PARTICLE KINEMATICS

Kinematics describes the motion using the concepts of space and time, without regard to the causes of the motion. **Any motion is characterized by the displacement, velocity and acceleration**.

At first we shall consider motion along a straight line that is one-dimensional motion.

One-dimensional motion

It is define as motion of the particle along a straight line, for example, along the x axis. If the particle moves along the x-axis from position x_i to x_f, its **displacement** is defined as $\Delta x = x_f - x_i$. From this equation we can see that Δx is positive if x_f is greater then x_i and negative if x_f is less then x_i. Starting with the concept of displacement we shall define the velocity and acceleration. The motion of a particle is completely known if its position in space is known at all time (Fig.2.1).

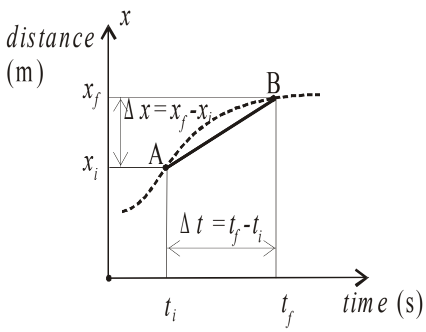

Fig. 2. 2

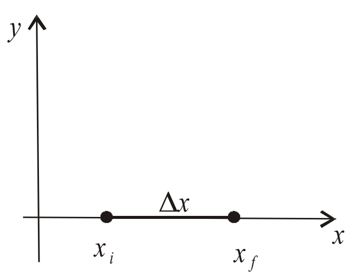

Fig. 2. 1

Consider the particle moving along the x-axis from point A to B. Let its position at point A is x_i at some time t_i, and let its position at point B be x_f at time t_f. This plot is called **position-time graph**.

Average velocity of a particle is defined as the ratio of its displacement, Δx, and time interval Δt

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} \qquad \text{(m/s).} \qquad (2.1)$$

From this definition we can see that the average velocity has dimension of length divided by time. The average velocity is proportional to the displacement, which in turn depends only on the initial and final coordinates of the particle. It can be interpreted geometrically by drawing a straight line between points *A* and *B*. This line forms hypotenuse of a triangle of height Δx and base Δt . The slope of this line is the ratio $\frac{\Delta x}{\Delta t}$ (see Fig.2.2).

Instantaneous velocity is defined as the limiting value of the average velocity as Δt approaches zero

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad (2.2)$$

This limit is called the derivative of *x* with respect to *t*. We see that it equals the slope of the tangent of the *x* versus *t*.

The particle is said to be accelerating when the velocity of the particle changes with time. Suppose a particle moving along *x*-axis has a velocity v_i and a velocity v_f at time t_f.

The average acceleration of the particle in the time interval $\Delta t = t_f - t_i$ is defined as a ratio $\frac{\Delta v}{\Delta t}$, where $\Delta v = v_f - v_i$ is the change in velocity in the time interval Δt (see Fig.2.3).

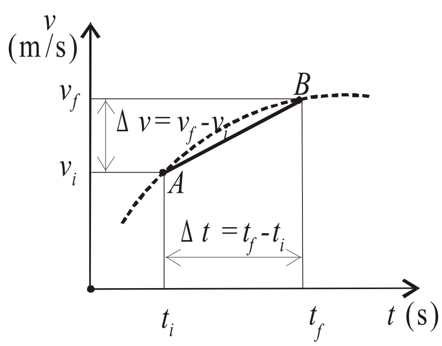

Fig. 2. 3

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \qquad (m/s^2).$$

(2.3)

In some situations, the value of the average acceleration may be different over different time intervals. Therefore, it is useful to define the instantaneous acceleration as the limit of the average acceleration as Δt approaches zero.

The instantaneous acceleration

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2}$$

(2.4)

Here $\frac{dv}{dt}$ is the derivative of *v* with respect to *t* and $\frac{d^2 x}{dt^2}$ is called the second derivative of *x* with respect to *t*.

