5. MEASUREMENT OF THE MOMENT OF INERTIA BY THE PHYSICAL PENDULUM

ASSIGNMENT

1. Measure the period of the physical pendulum
2. Calculate the moment of inertia
3. Verify parallel-axis theorem
4. Find the reduced length of the pendulum
5. Analyse the errors in experiments

THEORETICAL PART

The motions can be broadly categorized into two classes, according to whether the thing that is moving stays near one place or travels from one place to another. Examples of the first class are an oscillating pendulum, a vibrating violin string, electrons vibrating in atom, etc. Parallel examples of travelling motions are the electron beam in TV tube, a ray of light emitted at stars and detected at our eye etc.

We will busy about studying things that stay in on vicinity and oscillate or vibrate an average position. This motion is called periodic motion. The special case of the periodic motion is the motion of the physical pendulum.

The physical pendulum consists of any rigid body suspended from a fixed axis that does not pass through its centre of mass. Consider a rigid body pivoted at a point O (see Fig.5.1) that is the distance a from the centre of mass

Motion such as body gives by the formula

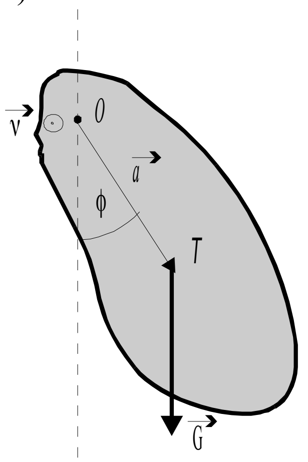

Fig.5.1

$$\tau = I\varepsilon \qquad (5.1)$$

where: τ is the torque of the force to the axis of rotation defined as $\tau = axF$

I is the moment of inertia about an axis through the pivot, defined by $I = \int a^2 dm$

a is the distance from the mass element dm to the axis of rotation

ε Is the angular acceleration given by $\vec{\varepsilon} = \dfrac{d^2\phi}{dt^2}$

Note eq. (5.1) is the rotational analogue of Newton's second law of motion $F = ma$. If we neglect friction force there are only one external force acting on the body. This force is the weight of body acting at the centre of mass.

At equilibrium the weigh vector passes through O, corresponding to $\phi=0$. The value of restoring torque τ about O when the system is displaced through an angle ϕ is by the definition $mga \sin \phi$. Recall that the torque is a vector quantity. This vector is perpendicular to the plane formed by vectors a and W and its direction is given by right-hand rule. The direction of ε is determined from its definition. From the definitions of torque and acceleration follows, that at any point during the motion of the pendulum the torque is in opposite direction with respect to the direction of the acceleration. Inserting the values of torque and acceleration into eq.5.1 valid for a body rotating about a fixed axis gives

$$-mga\sin\phi = J\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \qquad (5.2)$$

The mines sign indicates that torque and acceleration are in opposite directions to each other. Rewriting eq. 5.2 gives

$$\frac{mga}{I}\sin\phi + \frac{d^2\phi}{dt^2} = 0 \qquad (5.3)$$

We denote

$$\frac{mga}{J} = \omega^2 \qquad (5.4)$$

and we obtain the following relationship

$$\omega^2 \sin\phi + \frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = 0 \qquad (5.5)$$

If we assume that the angle ϕ is small (up to 5^0) we can take only first term in Taylor's expansion of $\sin\phi$ i.e. $\sin\phi \approx \phi$. Inserting this into eq.5.5 gives

$$\frac{d^2\phi}{dt^2} + \varpi^2\phi = 0 \qquad (5.6)$$

For example, if $\phi=5^0$ the value of $\sin\phi$ equals 0.0999. In some problems „0.0999=0.1“ is a poor approximation. eq.5.6 is the second order differential equation of a special kind of motion that is called the equation of the linear harmonic motion. The general solution of this equation is

$$\phi(t)=A\cos(\omega t+\alpha) \quad (5.7)$$

where: A is the amplitude of the motion (the maximum angular displacement)

α is the phase constant

A and α both must be chosen to meet the initial conditions of the motion, i.e. at the time t=0.

