RIGID BODY MECHANICS

Up to now we have been concerned with the motion of a single particle. We have assumed that our particle is approximated an ideal particle. It means we have assumed that it under went only translation motion. Real bodies, however, can undergo rotational motion as well. The mechanical system can be either a system of particles or an extended body. We describe the overall motion of any mechanical system in terms of a special point called **the center of mass.** We shall see that the mechanical system moves as if all its mass is concentrated at the center of mass.

Center of mass

Consider a mechanical system consisting of a pair of particles of masses m_1 and m_2 located along the *x* axis as in Fig.4.1. The *x* coordinate of the center of mass in this case is defined to be

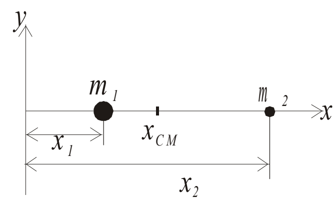

Fig.4. 1

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{4.1}$$

From this equation we can see that the center of mass lies closer to the more massive particle.

We can extend the center of mass concept to a system of *n* particles lying along *x* axis as

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m_i x_i}{\sum m_i} = \frac{\sum m_i x_i}{M}$$

Where x_i is the *x* coordinate of the *ith* particle and *M* is the total mass of the system of particle. This result may be extending to a system of many particles in three dimensions:

$$x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{M}$$

$$y_{CM} = \frac{\sum_{i=1}^{n} m_i y_i}{M} \tag{4.2}$$

$$z_{CM} = \frac{\sum_{i=1}^{n} m_i z_i}{M}$$

Where $M = \sum_{i-1}^{n} m_i$ is the total mass of the system, x_i, y_i, z_i are coordinates of the *i*th particle and *n* is number of particles.

The center of mass can be also located by its vector position

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M}, \tag{4.3}$$

Where r_i is the position vector of *it*h particle and *M* is the total mass of the system of particles.

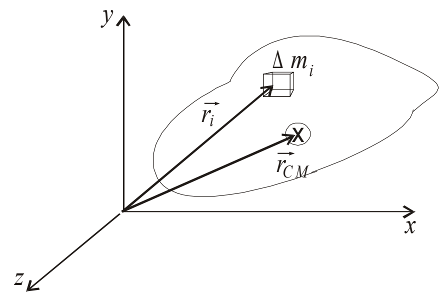

Fig.4. 2

A real body is made up of a large number of particles. The particle separation is very small and so the body can by considered to have a continuous mass distribution (Fig.4.2). By dividing the body into elements of mass Δm_i, with coordinates x_i, y_i, z_i we see that the *x*- coordinate of center mass is approximately

$$x_{CM} = \frac{\sum_{i=1}^{n} x_i \Delta m_i}{M}.$$

If we let the number of elements, *n*, approach infinity, then x_{CM} will be given precisely. In this limit, we replace the sum by an integral and replace Δm_i by differential element $\mathrm{d}m$, so that

$$x_{CM} = \lim_{\Delta m_i \to 0} \left(\frac{\sum_{i=1}^{n} x_i \Delta m_i}{M} \right) = \frac{1}{M} \int_M dm \tag{4.4}$$

and likewise for y_{CM} and z_{CM}

$$y_{CM} = \frac{1}{M}\int_M y\,dm$$

$$z_{CM} = \frac{1}{M}\int_M z\,dm.$$

We can express the vector position of center of mass of a rigid body in the form

$$\vec{r}_{CM} = \frac{1}{M}\int_M \vec{r}\,dm \qquad (4.5)$$

Where r is vector position of element $\mathrm{d}m$

Note: 1. The center of mass of various homogenous, symmetric bodies must lie on an axis of symmetry.

2. Since a rigid body is a continuous distribution of mass, each portion is acted upon by the force of gravity. The net effect of all of these forces is equivalent to the effect of a single force, Mg, acting through a special point, called **center of gravity**. If a rigid body is pivoted

at its center of gravity, it is balanced in any circulation.

Example

Determine the center of mass of a uniform hemisphere of radius *R*.

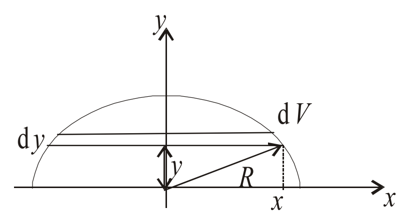

Fig.4. 3

<u>Solution:</u>

We choose the coordinate system so that the origin is at center of the sphere and *y* axis is along the axis of symmetry, i.e. along the *y* axis (seeFig.4.3). Then $x_{CM} = z_{CM} = 0$. To find y_{CM} we divide the hemisphere into the infinite number of infinitesimal cylinders. Each of the cylinders has the elementary volume

$$\mathrm{d}V = \pi x^2 \mathrm{d}y$$

Where *x* is the radius of cylinder, $\mathrm{d}y$ is the height of cylinder. From Pythagora`s theorem we have

$$R^2 = x^2 + y^2$$

or

$$x^2 = R^2 - y^2.$$

Inserting this expression into equation for determination of $\mathrm{d}V$ gives

$$\mathrm{d}V = \pi\left(R^2 - y^2\right)\mathrm{d}y.$$

By definition of *y* coordinate of the center of mass we have

$$y_{Cm} = \frac{1}{M}\int_0^R y\,\mathrm{d}m = \frac{\rho}{M}\int_0^R y\pi\left(R^2 - y^2\right)\mathrm{d}y = \frac{\pi\rho}{M}\left[\int_0^R R^2 y\,\mathrm{d}y - \int_0^R y^3\,\mathrm{d}y\right] = \frac{\pi\rho}{M}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi\rho}{M}\frac{R^4}{4}$$

Where ρ is the density of the hemisphere. Remember the density for a homogeneous body is defined by the expression $\rho = \frac{M}{V}$. Therefore, the total mass *M* of hemisphere is equal to the density ρ times the total volume V. It means

$$M = \rho V = \frac{1}{2}\left(\rho\frac{4}{3}\pi R^3\right).$$

Inserting this value into y_{Cm} gives

$$y_{CM} = \frac{3}{8}R.$$

Motion of a system of particles

Centre of mass and translation motion

We examine the motion of a system *n* particles of total mass *M*. From definition of the position vector of the center of mass we have (see eq.(4.3))

$$M\vec{r}_{Cm} = \sum_{i=1}^{n} m_i\vec{r}_i.$$

Now we differentiate this equation with respect to time and we give

$$M\frac{\mathrm{d}r_{Cm}}{\mathrm{d}t} = \sum_{i=1}^{n} m_i\frac{\mathrm{d}r_i}{\mathrm{d}t} \tag{4.6}$$

or

$$M\vec{v}_{CM} = \sum_{i=1}^{n} m_i\vec{v}_i \tag{4.7}$$

Where $\vec{v}_i = \frac{dr_i}{dt}$ is the velocity of *i*th particle of mass m_i, $\vec{v}_{CM} = \frac{dr_{CM}}{dt}$ is the velocity of the center of mass.