Uniformly accelerate motion

It is one–dimensional motion with constant acceleration. By the definition of the average acceleration is given by

$$a_{av} = a = \frac{v_f - v_i}{t_f - t_i} \tag{2.5}$$

To simply notation we assume the initial time equals $t_i = 0$ and initial velocity $v_i = v_0$ at time $t_i = 0$. If we designate $t_i = t_0 = 0$, $v_i = v_0$, $t_f = t$ and $x_i = x_0$, $x_f = x$, we can express the acceleration as

$$a = \frac{v - v_0}{t} \text{ or } v = v_0 + at \tag{2.6}$$

for $a = \text{const}$.

According to the definition of a velocity ($v = \frac{dx}{dt}$) we have

$$v dt = dx$$

and integration of this equation gives

$$\int_{x_0}^{x} dx = \int_0^t v dt = \int_0^t (v_0 + at) dt$$

$$x = x_0 + v_0 t + \frac{1}{2} a t^2, \tag{2.7}$$

Where x_0 and v_0 are initial position and initial velocity of an object at time t_0, respectively. **Note this formula is valid only for the uniformly accelerate motion!**

Examples

The velocity of the particle moving along the *x*-axis varies in time according to the expression $v = (40 - 5t^2)$ *m/s* (see Fig2.4). Find

The average acceleration in time interval $t = 0s$ to $t = 2s$

Determine the acceleration at $t = 2s$

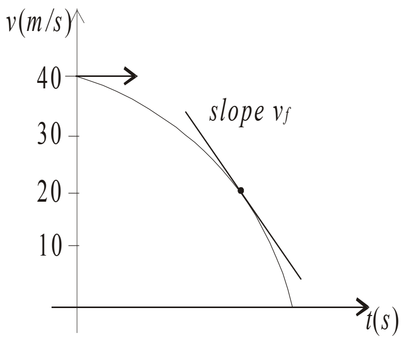

Fig. 2. 4

<u>Solution:</u>

$$v_i = (40 - 5t_i^2)\,m/s = (40 - 5.0^2)\,m/s = 40\,m/s$$

$$v_f = (40 - 5t_f^2)\,m/s = (40 - 5.2^2)\,m/s = 20\,m/s$$

Therefore, the average acceleration in $\Delta t = t_f - t_i$ is given

$$a_{av} = \frac{(20m/s - 40m/s)}{2s} = -10m/s^2$$

The velocity at time *t* is given by $v_i = (40 - 5t^2)\,m/s$ and the velocity at time $t + \Delta t$ is given by

$$v_f = (40 - 5(t + \Delta t)^2)\,m/s = 40 - 5t^2 - 10t\Delta t + 5(\Delta t)^2$$

Therefore, the change in the velocity equals

$$\Delta v = v_f - v_i = -10t\Delta t + 5(\Delta t)^2\; m/s$$

and the instantaneous acceleration at time *t*=2s has the value

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} (-10t + 5\Delta t) = -20m/s^2 .$$

This acceleration is not constant!

2.1 Motion in two dimensions

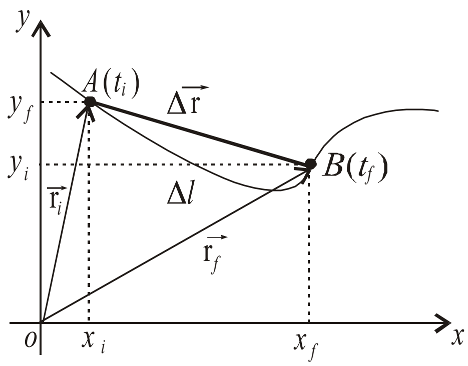

Fig. 2. 5

Let us extend this idea to the motion of particle in the (x, y) plane. Assume the particle is moving along the path from point A to point B (see Fig.2.5). Position vector at point A is $r_i = x_i i + y_i j$ and at point B is $r_f = x_f i + y_f j$.

Where i and j are the unit vectors along the x and y axes, respectively. During the motion from point A to point B the position vector changes from r_i to r_f. **The displacement vector** of the moving particle is defined as

$$\Delta r = r_f - r_i = (x_f i + y_f j) - (x_i i + y_i j) = (x_f - x_i)i + (y_f - y_i)j = xi + yj$$

Where $x = x_f - x_i$ and $y = y_f - y_i$.