The period T of the harmonic motion is the time to go the oscillating body or particle through one full cycle of its motion, i.e.

$$T = \frac{2\pi}{\omega} \qquad (5.8)$$

Therefore, inserting the eq.5.4 the period of the physical pendulum is

$$T=2\pi\sqrt{\frac{I}{mga}} \qquad (5.9)$$

If you can see eq.5.9 gives the method of the measurement of moment of inertia of the rotating body. . We note that I is the moment of inertia of the rotating body about the fixed axis that not pass through the centre of mass. When we use the parallel-axis theorem

$$I = I_{c.m.} + ma^2 \qquad (5.11)$$

where $I_{c.m.}$ is the moment of inertia of the rotating body to the axis of rotation passing through the centre of mass, m is the mass of the body and a is the perpendicular distance between both axis then eq.5.11 is in form

$$T = 2\pi\sqrt{\frac{I_{c.m.} + ma^2}{mga}} \qquad (5.12)$$

or

$$I_{c.m.} = \frac{mgaT^2}{4\pi^2} - ma^2 \qquad (5.13)$$

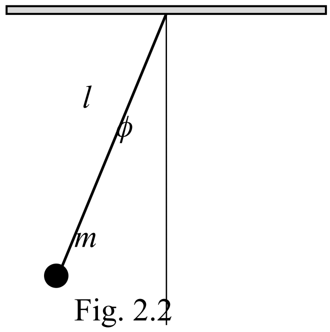

Fig. 2.2

Another mechanical system that exhibits periodic motion is **the simple pendulum**. It consists of a point of mass *m* suspended by a light string of length *l*, where the upper end of the string is fixed as in Fig. 5.2.

Moment of inertia that point of mass about the axis through *O* is

$J = ml^2$ (5.14)

The period of simple pendulum according to eq. (5.8) is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad (5.15)$$

As you can see from eq. (5.15) the period depends only on the length of the string and on the acceleration of gravity. We conclude that all simple pendulums of equal length at the same location oscillate with equal period. Eqs. (5.15) , (5.9) indicate that if the period of simple pendulum is equal to the period of physical pendulum we have

$$l_r = \frac{J}{ma} \qquad (5.16)$$

The l_r is called **the reduced length of the physical pendulum**. Therefore, applying this relation we obtain

$$T = 2\pi\sqrt{\frac{l_r}{g}} \qquad (5.17)$$

From the eq. (5.12) we see that the period of physical pendulum varies with the distance *a*.

Graph of T versus a for the physical pendulum of radius 0.1 m is shown in Fig.5.3

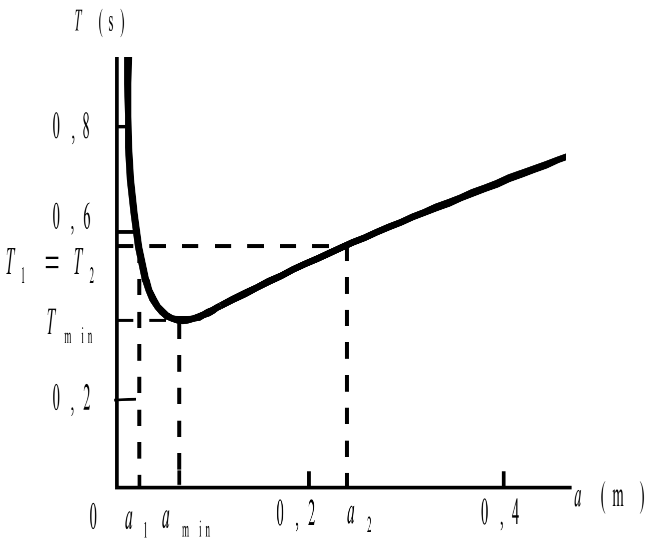

Fig.5.3

We can observe that there is the distance $a_{\min}$ of which is the period $T_{\min}$.

This minimum may be determine mathematically taking the derivative of the period given by eq. (5.12) with respect to distance a.

Then we have

$$a_{\min} = \sqrt{\frac{I_{c.m.}}{m}} \qquad (5.18)$$

Inserting eq. (5.18) into eq. (5.12) we have

$$T_{\min} = 8\pi\sqrt{\frac{a_{\min}}{g}} \qquad (5.19)$$

From graph in Fig 5.3 we see that for the period $T>T_{\min}$ we have two values of the distance a: a_1 and a_2.