We take the derivative of the expression 4.7 with respect to time again

$$M\vec{a}_{CM} = \sum_{i=1}^{n} m_i \vec{a}_i \quad (4.8)$$

Where $\vec{a}_i = \frac{dv_i}{dt}$ is the acceleration of the *i*th particle and the acceleration of the center of mass is $\vec{a}_{CM} = \frac{dv_{CM}}{dt}$ By the Newton`s second law the force acting on the particle of mass m_i equals

$$F_i = m_i a_i$$

Using this expression we can rewrite eq.4.8 into form

$$M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i \quad (4.9).$$

this equation says that **the vector sum all external forces acting on the system is equal to the total mass of the system times the acceleration of its centre of mass**. This equation concludes that **the centre of mass a system of particles or body of total mass *M* moves like a single particle of mass *M*, on which acts the same net force**.

Linear momentum and impulse. Law of conservation of linear momentum

There is another physical quantity describing the motion of a particle of mass *m* moving with the velocity *v.* This quantity is called the linear momentum**. The linear momentum** of a particle of mass *m* moving with velocity v is defined to be the product of its mass and its velocity:

$$p = mv \quad (4.10) \quad .$$

The linear momentum is vector quantity and its direction is along v .In the SI system, the linear momentum has the unit $kgms^{-1}$.

Using eq.4.10 we can rewrite the Newton's second law of motion as follows

$$\vec{F} = m\vec{a} = m\frac{dv}{dt} = \frac{d}{dt}(m\vec{v})$$

or

$$\vec{F} = \frac{dp}{dt} \quad (4.11)$$

The equation 4.11 is the more general form of the Newton's second law than eq(3.2) because it includes the situation in which the mass of the particle can be changed. The eq.(4.11) may be written as

$$dp = F dt \quad (4.12)$$

We can integrate this equation to find the change in the momentum of a particle. If the momentum of the particle changes from p_i at time t_i to p_f at time t_f, then integrating eq.(4.12) gives

$$\vec{\Delta p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F} \mathrm{d}t .$$

The quantity on the right side of equation is called **impulse of the force** F for the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

$$\vec{I} = \int_{t_i}^{t_f} \vec{F} \mathrm{d}t = \vec{\Delta p} . \qquad (4.13)$$

The unit of impulse is N.s.

This statement, known as **the impulse-momentum theorem**, is equivalent to Newton`s second law. **The theorem says that the impulse of force equals the change in momentum**. From the definition, we see that the impulse is a vector quantity having a magnitude equal to the area under the force- time curve (Fig.4.4 a).

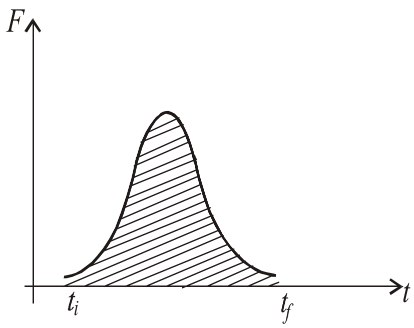

Fig.4. 4 a

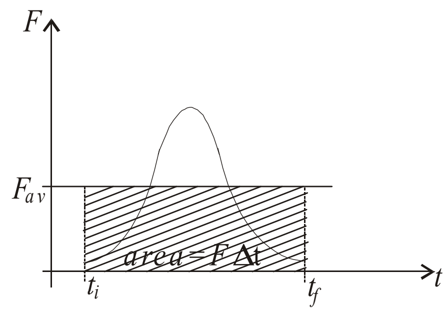

Fig.4. 4 b

Since the force can generally vary in time (as is shown in Fig.4.4 a), it is convenient to defined a time-average force F_{av}, given by (see Fig.4.4 b)

$$\vec{F}_{av} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F} \mathrm{d}t .$$

Therefore, we can express eq. (4.13) as

$$I = F_{av} \Delta t = \Delta p \qquad (4.14)$$

This average force can be thought of as the constant force in the time interval Δt. Remember the equation is applied to a single particle.

Let us now consider a system of *n* particles of total mass $M = m_1 + m_2 + ... + m_n$. Let us assume the particles have linear momentum $p_1 = m_1 v_1$, $p_2 = m_2 v_2$, ..., $p_n = m_n v_n$, where $v_1, v_2, ..., v_n$ are the velocities of the particles of the masses m_1, m_2, …

The total momentum p of our system is given as the sum of the moment of the particles of the system

$$\vec{p} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + ... + m_n \vec{v}_n = \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} \vec{p}_i \ . \tag{4.15}$$

Comparing this equation with eq. (4.7) leads to

$$p = M v_{CM} \ . \tag{4.16}$$

So, we may say, the **total linear momentum of a system of particles is equals to the product of the total mass *M* and the velocity of the center of mass of the system**. If we differentiate eq. (4.16) with respect to time, we obtained (if M=constant)

$$\frac{\mathrm{d}p}{\mathrm{d}t} = M \frac{\mathrm{d}v_{CM}}{\mathrm{d}t} = M \vec{a}_{CM}$$

or

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \vec{F} \ , \tag{4.17}$$

Where F is the net external force on the system. **This equation is Newton`s second law of motion for a system of particles**.

If the net external force acting on a system is zero, then we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0 \ . \tag{4.18}$$

from this equation follows that p =constant, i.e. the linear momentum is independent on the time. The equation (4.18) is called the **law of conservation of linear momentum**. **It stays that, when the net external force on a system is zero, the total linear momentum remains constant**.

A system on which no external force is *acting* on it is called **the isolated system.**

Collisions in one dimension

The collision process may be the result of the physical contact between two objects. We shall use the term collision to represent the event of two particles coming together for a short time, producing impulsive forces on each other. **The impulsive force due to the collision is assumed too much large than any external forces present**. These collisions are divided into two group **elastic** and **inelastic,** one. We shall take an interest in collisions in one dimension. These collisions are always called **head-on collisions**.