The Δr vector represents the displacement during the time interval $\Delta t = t_f - t_i$.

The average velocity vector

$$\vec{v}_{av} = \frac{\Delta r}{\Delta t} = \frac{r_f - r_i}{t_f - t_i} \neq \frac{\Delta l}{\Delta t}$$

(2.8)

Note that the magnitude of average velocity vector is not equal to the average speed which is the actual distance travelled Δl divided by Δt.

Instantaneous vector or velocity vector is defined as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(x\vec{i} + y\vec{j}) = v_x \vec{i} + v_y \vec{j} \qquad (2.9)$$

Where $v_x = \frac{\mathrm{d}x}{\mathrm{d}t}$, $v_y = \frac{\mathrm{d}y}{\mathrm{d}t}$, are the components of the velocity vector and $r = xi + yj$ is position vector of the moving particle at any moment. The direction of v is along the tangent to the path at that moment (see Fig.2.6). In similar way the **average acceleration vector** over time interval is defined as

$$\vec{a}_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i} \tag{2.10}$$

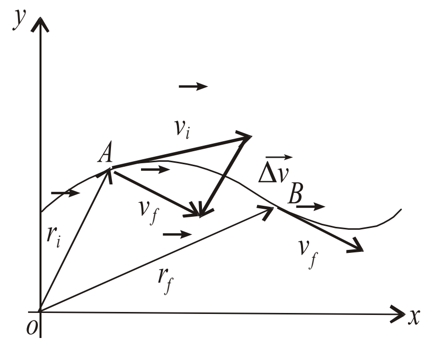

Fig. 2. 6

v_i and v_f may also have the same magnitudes but the directions are different. The difference of two such vectors will not be zero. Hence, **the acceleration can result from either a change in magnitude or direction of the velocity**, **or from a change in both**, the magnitude and the direction of the velocity vector.

The instantaneous acceleration vector is defined as

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_x \vec{i} + v_y \vec{j}\right) = a_x \vec{i} + a_y \vec{j} \tag{2.11}$$

Where

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t},\ a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t}$$

are components of the acceleration vector in plane (*x*, *y*) and i , j are units vectors along the *x* and *y* axes respectively. The definitions of a and v can be extended to the motion in (*x*, *y*, *z*). In general, position vectors can be expressed

$$r_i = x_i i + y_i j + z_i k$$

$$r_f = x_f i + y_f j + z_f k\ ,$$

where x_i , y_i , z_i are coordinates of the initial point, x_f , y_f , z_f are coordinates of the final point and i , j , k are units vectors dimensionless along *x*-, *y*-, *z*- axes, respectively. Similarly to two-dimensional motion is the change of position vector defined as $\Delta r = r_f - r_i = xi + yj + zk$. Therefore, the instantaneous velocity is given by

$$\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{\mathrm{d}x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \tag{2.12}$$

Where $v_x = \frac{\mathrm{d}x}{\mathrm{d}t}$, $v_y = \frac{\mathrm{d}y}{\mathrm{d}t}$ and $v_z = \frac{\mathrm{d}z}{\mathrm{d}t}$ are components of velocity vector. The magnitude of velocity vector can be expressed as

$$v = \left|\vec{v}\right| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{2.13}$$

The acceleration vector is

$$\vec{a} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(v_x\vec{i} + v_y\vec{j} + v_z\vec{k}\right) = \frac{\mathrm{d}v_x}{\mathrm{d}t}\vec{i} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\vec{j} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \tag{2.14}$$

Where $a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t}$, $a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t}$ and $a_z = \frac{\mathrm{d}v_z}{\mathrm{d}t}$ are components of acceleration vector. The magnitude of acceleration is

$$a = \left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2 + a_z^2}\ . \tag{2.15}$$

Example

The acceleration of a motion increases a uniform rate. The motion starts from rest and at the time t_1. The magnitude of its acceleration equals a_1. Determine the dependence of the velocity *v* and the trajectory *s* of the motion on time *t*.