Corresponding values a_1, a_2 we can obtain from eq. (5.12). Rewriting this equation gives

$$a^2 - g\frac{T^2}{4\pi}a + \frac{J^*}{m} = 0 \qquad (5.20)$$

Inserting eq. 5.16 into eq.5.20 gives

$$a^2 - l_r a + \frac{J^*}{m} = 0 \qquad (5.21)$$

This is a quadratic equation for the distance between the centre of mass and the location of the axis of rotation. The solution of this equation gives two values of a: a_1 and a_2. Therefore, by definition, we have

$$a_1 + a_2 = l_r \qquad (5.22)$$

$$a_1 . a_2 = \frac{J^*}{m} \qquad (5.23)$$

Next we proof the first formula. We shall make use of this important result in calculating the reduced length of the pendulum. If we find two axes corresponding to the same period T then the distance from each other equals to the reduced length of the physical pendulum.

THE METHOD - PRACTICAL PART I.

Eq. (5.13) gives the method of measurement of the moment of inertia to the axis of rotation passing through the centre of mass. I f we can see from this equation the moment of inertia can be obtained through a measurement of the period *T*, mass of the pendulum *m* and the distance *a* between the centre of mass of the pendulum and the axis of rotation.

1. MEASUREMENT

APPARATUS: measured body (disc), hinge for the body, stopwatch, balance, graph paper, meter- stick.

Measure the mass of physical pendulum (disc) *m* and the distance *a* between the pivot and the centre of mass. Fix the disc to the swinging. Displace the disk to one side through the angle of not more than 5^0. Measure the period *T* of the harmonic motion of the physical pendulum by the stopwatch. Record the measured values into Tab.5.1

CALCULATION THE MOMENT OF INERTIA

Calculate the moment of inertia $I_{c.m.}$ using eq. (5.13). The percentage error of the moment of inertia may be calculated from the expression

$$\frac{u_{J^*}}{J^*}100\% = \sqrt{\left(\frac{u_m}{m}\right)^2 + 4\left(\frac{gT^2 - 8a\pi^2}{gT^2 - 4a\pi^2}\right)^2\left(\frac{u_a}{a}\right)^2 + \left(\frac{2gT^2}{gT^2 - 4a\pi^2}\right)^2\left(\frac{u_T}{T}\right)}100\%$$

(5.24)

where u_m is accuracy of the balance

u_a is accuracy of the meter- stick

u_T is accuracy of the stopwatch

THE METHOD - PRACTICAL PART II.

Using eq. (5.21) we have

$$aT^2=q+ka^2 \tag{5.25}$$

where constants q and k are given by

$$q=\frac{4\pi^2 I_{c.m.}}{mg}, \qquad k=\frac{4\pi^2}{g} \tag{5.26}$$

The verification the parallel - axis theorem means the verification linearity of eq. (5.25)

2. MEASUREMENT

APPARATUS: measured body, stopwatch, meter-stick, balance

Measure the period of harmonic motion for small angular displacements using a few different distances a of the measured body and record these values into Tab.5.2.

CALCULATION

Make a plot of aT^2 versus a^2 and obtain a value for q and k from the slope of your best-fit straight-line graph by linear regression. Verify linearity of eq. (5.25). Using eqs.5.25, 5.18 and 5.19 calculate $I_{c.m.}$, g, $a_{\min}$ and $T_{\min}$ as

$$I_{c.m.} = \frac{q}{k}m \quad , \quad g = \frac{4\pi^2}{k}$$

$$a_{\min} = \sqrt{\frac{q}{k}} \quad , \quad T_{\min} = \sqrt[4]{4qk} \tag{5.27}$$

Determine the slope of the T versus a. The slope finds out $a_{\min}$ and $T_{\min}$. Compare these values with the values obtaining from eqs. 5.27. Value of acceleration of gravity g compare with accepted table's value. From the graph determine two distances a_1 and a_2 for which the period of pendulum is the same. Using eq.5.22 calculate the reduced length of the pendulum. Analyse the source of errors in your measurement.