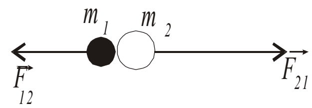

Fig.4.5

When the two particles of masses m_1 and m_2 (Fig. 4.5) collide the impulse forces may vary in time in a complicated way. If F_{12} is the force on m_1 due to m_2 , and we assume no external forces act on the particles, then the change in momentum of m_1 due to the collision is given by

$$\Delta\vec{p}_1 = \int_{t_i}^{t_f} \vec{F}_{12}\, dt .$$

Likewise, when F_{21} is the force acting on m_2 due to m_1. The change in momentum of m_2 is given by

$$\Delta\vec{p}_2 = \int_{t_i}^{t_f} \vec{F}_{21}\, dt .$$

However, Newton`s third law states

$$F_{12} = - F_{21}$$

and we conclude that

$$\Delta p_1 = - \Delta p_2 \qquad \text{or} \qquad \Delta p_1 + \Delta p_2 = 0$$

Since the total momentum of the system is $P = p_1 + p_2$, we conclude that the change in the momentum of the system of particles due to the collision is zero, that is

$$P = p_1 + p_2 = \text{constant} \qquad (4.19)$$

That is precisely what we expect if there are no external forces acting on the system. However, the result is also valid if we consider the motion just before and just after the collision. Since the impulsive forces due to the collision are internal, they do not affect the total momentum of the system. Therefore, we conclude that **for any type of collision, the total momentum of the system just before the collision equals the total momentum of the system just after collision.**

We have seen that the total momentum is always conserved in a collision. However, **the total kinetic energy is generally not conserved when a collision occurs** because some of the kinetic energy is converted into thermal energy and internal elastic potential energy when the bodies are deformed during the collision.

We define the **inelastic collision** as a collision in which momentum is conserved but kinetic energy is not.

Properties of inelastic and elastic collision:

1. An inelastic collision is one in which momentum is conserved, but kinetic energy not
2. A perfectly inelastic collision between two objects is an inelastic collision in which the two objects stick together after the collision, so their final velocities are the same.
3. An elastic collision is one in which both momentum and kinetic energy is conserved.

Rotation motion of a rigid body about a fixed axis.

We shall take an interest in rotational motion of rigid by rotational bodies. By **a rigid body** we mean a body that has a definite shape it means that does not deform under application of external forces. That is, all parts of a rigid body remain at a fixed separation with respect to each other when subjected to external forces. **By rotational motion about a fixed axis** we mean that all points in the body move in circle and that the centers of these circles lie on a line called **axis of rotation**.

Kinetics of rotational motion.

Consider a planar rigid body of arbitrary shape confined to the *xy* plane and rotating about a fixed axis through *O* perpendicular to the plane. A particle on the body at point *P* is at fixed distance *r* from the origin and rotates in a circle of radius *r* about *O* (Fig.4.6).

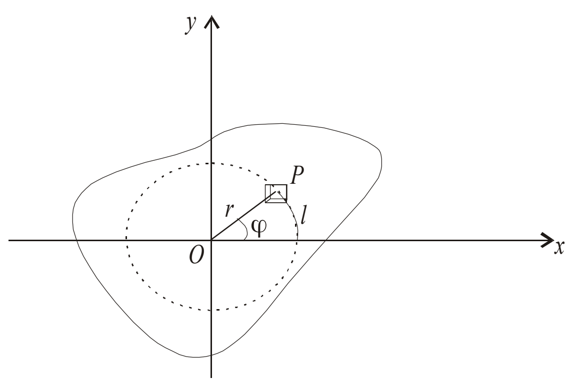

Fig.4. 6

It is convenient to represent the position of the point *P* with its polar coordinate (r, φ). In this representation, the only coordinate that changes in time is the angle φ, r remains constant. As the particle moves along the circle from the positive *x* axis to the point *P,* it moves through an arc length *l,* which is related to the angular position φ through the relation

$$\varphi = l / r$$

The angle φ is the ratio of an arc length and the radius of the circle, and hence is a pure number. However, we commonly refer to the unit of φ as a **radian** (rad). One rad is the angle subtended by an arc length equal to the radius of the arc. (To convert an angle in degrees to an angle in radians, we can use the expression $\varphi(\text{rad}) = \frac{\pi}{180}\varphi(\text{deg})$).

As the particle travels from one point to another in the time Δt, the radius vector sweeps out an angle $\Delta\varphi = \varphi_2 - \varphi_1$, which equals the angular displacement. The average angular velocity of the moving particles is by the definition

$$\omega_{av} = \frac{\Delta\varphi}{\Delta t}$$

and the instantaneous angular velocity equals

$$\omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t},$$

If the instantaneous angular velocity of a particle changes from ω_1 to ω_2 in the time interval Δt, the particle has an average angular acceleration given by

$$\varepsilon_{av} = \frac{\Delta\omega}{\Delta t}$$

Therefore, the instantaneous angular acceleration is given by

$$\varepsilon = \frac{\mathrm{d}\omega}{\mathrm{d}t}.$$

For rotation about a fixed axis every particle on the rigid body has the same angular velocity and the same angular acceleration. These quantities are relate to the velocity and acceleration (see eqs.(2.27, 2.28, 2.29) as

$$v = r\omega$$
$$a_t = r\varepsilon \qquad (4.20)$$
$$a_r = \omega^2 r$$

Where r is the perpendicular distance between any point and the axis of rotation It is clear, that v, a_r, a_t are different for particles at different distances *r* from the axis of rotation.

The frequency of rotation *f* is related to the angular velocity as

$$\omega = 2\pi f.$$

Rotational kinetic energy. Moment of inertia.

Let us consider a rigid body as a collection of small particles and let us assume that the body rotates about the fixed axis with the angular velocity ω. Each particle has some kinetic energy, determined by

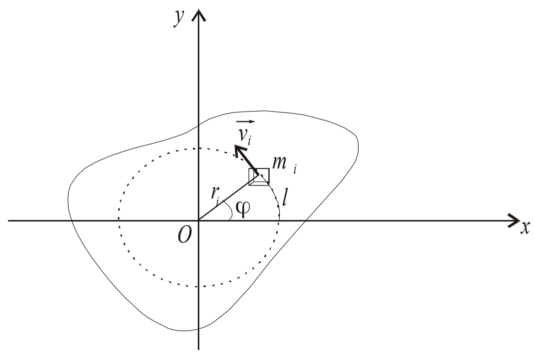

Fig.4. 7

its mass and velocity. If the mass of *i*th particle is m_i and its speed is v_i the kinetic energy of this particle is

$$K_i = \frac{1}{2} m_i v_i^2 \tag{4.21}$$

To proceed further, we must recall that every particle in a rigid body has the same angular velocity ω, the individual linear velocity depends on the distance r_i from the axis of rotation according to expression

$$v_i = r_i \omega.$$

The total kinetic energy of the rotating rigid body is the sum of the kinetic energies of the individual particles

$$K = \sum K_i = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum m_i r_i^2 \omega^2 = \frac{1}{2} \left(\sum m_i r_i^2 \right) \omega^2 , \tag{4.22}$$

Where we have factored ω^2 from the sum.

