PARTICLE DYNAMICS.

The laws of motion

Dynamics is the science, which describes the change in motion of particle using the concepts of force and mass. We shall discuss the three basic laws of motion, which are based on experimental observations and were formulated by Sir Isaac Newton (1642-1727) three centuries ago.

The first law of motion states that a body remains in its state of rest or uniform motion unless acted on by external force. The tendency of a body to maintain its state of rest or uniform motion in a straight line is called inertia. As a result, this law is often called **the law of inertia**. That is, when the sum of all external forces equals zero. It means

$$\sum_{i=1}^{n} \vec{F_i} = 0 \tag{3.1}$$

Newton's second law defines a special set of reference frames called **inertial frames**. A reference frame that moves with constant velocity relative to the distant stars is the best approximation of inertial frame. The Earth is not inertial frame, because of its orbital motion about the Sun and rotational motion about its own axis. As the Earth travels in its nearly circular motion around the Sun, it experiences the centripetal acceleration of about 4.4×10^{-3} m/s^2 toward the Sun. In addition, since the earth rotates about its own axis once every 24 *h*, a point on the equator experiences an additional centripetal acceleration of 3.37×10^{-2} *m/s*$^{-2}$ toward the center of the earth. These quantities are small compared to $g = 9.81$ m/s^{-2} and can be neglect. It means in most situations we can use the earth as an inertial frame.

The second law of motion. Newton`s second law makes use of the concept of mass. This law answers the question of what happens to an object that has nonzero resultant force acting on it. The observations show that the object moves with some acceleration a . **The Newton's second law states that the acceleration of an object is directly proportional to the resultant force acting on it and inversely proportional to its mass.**

$$\vec{a} \approx \frac{F}{m}.$$

The choice of a constant of proportionality can be arbitrary since us relating quantities with different unit. If this constant equals one then the second law has the form

$$\sum F = ma \tag{3.2}$$

We should note that equation (3.2) is a vector expression and hence is equivalent to the following three component equations:

$$\begin{aligned} F_x &= ma_x \\ F_y &= ma_y \end{aligned} \tag{3.3}$$

$F_z = ma_z$

Where F_x, F_y, F_z are components of force *F* along the *x*, *y*, *z* axes, respectively and a_x, a_y, a_z are components of acceleration along the *x*, *y* and *z* axes, respectively. The unit of force in SI units is 1N=1 kg.m/s^2.

The third law of motion. Newton`s third law states that if two bodies **interact, the force exerted on body 1 by body 2 is equal to an opposite the force exerted on body 2 by body 1**. That is

$$F_{12} = - F_{21} \qquad (3.4)$$

This law is equivalent to stating that **the forces occur in pairs**, or that a single isolated force cannot exist.

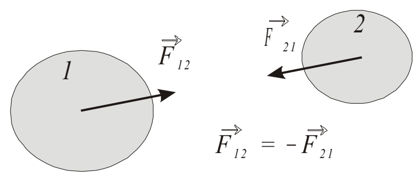

Fig.3.1

The force of gravity-weight.

We know that objects dropped near the surface of the earth will fall with the same acceleration g, if air resistance is neglected. If we applied the second Newton`s law to the force of gravity we give

$$F_g = W = mg$$

The direction of this force is down toward the center of the earth. This force is called **weight of the body.** Since the weight depends on the acceleration of gravity g, it varies with geographic location. Bodies weight less at higher altitudes that at sea level. This is because *g* decreases with increasing distance from the center of the earth. Hence, weight, unlike mass, is not inherent property of a body.

The normal force.

The force of gravity acts on an object when it is falling. When the object is at rest on the earth this force does not disappear. From the second law of motion the resultant force on object at rest is zero. There must be another force acting on this object to balance the force of gravity. Situation is shown in figure 3.2. For an object resting on the table, the table exerts this upward force. The table is compressed slightly by the object and due to its elasticity it pushes up on the object. The force exerted by the table is often called **a contact**t **force**, since it occurs when two objects are in contact. When the contact force acts perpendicular (normal) to the surface of contact, it is called as **normal force** F_N. Now we have two forces acting on the object and the object remains at rest. It means

$$F_N + W = 0$$

We must remember that upward force F_N on the object is exerted by the table.