Solution:

The given motion is nonuniform one, but its acceleration varies uniformly with time as $a(t) = kt$, where constant *k* can be given from final conditions $k = \frac{a_1}{t_1}$. Using the definition of $a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t}$ we can calculate the velocity of the particle as

$$v(t) = \int_0^t a(t)\mathrm{d}t = \int_0^t kt\mathrm{d}t = \frac{1}{2}kt^2$$

To calculation of the acceleration we start from the definition of the velocity $v(t) = \frac{\mathrm{d}s(t)}{dt}$. Rearranging this expression gives

$$s(t) = \int_0^t v(t)\mathrm{d}t = \int_0^t \frac{1}{2}kt^2\mathrm{d}t = \frac{1}{6}kt^3$$

or

$$s(t) = \frac{1}{6}\frac{a_1}{t_1}t^3$$

Uniform circular motion

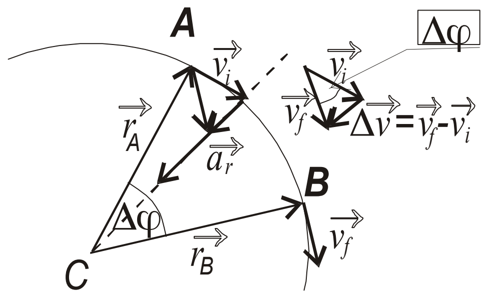

Fig. 2.7

An object that moves in a circle at constant speed *v* is said to undergo uniform circular motion (see Fig.2.7). Although the magnitude of the velocity vector remains constant, its direction is continually changing. **At each point the instantaneous velocity vector is in tangent direction to the circular path**. Consider the particle moving along the circle. We denote the velocity at point *A* as v_i and at point *B* as v_f. Then the change of velocity from point *A* to point *B* equals $\Delta v = v_f - v_i$.

The acceleration is defined by

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{\mathrm{d}v}{\mathrm{d}t} \tag{2.16}$$

Where Δv is a change in velocity vector during the time Δt. toward to the centre of the circle. Since the acceleration a is in the same direction as Δv, it must point **The change in velocity vector** Δv **is pointed toward the centre of the circle**, too. Therefore, this acceleration is called **centripetal** or **radial acceleration** a_r **.**

Now we determine the magnitude of the centripetal acceleration. The vector v_i, v_f and Δv form a triangle that is similar to the triangle *ABC* (two triangles are similar if the angle between any two sides is the same for both triangles and if the ratio of these sides is the same). From the Fig.2.7 follows:

$$\frac{\Delta v}{v} = \frac{\Delta l}{r} \qquad \text{or} \qquad \Delta v = \frac{\Delta l}{r}v$$

Inserting this value into definition of acceleration gives

$$a_r = \lim_{\Delta t \to 0} \frac{v}{r} \frac{\Delta l}{\Delta t} = \frac{v}{r} v = \frac{v^2}{r}$$

Since $\frac{\Delta l}{\Delta t} = v$ (2.17)

To summarise, a particle moving in a circle of the radius *r* with constant speed *v* has an acceleration directed toward the centre of the circle. For this, motion velocity and acceleration vectors are perpendicular to each other.

Nonuniform circular motion. Tangential and radial acceleration.

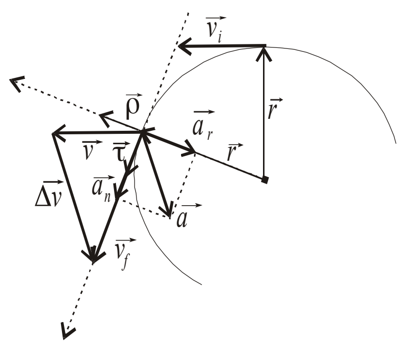

Fig. 2. 8

Let us consider the motion of a particle along a circle where the velocity changes both the direction and the magnitude, too. (Fig.2.8).

There will be a tangential acceleration a_t, as well as the radial (centripetal) acceleration a_r.