The quantity in parentheses is called **the moment of inertia**

$$I = \sum m_i r_i^2 \tag{4.23}$$

Using this notation, we can express the kinetic energy of the system of particles as

$$K = \frac{1}{2} I \omega^2 \tag{4.24}$$

Dimension of the moment of inertia is kgm^2. It plays the role of mass in all rotational equations.

Now we can use the definition (4.23) for the body, which is divided into volume elements Δm. If we take the limits of this sum as $\Delta m \to 0$ then

$$I = \lim_{\Delta m_i \to 0} \sum r^2 \Delta m = \int r^2 \mathrm{d}m . \tag{4.25}$$

Generally, the volume density of the body is defined as

$$\rho = \frac{\mathrm{d}m}{\mathrm{d}V}$$

Rearrangement of this equation gives

$$\mathrm{d}m = \rho \mathrm{d}V \,.$$

Therefore, the moment of inertia can be expressed in the form

$$I = \int_V \rho r^2 dV \,. \qquad (4.26)$$

If the body is homogeneous i.e. $\rho = \text{constant}$ the expression for determination the moment of inertia can by rewrite into form

$$I = \rho \int_V \rho r^2 dV \,. \qquad (4.27)$$

The eq.4.27 defines the moment of inertia of a solid body about the fixed axis of rotation. Note that *r* is the distance of the element $\mathrm{d}V$ from the axis of rotation.

Example

A uniform solid cylinder has a radius *R*, mass *M* and length *l*. Calculate the moment of inertia of the cylinder rotating about its axis of the symmetry (Fig.4.8).

Solution:

It is convenient to divide the cylinder into cylindrical shells of radius *r*, thickness $\mathrm{d}r$ and length *l*. In this case, cylindrical shells are chosen because one wants all mass elements d*m* to have a single value for *r*.

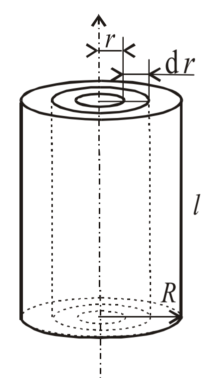

Fig.4. 8

The volume of each shell is

$$\mathrm{d}V = l\mathrm{d}A = (2\pi r \mathrm{d}r)l$$

Where $\mathrm{d}A = (2\pi r \mathrm{d}r)$ is its cross-sectional area of the shell

Using the definition of the volume density we can express the mass element of the cylinder in form

$$\mathrm{d}m = \rho \mathrm{d}V = \rho 2\pi r l \mathrm{d}r \,.$$

Substituting this into equation for the moment of inertia *I*, we get

$$I = \int r^2 \mathrm{d}m = \int_0^R \rho 2\pi l r^3 \mathrm{d}r = 2\pi\rho l \int_0^R r^3 \mathrm{d}r = 2\pi \frac{M}{\pi R^2 l}\left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}MR^2$$

If you can see the moment of inertia of a uniform solid cylinder depends of its mass M and radius R, only.

Parallel-axis theorem.

Suppose a body rotates in the *x*, *y* plane about an axis through *O* and the coordinates of the center of mass are x_{CM}, y_{CM}. The moment of inertia about the *z* axis through *O* is

$$I = \int r^2 \mathrm{d}m = \int \left(x^2 + y^2\right)\mathrm{d}m,$$

Where $r^2 = x^2 + y^2$ is the distance of the element $\mathrm{d}m$ from *z* axis.

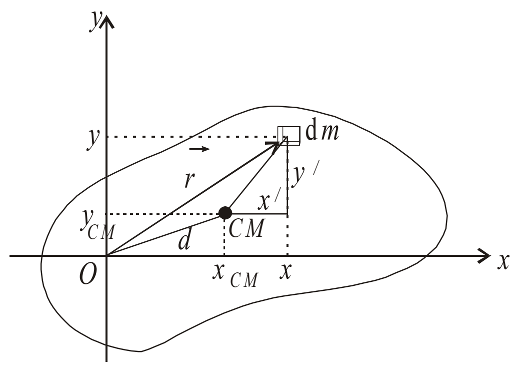

Fig.4. 9

We can relate the coordinates *x*, *y* to the coordinates of the center of mass x_{CM}, y_{CM}. From the Fig. 4.9. follows

$$x = x' + x_{CM}$$

$$y = y' + y_{CM}.$$

Therefore, the moment of inertia of the body about the axis through *O* is given by

$$I = \int\left[\left(x' + x_{CM}\right)^2 + \left(y' + y_{CM}\right)^2\right]\mathrm{d}m = \int\left(x'^2 + y'^2\right)\mathrm{d}m +$$
$$+ 2x_{CM}\int x'\mathrm{d}m + 2y_{CM}\int y'\mathrm{d}m + \left(x_{CM}^2 + y_{CM}^2\right)\int \mathrm{d}m$$

The first term on the right is, by the definition, the moment of inertia about an axis parallel to the *z* axis. The second two terms on the right are zero, since by the definition of the center of mass $\int x'\mathrm{d}m = \int y'\mathrm{d}m = 0$ (x' and y' are the coordinates of the mass element relative to center of mass). The last term on the right equals Md^2, where $d^2 = x_{CM}^2 + y_{CM}^2$.

Therefore, the moment of inertia equals

$$I = I_{CM} + Md^2 \qquad (4.28)$$

This equation is called **parallel-axis theorem**. **It states that the moment of inertia about any axis that is parallel to and a distance *d* away from the axis that passes through the center of mass is given by eq.(4.28).**

Example

Consider a uniform rigid rod of mass M and length l (see Fig.4.10). Find the moment of inertia of the rod about an axis perpendicular to the rod passes

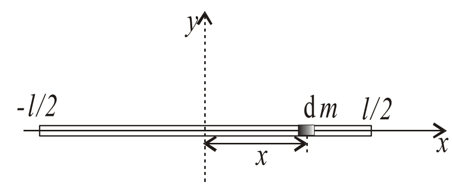

Fig.4. 10

a) Through the center of mass

b) Through the end of the rod.