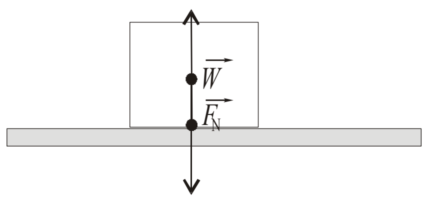

Fig.3.3

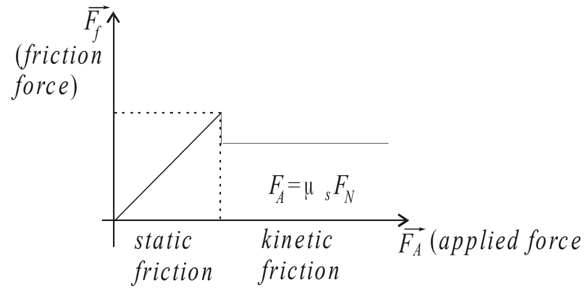

Fig.3.2

The friction force.

Until now we have ignored friction, which must be, however, taken into account in most practical problems.

When the body is in the motion along a rough surface, the force acting on the body is called **kinetic friction**. This friction acts opposite to the direction of the body`s motion. The magnitude of this force depends on the nature of the two sliding surfaces. Its value is expressed by the equation

$$F_f = \mu_K F_N , \tag{3.7}$$

Where F_N is the normal force between the two surfaces, μ_K is coefficient of proportionality between friction force F_f and normal force F_N . This coefficient called the **coefficient of kinetic friction**. If you can see from this expression the coefficient of kinetic friction is dimensionless quantity and its value is different for various contact surfaces.

Now, suppose an object such as a desk is resting on a horizontal floor. If no horizontal force is exerted on a desk, there is no friction force. Now suppose we try to push the desk, but it does not move. We are exerting a horizontal force, but the desk is not moving. There must be another force acting on the desk keeping it in rest. This force is called **the static friction force**. If we push with a greater force, the desk will finally start to move. At this moment we have exceed the maximum force of static friction, which is given by

$$F_f = \mu_s F_N , \tag{3.8}$$

Where μ_s is the dimensionless quantity that is called **the coefficient of the static friction**. Since the static friction varies from zero to its maximum value, we can write

$$F_f \le \mu_s F_N \tag{3.9}$$

Graph of friction force F_f , versus applied force F_A is shown in Fig. 3.3.

Work and Energy

Work done by a constant force.

The work done by the constant force is defined as a product of the component of the force in the direction of the displacement and magnitude of the displacement.

Since the component of F in the direction x is $F\cos\alpha$ (see Fig. 3.4), the work done by F is given by

$$W = (F\cos\alpha)x = F\cdot x. \tag{3.10}$$

where is used the dot product between two vectors.

According to this definition, work is done by F on an object under following conditions:

1. The object must undergo a displacement
2. The force F must have a nonzero component in the direction x
3. If an applied force F acts along the direction of the displacement, the $\alpha = 0$ and $\cos 0^\circ = 1$.

   In this case the definition of work gives $W = Fx$.
4. Work is scalar quantity and its unit is 1N.m=1J (joule).

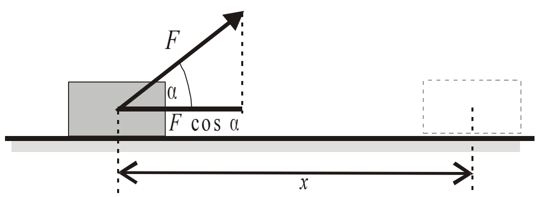

Fig.3. 4

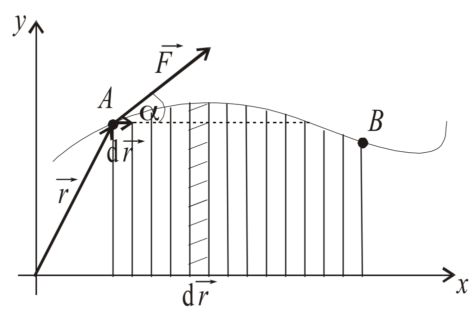

Fig.3. 5

Work done by a nonconstant force.