The radial acceleration arises from the change in direction of the velocity and has a magnitude

$$a_r = \frac{v^2}{r} \quad (2.18)$$

The tangential acceleration arises from the change in the magnitude of the velocity. It is defined as

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} \quad (2.19)$$

The tangential acceleration always points in a direction tangent to the circle, and it is in the direction of motion (parallel to v) when the speed is increasing. a_t is antiparallel to v when the speed of the particle decreases. The accelerations a_t and a_r are always perpendicular to each other, and their directions change continually as the particle moves along its circular path. **The total vector acceleration** a is the sum of these two components:

$$a = a_r + a_t \quad (2.20)$$

and its magnitude is

$$a = \sqrt{a_r^2 + a_t^2} \; . \qquad (2.21)$$

It is convenient to write the acceleration of a particle moving in circular path in term of unit vectors. We can do this by defining the unit vector ρ and τ . The unit vector ρ is directed radially outward along the radius vector from the centre of circle and τ is a unit vector tangent to the circular path pointed in the direction of the moving particle. Using this notation, we can expressed the total acceleration as

$$\vec{a} = \vec{a}_t + \vec{a}_r = \frac{dv}{dt}\vec{\tau} - \frac{v^2}{r}\vec{\rho} \qquad (2.22)$$

The negative sign for a_r indicates that it is always directed radially inward, opposite to vector ρ .

Example

A ball tied to the end of the string of length *l*=0.5m swings in a vertical circle under influence of gravity. When the string makes an angle $\alpha = 20^\circ$ with the vertical, the ball has speed of 1.5 m/s (see Fig. 2.9)

Find the radial component of acceleration at this distant

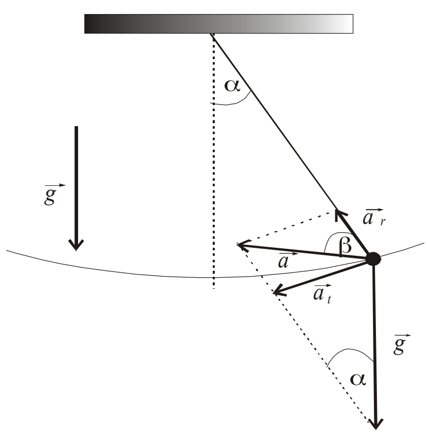

Fig 2.9

a. Find the radial component of acceleration at this distant

b. Find the total acceleration

c. Find the angle between the total acceleration and string

Solution:

a. $a_r = \frac{v^2}{r} = \frac{(1.5m/s)^2}{0.5m} = 4.5m/s^2$

b. $a_t = g \sin\alpha = 9.80m/s^2 \sin(20^\circ) = 3.36m/s^2$

$a = \sqrt{a_t^2 + a_r^2} = 5.62m/s^2$

c. $\mathrm{tg}\beta = \frac{a_t}{a_r} = \frac{3.36 m/s^2}{4.5 m/s^2} \Rightarrow \beta = 36.7°$

Angular variables of circular motion

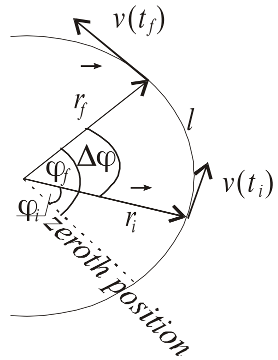

Fig. 2. 10

We consider a particle rotating in circle of radius *r*. The particle has moved along the circle a distance *l* and its angular position have changed by angle φ (see Fig. 2.10). Angle φ is in radians. One radian is defined as the angle subtended by arc whose length *l* is equal to the radius *r*. So, in general, any angle φ is related to radians by $\varphi = l/r$. **Arc** is a part of circumference of a circle or part of any other curve.

Let φ_i and φ_f represent the angular position at times t_i and t_f, respectively (Fig.2.10). **Average angular velocity** is defined as

$$\omega_{av} = \frac{\varphi_f - \varphi_i}{t_f - t_i} = \frac{\Delta\varphi}{\Delta t} \qquad (s^{-1}) \qquad (2.23)$$

Where $\Delta\varphi$ is the angular displacement and Δt is time interval.

The magnitude of instantaneous velocity is

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt} \qquad (2.24)$$

Average angular acceleration is defined as the change in angular velocity divided by the time required to make this change. Let ω_i and ω_f represent the instantaneous angular velocities at times t_i and t_f. Then

$$\varepsilon_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad (s^{-2}) \qquad (2.25)$$

the instantaneous angular acceleration is defined as the limit of this ratio as Δt approaches zero

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \quad (2.26)$$

Now we can relate the angular quantities ω and ε to the linear velocity and tangential acceleration. By the definition

$$v = \frac{dl}{dt} = \frac{d}{dt}(r\varphi) = r\frac{d\varphi}{dt} = r\omega \quad (2.27)$$

since the radius of circle is constant, i.e. it is independent of the time.