<u>Solution:</u>

a) By the definition of the moment of inertia is

$$I_{CM} = \int x^2 \mathrm{d}m = \rho \int x^2 \mathrm{d}V = \rho A \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \mathrm{d}x = \rho A \left[\frac{x^3}{3} \right]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{1}{12} M l^2$$

Where $\mathrm{d}V = A\mathrm{d}x$, A is the cross-sectional area of the rod, x is the distance between mass element $\mathrm{d}m$ and

the axis of rotation.

b) We shall use the parallel axis theorem to calculation of the moment of inertia of the rod about an axis perpendicular to the rod passes through the its end

$$I_{end} = I_{CM} + Md^2 = I_{CM} + M\frac{l^2}{4} = \frac{1}{12}Ml^2 + \frac{1}{4}Ml^2$$

or

$$I_{end} = \frac{1}{3}Ml^2$$

Where $d = l/2$ is the distance between two parallel axes.

Torque.

When a force is exerted on a rigid body pivoted about some axis, the body will tend to rotate about that axis. **The tendency of a body to rotate a body about some axis is measured by a quantity called torque**. Consider a force F acting on a rigid body at the vector position r as is shown in Fig.4.11. The torque is defined as

$$\tau = r \times F \tag{4.29}$$

The magnitude of the torque due to the force relative to the origin *O* is $rF\sin\phi$, where ϕ is the angle between r and F. .The axis about which the body would tend to produce rotation is perpendicular to the plane formed by r and F.Because the direction of the torque is given by the right-hand rule the torque tends to rotate the body counterclockwise. The quantity $d = r\sin\phi$ is called **the arm of the force** F and represents the perpendicular distance from the rotation axis to the line of action of F.

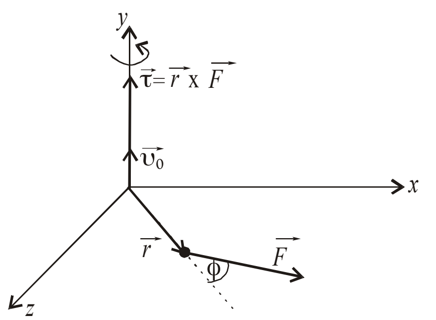

Fig.4. 11

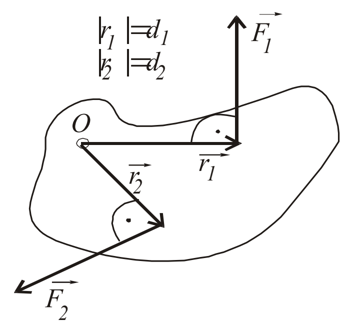

Fig.4. 12

The torque is vector quantity and its unit is N.m in SI units. If there are two or more forces acting on the rigid body, then each has the tendency to produce a rotation about the pivot *O* (Fig.4.12). We shall use the convention that the sign of the resulting torque of the force is positive if its turning tendency is counterclockwise and negative if its turning tendency is clockwise. Hence, the net torque acting on the rigid body about *O* is

$$\tau_{net} = \tau_1 + \tau_2 \Rightarrow F_1 d_1 - F_2 d_2 = \tau \tag{4.30}$$

The rotating tendency increases as F increases and *d* increases as follows from the definition of the torque.

If we have a system of particles (which could be the particles making up the rigid body) the total torque is the sum of the torque on the individual particles as

$$\tau = \sum r_i \times F_i \tag{4.31}$$

Where r_i is the position vector of the *i*th particle, F_i is the net force on the *i*th particle.

Torque and rotational inertia.

Let us consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force F_t, as is shown Fig.4.13.

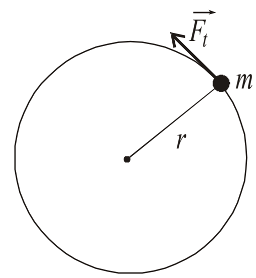

Fig.4. 13

The torque which gives rise to its angular acceleration ε is

$$\tau = r \times F_t \tag{4.32}$$

or

$$\tau = rF_t ,$$

Since the angle between vectors r and F_t is 90° and $\sin 90^\circ = 1$. If we use the relation between tangential and angular acceleration in form $a_t = r\varepsilon$, we can write the Newton's second law of motion for our particle as

$$F = ma_t = mr\varepsilon$$

Therefore, the torque is now given by

$$\tau = rF = mr^2\varepsilon . \tag{4.33}$$

This equation represents a relation between torque and angular acceleration for a single particle. The quantity mr^2 represents the moment of inertia for this particle.

Now we consider a rotating rigid body. We can write the torque on *i*th particle of the body as

$$\tau_i = m_i r_i^2 \varepsilon , \tag{4.34}$$

Where m_i and r_i are the mass of the *i*th particle and its distance from the axis of rotation, respectively and ε is the angular acceleration which is the same for all particles.

Then the net (total) torque on the body equals

$$\tau = \sum_{i=1}^{n} \tau_i = \varepsilon \sum_{i=1}^{n} m_i r_i^2 \tag{4.35}$$

Where *n* is the number of particles.

Where $\sum_{i=1}^{n} m_i r_i^2$ is moment of inertia of the system of particles. Therefore, we can rewrite eq.4.35 into form

$$\tau = I\varepsilon . \qquad (4.36)$$

We can evaluate this equation for an extended body by imagine that the body is divided into volume element, each of mass $\mathrm{d}m$ as is shown in Fig.4.14. Each element rotates in a circle about the origin and has a tangential acceleration a_t produced by external tangential force $\mathrm{d}F_t$.

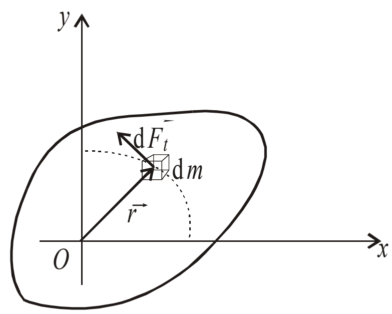

Fig.4. 14

From Newton`s second law follows that the tangential force acting on this element is $\mathrm{d}F_t = (\mathrm{d}m)a_t$. Therefore, the torque $\mathrm{d}\tau$ associated with the force $\mathrm{d}F_t$ equals

$$\mathrm{d}\tau = r\mathrm{d}F_t = (r\mathrm{d}m)a_t .$$

Since the tangential acceleration is $a_t = r\varepsilon$ the elementary torque on the mass element is

$$\mathrm{d}\tau = (r^2\mathrm{d}m)r\varepsilon .$$

It is recognizing that although each point of rigid body may have a different a_t , **all mass elements have the same angular acceleration,** ε . The above equation can be integrating to obtain the net torque of the external forces about *O*:

$$\tau = \int_M r^2 dm\varepsilon = \varepsilon \int_M r^2 dm . \qquad (4.37)$$

Since the moment of the inertia of the body about the rotation axis through *O* is defined as

$$I = \int_M r^2 dm ,$$

then eq.(4.37) becomes

$$\tau = I\varepsilon . \qquad (4.38)$$

We see again that **the net torque about the axis of rotation** is **proportional to the angular acceleration of the body with the proportionality factor being *I*. This equation is valid for the rotation of a rigid body about a fixed axis**.