In many cases a force may vary both the magnitude and the direction. Let us assume a particle moving from point *A* to point *B* along the path under the varying force as is shown in Fig.3.5. The path can be divided into infinitesimal intervals. During each interval the force is assumed to be approximately constant.

Assume that r is position vector of any infinitesimal interval. Therefore, this interval can be described by the infinitesimal vector displacement $\mathrm{d}r$. The direction $\mathrm{d}r$ is along the tangent to the curve at that point, which has the position vector r . So, α is the angle between F and dr . If we denote $dl = |dr|$ as the magnitude of an infinitesimal displacement vector dr , the work done by the force F along the infinitesimal displacement of the path is with respect to eq. 3.10 equal

$$\mathrm{d}W = F\mathrm{d}l\cos\alpha = \vec{F}\cdot\mathrm{d}\vec{r} \tag{3.11}$$

Where is used the scalar product of two vectors. Then the net work done along the path from point *A* to *B* equals

$$W = \int_A^B \vec{F}\cdot\mathrm{d}\vec{r} \tag{3.12}$$

Note that the factor $F\cos\alpha$ represents the component of the force parallel to the curve tangent at any point. Only the component of F parallel to the velocity vector, $F\cos\alpha$, contributes the work. A force acting perpendicular to the velocity vector does no work. Such a force changes only the direction of velocity, it does not affect the magnitude of the velocity. Example of this is a uniform circular motion with constant speed. Centripetal force does no work on the object.

The important notes:

1. The work is a scalar quantity.
2. The expression (3.12) is the most general definition of the work
3. To calculate the integral in eq.3.12 we must be able to express a nonconstant force F as a function of position.

Power

Power is defined as the rate at which work is done. **The average power** P_{av} when amount of work ΔW is done in time Δt is

$$P_{av} = \frac{\Delta W}{\Delta t} \tag{3.13}$$

The instantaneous power is then

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{3.14}$$

The unit of power in the SI systems is J/s=W (watt).

Now we derive the relation between power and velocity of the moving body. We know that the elementary work is defined by the equation (3.11). Inserting this definition into definition of the power gives

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{F\cdot\mathrm{d}r}{\mathrm{d}t} = \vec{F}\cdot\vec{v}. \tag{3.15}$$

Kinetic energy.

The energy is one of most important concepts in science. There are various types of energy. At first we shall take an interest in kinetic energy of the translation motion.

Suppose the net force F on the object varies in both magnitude and direction and the path is the curve. By the definition the work done by this force is

$$W = \int F\cos\alpha\mathrm{d}r \tag{3.16}$$

Because $F\cos\alpha$ represents the component of force F parallel to the curve tangent at any point. By the Newton's second law of motion we can write

$$F\cos\alpha = ma_t$$

Where a_t is the component of acceleration parallel to the curve tangent at any point, i.e. tangential acceleration. This acceleration equals the rate of change of speed given by eq. 2.19 . We can assume velocity as the function of position vector, and using **the chain rule for derivatives** we have

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}r} \qquad (3.17)$$

Since $\frac{\mathrm{d}r}{\mathrm{d}t} = v$ is the speed.

After substituting $F\cos\alpha = mv\frac{\mathrm{d}v}{\mathrm{d}r}$ into expression (3.16) we have

$$W = \int_{v_1}^{v_2} mv\frac{\mathrm{d}v}{dr}\mathrm{d}r = m\int_{v_1}^{v_2} v\mathrm{d}v = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

(3.18).

Where v_1 and v_2 are initial and final velocities of the particle, respectively.

The product of one half of the mass and the square of speed is defined as kinetic energy of the particle K

$$K = \frac{1}{2}mv^2$$

The kinetic energy is a scalar quantity and has the same unit as the work. It is convenient to write the equation (3.18) into form

$$W = K_2 - K_1 = \Delta K \qquad (3.19)$$

ΔK represents the change of kinetic energy. **The equation states that the net work done by the resultant force in displacing a particle equals the change in kinetic energy of the particle.** This equation is known as the **work-energy theorem**.

