the body of mass *M* gives

$$G\frac{Mm}{r^2} = m\frac{v^2}{r} \tag{7.22}$$

Multiplication of this expression by the factor $\frac{1}{2}$ gives

$$\frac{1}{2} G\frac{Mm}{r^2} = \frac{1}{2} m\frac{v^2}{r}.$$

This expression gives the value for the kinetic energy of the bound system as

$$K = \frac{1}{2} m v^2 = \frac{1}{2} G\frac{Mm}{r} \tag{7.23}$$

Substituting this into eq.(7.20) gives

$$E = -G\frac{Mm}{2r}. \tag{7.24}$$

This expression clearly shows that **the total energy must be negative in the case of circular orbit**. Note that the kinetic energy is positive and equal to one half the magnitude of the potential energy (see7.23).

Example

Calculate the work required to move an earth satellite of mass *m* from a circular orbit of radius $2R_e$ to one of radius $3R_e$ (Fig.7.9).

Solution:

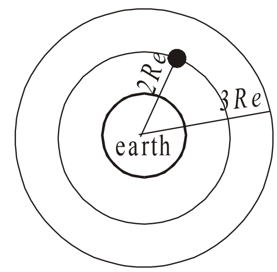

Fig.7. 7

Applying eq.(7.24) we can calculate the initial and final energies as

$$E_i = -G\frac{M_e m}{4R_e}$$

$$E_f = -G\frac{M_e m}{6R_e}.$$

Therefore, the work required to increase the energy of the system is

$$W = E_f - E_i = G\frac{M_e m}{12R_e}.$$

Notes: 1. If we take $m = 10^3$kg, we find that the work required is $W = 5.2\text{x}10^9$ J, which is the energy of equivalent of 39 gal gasoline

2. It is interesting to point out that the process of orbit injection consists of two stages: First, the satellite is placed in an elliptical orbit. At second, its potential energy is maximized, giving it additional kinetic energy.

Escape velocity

Suppose an object of mass *m* is projected vertically upward from the earth`s surface with an initial velocity $v_i$, as is shown in Fig.7.10. At the surface of the earth is $v_i = v$, $r_i = R_e$. We can use the law of conservation energy to find the minimum value of initial speed such that the object will escape the earth`s gravitational field. When the object reaches its maximum altitude, $v_f = 0$ and $r_f = r_{\max} = R_e + h$, where h is the height of the object above the earth`s surface. Because the total energy is conserved, substitution of these conditions into law of conservation of energy gives (see eq.(7.21))

$$\frac{1}{2}v^2 - G\frac{M_e}{R_e} = -G\frac{M_e}{r_{\max}}. \tag{7.25}$$

Solving for v^2 gives

$$v^2 = 2GM_e\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right). \tag{7.26}$$

If the initial speed is known, this expression can be used to calculate the maximum altitude $h = r_{\max} - R_e$.To calculation **the minimum speed** the object must have at the earth`s surface in order to escape from the influence of the earth`s gravitational field. This corresponds to the situation where the object can just reach infinity with a final velocity of zero. Setting $r_{\max} = \infty$ into eq.(7.26) and taking $v_i = v_{\text{esc}}$ we get

$$v_{\text{esc}} = \sqrt{\frac{2GM_e}{R_e}} = 1.12\times10^4 \text{ m/s} = 11.2 \text{ km/s}. \tag{7.27}$$

Note that this expression for v_{esc} is independent of the mass of the object projected from the earth.

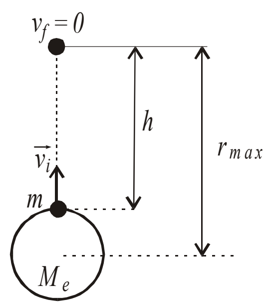

Fig.7. 8