We see that the moment of inertia of a body plays the same role for rotational motion that mass does for translation motion**. Equations (4.36), (4.38) are rotational analogy of the Newton`s second law for translational motion**.

Angular momentum and its conservation

The instantaneous angular momentum L of the particle relative to the origin O is defined as

$$L = r \times p \tag{4.39}$$

The unit of angular momentum is kg.m^2/s. It is important to note that the magnitude and direction of angular momentum depends on the origin choose. The direction of L (Fig.4.15) is perpendicular to the plane formed by r and p and it since is given by the right-hand rule.

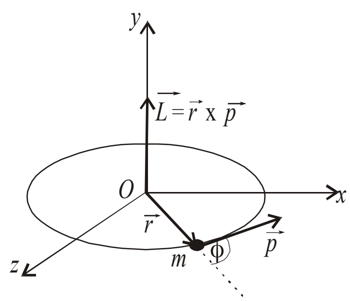

Fig.4. 15

We shall now show that Newton`s second law implies that the resultant torque acting on the particle equals the time rate of change of its angular momentum. The torque is defined as

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{\mathrm{d}p}{\mathrm{d}t} \tag{4.40}$$

where we use the Newton's second law in general form (see eq.4.11).
Let us differentiate equation (4.40) with respect to time

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t} \times \vec{p} \tag{4.41}$$

The last term on the right equals zero, since $\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t}$ is parallel to p, i.e. the angle between v and p is zero. Therefore

$$\vec{\tau} = \frac{\mathrm{d}L}{\mathrm{d}t}. \tag{4.42}$$

This result can be extended for a system of particles about some point. Let us consider

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_n = \sum_{i=1}^{n} \vec{L}_i . \tag{4.43}$$

Then **the net external torque on the system equals**

$$\vec{\tau}_{ext} = \sum_{i=1}^{n} \frac{dL_i}{dt} = \frac{dL}{dt}. \quad (4.44)$$

That is, the **time rate of change of the total angular momentum of the system about some origin in a inertial frame equals the net external torque acting on the system about the origin.**

Conservation of angular momentum

If the resultant external torque acting on the system is zero then

$\tau_{ext} = 0$

Using eq.4.44 we have

$$\frac{dL}{dt} = 0 \quad (4.45)$$

The angular momentum L is independent on the time. as follows from this expression.

For a system of particles we write

$$\sum_{i=1}^{n} \vec{L}_i = \vec{L} = \text{constant}.$$

This equation says that if the system is isolated, that is, not subjected to any external torques, its angular momentum will remain unchanged. This law is called **law of conservation of the angular momentum**.

The conditions for equilibrium

In general, an object will be in rotational equilibrium only if its angular acceleration $\varepsilon = 0$. Since y $\tau_{net} = I\varepsilon$ for rotation about a fixed axis, a necessary condition of equilibrium for an object is that **the net torque about any origin must be zero.**

We now have two necessary conditions for equilibrium of an object, which can be stated as follows:

1. The resultant external force must equal zero

$$\sum_{i=1}^{n} \vec{F}_i = 0. \quad (4.46)$$

2. The resultant external torque must be zero about any origin

$$\sum_{I=1}^{N} \vec{\tau}_I = 0. \quad (4.47)$$

The first condition is a statement of a **translation equilibrium that** is the linear acceleration of the center of mass of the object must be zero when viewed from an inertial reference system. The second condition is a statement of **rotational equilibrium**, that is, the angular acceleration about any axis must be zero. In the special case of **static equilibrium,** which is the main subject of this case, the object is at rest so that it is no linear or angular velocity.

Example

A uniform ladder of length *l* and weight *W=50N* rests against a smooth, vertical wall (Fig.4.16). If the coefficient of static friction between the ladder and ground is $\mu = 0.04$, find the minimum angle $\alpha_{\min}$ such that the ladder will not slip.

<u>Solution:</u>

All the external forces acting on the ladder are showed on the free-body diagram. From the first condition for equilibrium applied to the ladder, we have

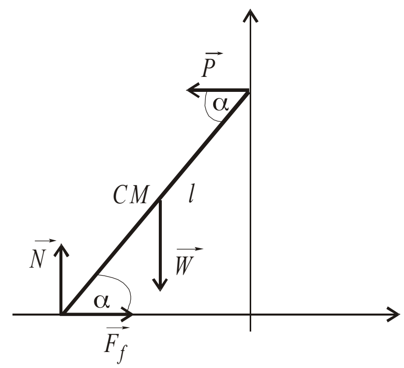

Fig.4. 16

$$\sum F_x = F_f - P = 0 \Rightarrow F_f = P$$

$$\sum F_y = N - W = 0 \Rightarrow N = W$$

Where $F_f = \mu N$ is force of friction.

To find the value of $\alpha_{\min}$, we must find the second condition for equilibrium. When the torques are taken about the origin *O* at the bottom of the ladder we get

$$\sum \tau_O = Pl \sin\alpha_{\min} - W\frac{l}{2}\cos\alpha_{\min} = 0$$

This expression gives

$$\text{tg}\alpha_{\min} = \frac{W}{2P} = \frac{W}{2\mu W} = \frac{1}{0.8} \Rightarrow \alpha_{\min} = 51.3^\circ$$

It is interesting to note that the result does not depend on the length and the weight of the ladder.