Thus the magnitude of the linear velocity of a particle moving in a circle is equal to the radius of the circle times the magnitude of the angular velocity ω.Using the definition of the tangential acceleration we can find the relation between the magnitude of the angular acceleration ε and tangential acceleration a_t :

$$a_t = \frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\varepsilon \quad (2.28)$$

For the radial acceleration we have

$$a_r = \frac{v^2}{r} = \frac{r^2\omega^2}{r} = r\omega^2 . \quad (2.29)$$

It is useful consider the frequency of rotation. **Frequency** is defined as number of revolution per second

$$f = \frac{N}{t} = \frac{\omega}{2\pi} \quad \text{or} \quad \omega = 2\pi f \quad (2.30)$$

Where N is number of revolution during the time t.

The time required for one revolution is called the **period *T***

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi r}{v} . \quad (2.31)$$

Example

A flywheel rotates with frequency $N = 1500$ revolutions per minute. Due to breaking its motion becomes a uniformly retarded one and it finishes during the time $t_1 = 30$ seconds after the breaking started. Determine

a. Angular acceleration

b. Number of revolutions performed from the beginning of breaking till stop of the motion.

<u>Solution:</u>

a. Instantaneous acceleration is defined as

$$\varepsilon = \frac{d\omega}{dt}$$

Because the value of the infinitesimal time interval dt is not equal zero we can rearrange this expression into form $d\omega = \varepsilon dt$. To finding the angular velocity as a function of the time we have to integrate this expression:

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \varepsilon dt$$

Since the angular acceleration ε is constant the calculation of these integrals gives the value of the angular velocity as

$$\omega - \omega_0 = \varepsilon t \text{ .}$$

or

$$\omega = \omega_0 + \varepsilon t$$

Where ω_0 is initial angular velocity of the uniformly accelerated motion. For our case

$$\omega_0 = 2\pi f = 2\pi \frac{N}{t} = 2\pi \frac{1500}{60} = 50\pi \text{ s}^{-1}.$$

At the time t_1 , must $\omega(t_1)$ be equal zero. So,

$$\omega(t_1) = \omega_0 + \varepsilon t_1 = 0 \text{ .}$$

Thus the angular acceleration

$$\varepsilon = -\frac{\omega_0}{t_1} = -\frac{50\pi}{30} = -\frac{5\pi}{3} s^{-1}$$

The negative sign means that this motion is deaccelerated..

b. The number of revolutions performed during time $t_1 = 30s$ equals

$$N = \frac{\varphi_1}{2\pi}$$

Where φ_1 is the angle subtended during time t_1 .By the definition of the angular velocity we have

$$\omega = \frac{d\varphi}{dt} \Rightarrow \int_0^{\varphi_1} d\varphi = \int_0^{t_1} \omega dt$$

Since $\omega = \omega_0 + \varepsilon t$ we give the value for the angle subtended during the time t_1 as

$$\varphi_1 = \int_0^{t_1} (\omega_0 + \varepsilon t) dt = \omega_0 t_1 + \frac{1}{2}\varepsilon t_1^2 = 50\pi 30 - \frac{1}{2}\frac{5\pi}{3}(30)^2 = 750\pi \text{ .}$$

Therefore, the number of revolutions performed during time $t_1 = 30s$ equals

$$N = \frac{\varphi_1}{2\pi} = 375 .$$

Projectile motion

It is two dimensional motions with constant acceleration: $a = g$**.** This very common form of the motion is surprisingly simple analyze if the following two assumptions are made

1. The acceleration due to gravity, *g*, is constant over the range of motion and directed downward
2. The effect of resistance is negligible.