Examples

A block of mass m is pushed up a rough incline by constant force F acting parallel to the incline. The block is displaced a distance d up the incline (Fig.3.6).

a) Calculate the work done by the force of gravity for this displacement.

b) Calculate the work done by the applied force F .

c) Find the work done by the force of kinetic friction if the coefficient of friction is μ .

d) Find the net work done on the block for this displacement.

Solution:

a) The force of gravity is oriented downward but has a component down the plane. This component has the value (see Fig.3.6) given by

$$F_x = -mg\sin\alpha$$

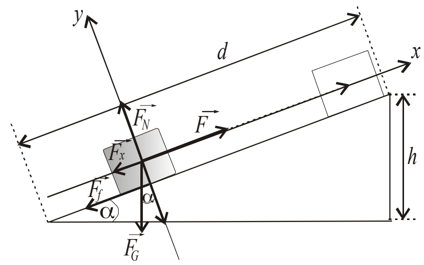

Fig.3. 6

since the positive x axis is chosen to be up the incline plane.

The work done by the gravity for this displacement d is

$$W_{F_G} = -mgd\sin\alpha = -mgh$$

Where $h = d\sin\alpha$ is the vertical displacement. From this equation we can see that the work done by the gravity has magnitude equal to the force of gravity multiplied by the upward vertical displacement. That is, the work done by the gravity depends only on the initial and final coordinates and it is independent of the path taken between the points.

b) Since the applied force, F , is in the same direction as the displacement, we get

$$W_F = Fs = Fd$$

c) The magnitude of the friction force equals

$$F_f = \eta F_N = \mu mg\cos\alpha$$

Since direction of this force is opposite the direction of the displacement, then the work of this force is given by

$$W_{F_f} = -F_f d = -\mu mgd\cos\alpha$$

d) Using those results we get we can calculate the total work on the block for the displacement d as

$$W_{net} = W_g + W_F + W_{F_f} = Fd - mgd(\sin\alpha + \mu\cos\alpha).$$

Conservative and nonconservative forces.

It is useful to divide forces into two kinds: conservative and nonconservative, one. The force is called a **conservative force** if the force depends only on position and if the work done by this force on a particle moving between any two points depends only on the initial and final position of these points. Such a conservative force is, for example, the force of gravity, electric force and so on. Now we show that the force of gravity is conservative force. Suppose an object moving along some arbitrary path in xy plane from point A to B.

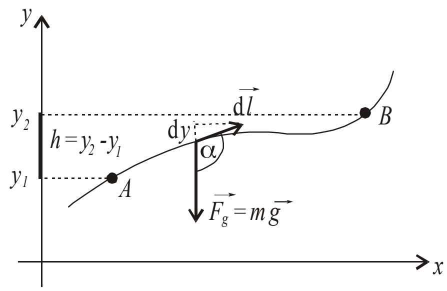

Fig.3 7

The work done by gravity is

$$W_g = \int_A^B \vec{F}_g \, d\vec{l} = \int_A^B mg\cos\alpha dl \tag{3.20}$$

Where dl is the infinitesimal length of the path and α is the angle between the force of gravity and

dl

From the Fig.3.7 we can see that $dy = dl\cos(\pi - \alpha) = -\cos\alpha dl$. Inserting this value into eq.(3.20) for W_g gives

$$W_g = -\int_{y_1}^{y_2} mg dy = -mg(y_2 - y_1) = -mgh \tag{3.21}$$

We see that the work depends only on the vertical height $h = y_2 - y_1$ and does not depend on the path gone. Note that in our case is $y_2 \rangle y_1$ and therefore the work done by gravity are negative. If $y_2 \langle y_1$ (object is falling), the work done by gravity would be positive. We can also say that the force is conservative if the work done by the force is zero if a particle moves along any closed path that returns it to the original position. Situation is shown in Fig.3.8.

$$W_{g(1)} = -\int_{A(y_1)}^{B(y_2)} mg dy = -mg(y_2 - y_1)$$

$$W_{g(2)} = -\int_{B(y_2)}^{A(y_1)} mg dy = -mg(y_1 - y_2)$$

Where $W_{g(1)}$ is the work done by the conservative force along the path (1), $W_{g(2)}$ is the work done by the conservative force along the path (2).