Simple harmonic motion

If the force varies in time, the velocity and acceleration of the body will also change with time. A very special kind of motion occurs when the force on the body is proportional to the displacement of the body from equilibrium. If the force always acts toward the equilibrium position of the body, a respective back-and-forth motion will result about this position. The motion is called **periodic** or **oscillatory motion**. Such a motion is the motion of a pendulum, the motion of the molecules in a solid about their equilibrium position etc. Special case of the motion is **simple harmonic motion** in which an object oscillates between two positions for an indefinite period of time, with no loss in mechanical energy. In real mechanical systems, retarding (frictional) forces are always prevented. Such forces reduce the mechanical energy of the system and the oscillations are said to be damped. If an external driving force is applied such the energy balanced by the energy input, we call the motion a **forced oscillation.**

At this time we shall take interest in simple harmonic motion in which a particle moves along the *x* axis. This displacement varies in time according to relationship

$$x = A\cos(\omega t + \delta) \qquad (4.48)$$

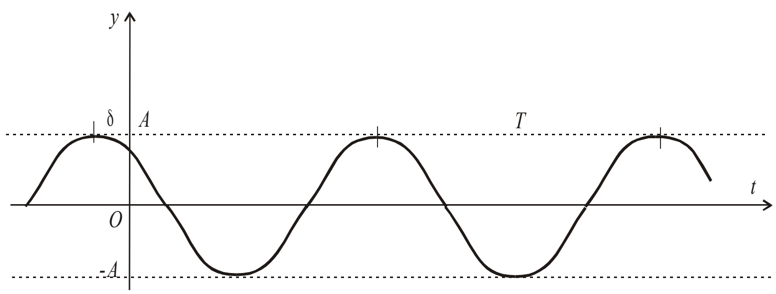

Fig.4. 17

Where A, ω and δ are constant of motion. *A* is called the **amplitude of motion**, it is the maximum displacement of the particle in either the positive or negative *x* direction, ω is called **angular frequency** and δ is called the **phase constant**. In order to give physical significance to these constants it is convenient to plot *x* as function of *t*, as is shown in Fig. 4.17.

The constants δ and *A* tell us what the displacement was at time $t = 0$. The quantity $(\omega t + \delta)$ is called the **phase of the motion** and is useful in comparing the motion of two systems of particles. Note that the function *x* is periodic and repeats itself when ωt increases by 2π radians.

The period, *T*, is the time to go the particle through one full cycles of its motion. That is, the value of *x* at time *t* equals the value of *x* at time $t + T$. We can show that period of the motion is given by $T = \dfrac{2\pi}{\omega}$ by using the fact that the phase increases by 2π radians in time T:

$$\omega t + \delta + 2\pi = \omega(t + T) + \delta .$$

Hence $\omega T = 2\pi$.

The inverse of the period is called **frequency of the motion**, *f*. The frequency represents the number of oscillations the particle makes per unit time

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \cdot$$

The unit is sec^{-1} and it is called Herz (Hz).

Rearranging this equation gives

$$\omega = 2\pi f = \frac{2\pi}{T} .$$

The constant ω has a unit of rad/s.

We can obtain the velocity of a particle undergoing simple harmonic motion by differentiating equation (4.48) with respect to time

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega A \sin(\omega t + \delta) \tag{4.49}$$

The acceleration of the oscillating particles defined as the first derivative of the velocity of this particle, that is

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -\omega^2 A \cos(\omega t + \delta) \tag{4.50}$$

Since $x = A\cos(\omega t + \delta)$ we can rewrite this equation into form

$$a = -\omega^2 x$$

or

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 x \tag{4.51}$$

From eq.(4.49) we see that the sine and cosine function oscillate between ± 1 . The extreme values of v are equal $\pm\omega A$. Similar, eq(4.50) tell us that the extreme values of acceleration are $\pm\omega^2 A$. Therefore, the maximum values of the velocity and acceleration are given by

$$v_{\max} = \pm\omega A \tag{4.51}$$

$$a_{\max} = \pm\omega^2 A \tag{4.53}$$

We conclude this by pointing out the following important properties of a particle moving in simple harmonic motion:

1. The displacement, velocity and acceleration vary sinusoidally with time but are not in phase.
2. The acceleration of the particle is proportional to the displacement, but in opposite direction.
3. The frequency and period of motion are independent on the amplitude.

Now we shall apply these results in the special kind of the periodic motions, on the motion of the simple and physical pendulum, respectively.

The simple pendulum

The simple pendulum is the mechanical system that exhibits periodic motion. It consists of a point mass *m* suspended by a light string of length *l*, where the upper end of the string is fixed (Fig.4.18). The forces acting on the mass are the tension, T , acting along the string and weight of the mass mg . If we displace the point mass from equilibrium position about the angle φ then the torque about pivot *O* is equal ($-mgl\sin\varphi$). Note that the component of weight $mg\cos\varphi$ is balanced by the tension of the

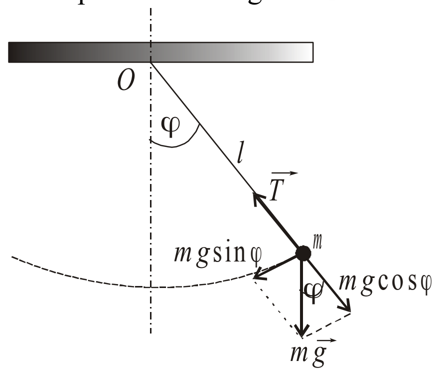

Fig.4. 18

string. Using the equation of motion for rotational motion:

$$\tau = I\varepsilon \tag{4.54}$$

We get

$$-mgl\sin\varphi = I\frac{d^2\varphi}{dt^2} \tag{4.55}$$

Where the moment of inertia of the point mass is I and the value is $I = ml^2$. $\varepsilon = \frac{d^2\varphi}{dt^2}$ is the angular acceleration. **The mines sign in this equation means that the tangential component, always act to equilibrium position**. Inserting the value of *I* into equation (4.55) and the rearrangement of this expression gives

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\sin\varphi \tag{4.56}$$

If we assume that angle φ is small, then the approximation $\sin\varphi \approx \varphi$ is valid and the equation reduces to the form

$$\frac{d^2\varphi}{dt^2} = -\frac{g}{l}\varphi \tag{4.57}$$

This equation is similar to the equation of simple harmonic motion given by eq.(4.51)

$$\frac{d^2\varphi}{dt^2} = -\omega^2\varphi \tag{4.58}$$

From eq.4.58 we conclude **that the motion of simple pendulum is simple harmonic motion**. Therefore, the solution of this equation can be written as

$$\varphi = \varphi_0 \cos(\omega t + \delta) \tag{4.59}$$

Where φ_0 is the maximum angular displacement, ω is the angular frequency and δ is the phase constant. Comparing eqs.(4.57) and (4.58) gives the value of the angular frequency in form

$$\omega = \sqrt{\frac{g}{l}}\ . \tag{4.60}$$

The period of motion of the simple pendulum is by the definition

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \tag{4.61}$$

or $$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{l}{g}}$$

From this expression we see that the period and the frequency of a simple pendulum depend only on the length *l* of the string and the acceleration of gravity *g*.