If we choose our reference frame such that the *y* direction is vertical and positive upward, then $a_y = - g$` and $a_x = 0$ (since air resistance is neglected). Furthermore, let us assume that at the time $t = 0$, the projectile leaves the origin with the velocity v_0. If the vector of velocity makes an angle α with the horizontal, then from the definition of the cosine and sine functions we have (see Fig. 2.11)

$$v_{0x} = v_0 \cos\alpha \tag{2.32}$$

$$v_{0y} = v_0 \sin\alpha \tag{2.33}$$

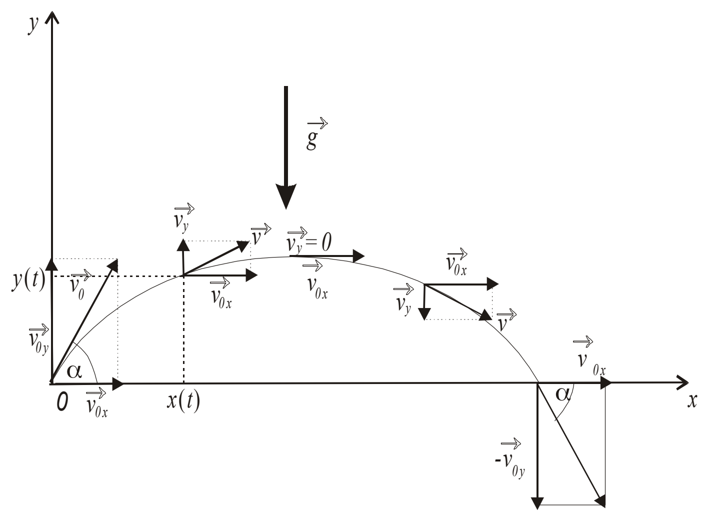

Fig. 2. 11

. Now we apply the equation for one dimensional motion along *x* and *y* axes (eq.2.7), respectively. Because along the *x* axis is the acceleration $a_x = 0$, the equation of the motion along this axis is given by

$$x = x_0 + v_{0x} t \tag{2.34}$$

and velocity at any time is

$$v_x = v_{0x} . \quad (2.35)$$

The projectile starts from the origin, i.e. at the time $t = 0$ its position is $x_0 = 0$. Therefore, we have

$$x = v_{0x}t = tv_0 \cos\alpha \quad (2.36)$$

$$v_x = v_0 \cos\alpha . \quad (2.37)$$

And along the *y*-axis

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \quad (2.38)$$

$$v_y = v_{0y} + at . \quad (2.39)$$

Because $a = -g$ and $y_0 = 0$ for the motion along the *y* axis are valid these expressions

$$y = v_0 \sin\alpha - \frac{1}{2}gt^2 \quad (2.40)$$

$$v_y = v_0 \sin\alpha - gt . \quad (2.41)$$

The trajectory of projectile is the parabola. If we solve for *t* expression of motion along *x*-axis and substitute this expression for *t* into equation for motion along *y*-axis we have

$$y = v_0 \sin\alpha \frac{x}{v_0 \cos\alpha} - \frac{1}{2}g\frac{x^2}{v_0^2 \cos^2\alpha} \quad (2.42)$$

or

$$y = \tan\alpha \cdot x - \left(\frac{1}{2}\frac{g}{v_0^2 \cos^2\alpha}\right) \cdot x^2 .$$

Now we determine the **maximum height, *h*,** of the projectile. If we can see from the figure 2.11 at peak is the *y* component of the velocity equal $v_y = 0$. Therefore, using eq.2.41 we can calculate the time, t_1, in which the projectile reaches the maximum height:

$$t_1 = \frac{v_0 \sin\alpha}{g}$$

Inserting this value into equation of motion along the *y* axis:

$$y_{\max} = h = v_0 \sin\alpha t_1 - \frac{1}{2}gt_1^2$$

gives

$$h = v_0 \sin\alpha \frac{v_0 \sin\alpha}{g} - \frac{1}{2}g\frac{v_0^2 \sin^2\alpha}{g^2} \quad (2.43)$$

or

$$h = \frac{1}{2} \frac{v_0^2 \sin^2 \alpha}{g} . \tag{2.44}$$

The range, $x_{\max}$ **=R,** is the horizontal distance travelled in twice the time it takes to reach the peak, that is, in time $t_2 = 2t_1$. We find that

$$R = x_{\max} = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} . \tag{2.45}$$