$$W_{g(1)} + W_{g(2)} = -mg(y_2 - y_1) - mg(y_1 - y_2) = -mg(y_2 - y_1 + y_1 - y_2) = 0$$

Nonconservative force is one if the work done by this force depends on the path length. For example, such a force is force of friction since the work done by the friction force is equal to the product

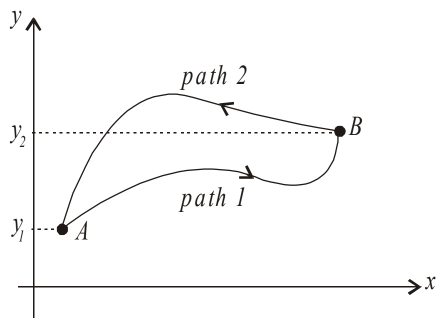

Fig.3. 8

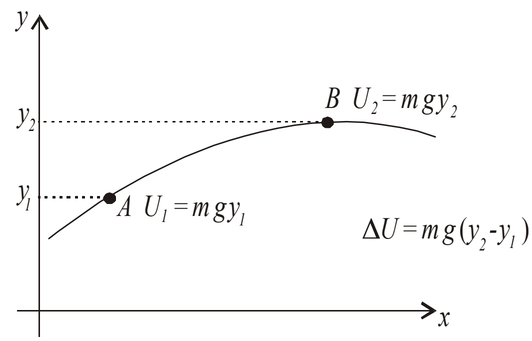

Fig.3. 9

of this force and distance travelled. (It is due to the fact that the force of friction depends on the nature of the two sliding surfaces).

Potential energy. Principle of conservation of mechanical energy.

The potential energy is connected with the position of a body and which can be defined only in relation to a conservative force and which is closely related to the concept of work. We define the **potential energy** as the change in gravitation potential energy *U* when the object moves from a height y_1 to height y_2 relative to any horizontal surface as

$$\Delta U = U_2 - U_1 = mg(y_2 - y_1) \tag{3.22}$$

This equation defines the change in potential energy between two points. We know if the object moves from point *A* to point *B* (see Fig.3.9); work done by force of gravity is given by (3.21). By comparison eqs. (3.20), (3.21), (3.22) we see that the change in gravitational potential energy equals to the negative of work done by the gravity

$$\Delta U = -W_g = -\int_A^B \vec{F}_g \cdot \mathrm{d}\vec{l} \ . \tag{3.23}$$

Generally, we define the change in potential energy associated with a conservative force, F , as the negative of the work done by that force.

Let us consider a conservative system in which energy is transformed from kinetic to potential or vice versa. According to work-energy theorem the work W done on the particle equals to the change in kinetic energy it means

$$W = \Delta K$$

Since we assume a conservative system, the net work done can be written in terms of the change in gravitational energy as

$$W = -\Delta U$$

From this equation follows

$$\Delta K + \Delta U = 0 \tag{3.24}$$

From eq.(3.24) we can see that if the kinetic energy of the system increases, the potential energy decreases and vice versa.

We now define a quantity E, called **total mechanical energy** of system as a sum of the kinetic and potential energy

$$E = K + U$$

Using eq.3.24 we give the important result valid only for the system of conservative forces, especially for the motion of the particles or bodies in gravitational field

$$E = K + U = \text{constant}. \tag{3.25}$$

This expression says that **the total mechanical energy of a conservative system remains constant**. This is called **the principle of conservation of mechanical energy for conservative forces**.

Example 1

A pendulum bob is pulled from the vertical through an angle φ and released (see Fig.3.10). Find the speed of the bob and tension of the string at the lowest point assuming $l = 0.3\, m$, $\varphi = 30^\circ$ and $m = 0.5\, kg$.