The physical pendulum

Physical pendulum consists of any rigid body suspended from a fixed axis that does not pass through the body`s center of mass. The system will oscillate when is displaced from its equilibrium position.

Let us consider a rigid body pivoted at a point *O* that a distance *d* from the center of mass as is shown in Fig.4.19. The torque about *O* is provided by the force of gravity, and its magnitude is $mgd\sin\varphi$.

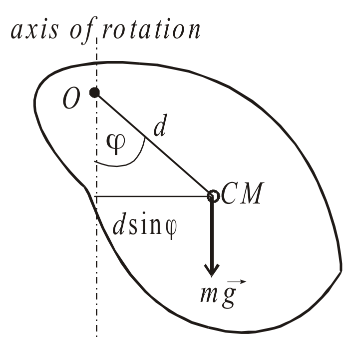

Fig.4. 19

Using equation of motion (4.36), where I is the moment of inertia about the axis through O , we get

$$- mgd\sin\varphi = I\frac{d^2\varphi}{dt^2}. \quad (4.62)$$

The mines sign on the left side indicates that the torque about O tends to decrease φ. If we again assume that φ is small (up to 5°), then the approximation $\sin\varphi \approx \varphi$ is valid and the equation of motion reduces to

$$\frac{d^2\varphi}{dt^2} = -\frac{mgd}{I}\varphi. \quad (4.63)$$

We note that this equation is of the same form as eq.(4.51) for simple harmonic motion. So, the motion is simple harmonic motion. That is, the solution of this equation is

$$\varphi = \varphi_0\cos(\omega t + \delta), \quad (4.64)$$

Where φ_0 is the maximum angular displacement and ω is the angular frequency given by

$$\omega = \sqrt{\frac{mgd}{I}}. \quad (4.65)$$

The period, T, of its motion equals

$$T = 2\pi\sqrt{\frac{I}{mgd}}. \quad (4.66)$$

One can use this result to measure the moment of inertia of a planar rigid body. If the location of the centre of mass, and hence of d, are known, the moment of inertia can be obtained through the measurement of the period. Finally, note that this equation reduces to the period of a simple pendulum when $I = md^2$. That is the case when all the mass is concentrated at the center of mass.

Example

The physical pendulum in the form of a uniform rod of mass $M=1kg$ and length $l=1m$ is pivoted about one end and oscillates in a vertical plane (see Fig.4.20). Find the period T of oscillations if the amplitude of the motion is very small.

Solution:

A rigid rod oscillating about the pivot through one end is physical pendulum with $d = \frac{1}{2}l$.

The moment of inertia of the rod about an axis through its end is (see example on the page 48)

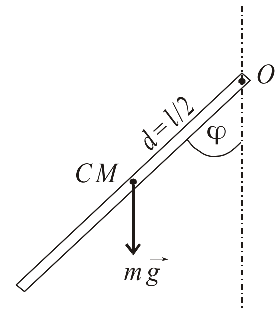

Fig.4. 20

$$I = \frac{1}{3} M l^2 .$$

Substituting these quantities into eq.(4.63) gives

$$T = 2\pi \sqrt{\frac{\frac{1}{3} M l^2}{M g \frac{l}{2}}} = 2\pi \sqrt{\frac{2l}{3g}} .$$

If you can see from this result, the period T depends only on the length of the rod and the acceleration of gravity. This result can be used to measurement of the acceleration of gravity:

$$g = \frac{8\pi^2 l}{3T^2}$$

Rolling motion

In the previous chapter we learned how to tread the rotation of a rigid body about a fixed axis. Now we shall take an interest in more general case, where **the axis of rotation is not fixed in space**. The general motion of a rigid body in space is very complex. However, we can simplify matter by the motion of a rigid body having a high degree of symmetry, such as cylinder or sphere. We shall assume that the body undergoes rolling motion in a plane.

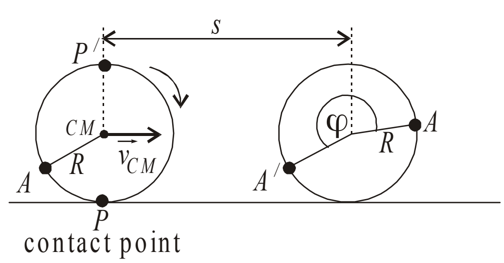

Fig.4. 21

Consider uniform cylinder of radius R rolling on rough, horizontal surface as is shown in Fig.4.21. The center of mass moves in a straight line, while the point on the rim moves in a more complicated path, which corresponds to the path of a cycloid. We shall see this motion is a combination of rotation about the center of mass and the translation of the center of mass.

Consider a uniform cylinder of radius R rolling on the rough, horizontal surface. As the cylinder rotates through an angle φ, its center of mass moves a distance $s = R\varphi$. Therefore, the velocity of the center of mass for pure rolling motion is given by the definition as

$$v_{CM} = \frac{\mathrm{d}s}{\mathrm{d}t} = R\frac{\mathrm{d}\varphi}{\mathrm{d}t} = R\omega \tag{4.67}$$

and the acceleration of the center of mass

$$a_{CM} = \frac{\mathrm{d}v_{CM}}{\mathrm{d}t} = R\frac{\mathrm{d}\omega}{\mathrm{d}t} = R\varepsilon\,. \tag{4.68}$$

Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point. A general point on the cylinder, such as *A*, has both horizontal and vertical components of velocity. However, contact point P and the point P' and the point at the center of mass are unique and of special interest. Relative to surface on which cylinder is moving, the center of mass moves with the velocity

$$v_{CM} = R\omega \tag{4.69}$$

whereas the contact point has zero velocity. The upper point P' has a velocity equal

$$2v_{CM} = 2R\omega \tag{4.70}$$

Since all points on the cylinder have the same angular velocity.

We can express the total kinetic energy of the rolling cylinder as

$$K = \frac{1}{2}I_P\omega^2 \tag{4.71}$$

Where I_P is the moment of inertia about the axis through *P*. Applying parallel-axis theorem, we can substitute

$$I_P = I_{CM} + MR^2$$

into eq.(4.71) to get

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}MR^2\omega^2$$

or

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2\,, \tag{4.72}$$

Where is used eq.(4.69). From this equation follows that **the total kinetic energy of an object undergoing rolling motion is the sum of rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**