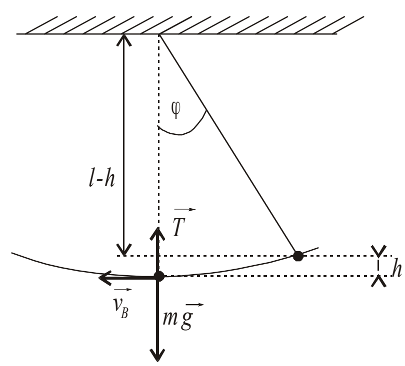

Fig.3. 10

<u>Solution:</u>

A total energy of the bob at point A, just before being released is

$$E_A = K + U = mgh$$

Where h is the vertical height above the point B. Using geometry we see that $h = l - l\cos\varphi$ and so the initial energy equals

$$E_A = mgl(1 - \cos\varphi)$$

The total energy at the point *B* is purely kinetic energy, because $h = 0$. Therefore

$$E_B = \frac{1}{2} m v_B^2 .$$

The law of conservation energy implies

$$mgl(1 - \cos\varphi) = \frac{1}{2} m v_B^2 .$$

Solving for the speed v_B we find

$$v_B = \sqrt{2gl(1 - \cos\varphi)} = 0.9\ m/s .$$

To find the magnitude of the tension, T, of the string we use Newton`s second law as

$$\sum F_{ext} = T - mg = ma_r$$

Where $a_r = \frac{v_B^2}{l}$ is the radial acceleration (centripetal) of the bob at point *B*.

Therefore the tension of the string equals

$$T = mg + m\frac{v_B^2}{l} = m\left(g + \frac{v_B^2}{l}\right)$$

Taking the values of v_B, *l*, *m* and *g* gives the value of the tension as $T = 6.2\ N$.

Example 2.

A block lying on a smooth, horizontal surface is connected to a spring with a force constant of $80\ N/m$. The spring is compressed a distant of $3.0\ cm$ from equilibrium. Calculate the work done by the spring force as the block moves from $x_i = -3.0\ cm$ to its unstretched position $x_f = 0$ (see Fig.3.11)

Solution:

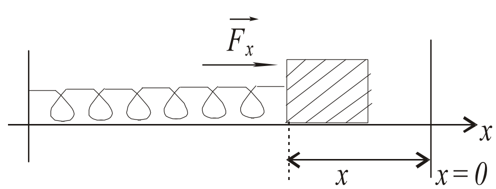

Fig.3. 11

A common physical system for which the force varies with position is shown in Fig.3.11. A body is a horizontal, smooth surface is connected to a helical spring. If the spring is attached or compressed a small distance from its equilibrium, the spring will exert a force on the block given by

$$F_s = -kx$$

Where *x* is the displacement of the block from its unstreched position, ***k* constant is called the elastic constant of the spring or force constant of the spring.** This force law for spring is known as Hooke`s law. The negative sign signifies that the force exerted by the spring is always directed opposite the displacement. Since the spring force always acts toward the equilibrium position, it is sometimes called a **restoring force**. The work done by the spring force as the body moves from $x_i = -x_m$ to $x_f = 0$ is given by

$$W_s = \int_{x_i}^{x_f} F_s \mathrm{d}x = \int_{x_i}^{x_f} (-kx)\mathrm{d}x = -k\int_{x_m}^{0} x\mathrm{d}x = \frac{1}{2}kx_m^2 .$$

This work is always positive since the spring force is in the same direction as the displacement.

Then

$$W_s = \frac{1}{2}kx_m^2 = \frac{1}{2}80\ N/m\left(-3\times10^{-2}\right)^2 m^2 = 3.6\times10^{-2}\ J .$$

<u>Remarks:</u>

1. To describe the potential energy stored in the spring we have to calculate the work done by the spring on the block as the block moves from $x = x_i$ to $x = x_f$:

$$W_s = \int_{x_i}^{x_f} (-kx)\mathrm{d}x = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

2. From this equation we can see that this force is another example of conservative force because W_s depends only on the initial and final *x* coordinates, respectively.

3. The quantity $\frac{1}{2}kx^2$ is defined as **the elastic potential energy stored in the spring.** It is denoted

by the symbol U_s

$$U_s = \frac{1}{2}kx^2 .$$

Where U_s is the maximum of potential energy (for the spring in the maximum compression)?

4. The elastic potential energy stored in the spring is zero when the spring is not unstretched (*x*=0). Furthermore, U_s, is a maximum when the spring has reached its maximum compression. Finally, U_s, is always positive since it is proportional to x^2.