VECTORS

For mathematical description of physical systems deals with two types of quantities: one type is called **scalar** which is fully described by a number. Such quantities are, for example, mass, time, temperature, work, energy, etc. Another type is called **vectors**. These quantities are specified by magnitude, direction and sense that may be represented in some reference system by an orientated arrowed line segment whose length is the simple function of the magnitude. The displacement, acceleration, velocity or the force on the body, are examples of vectors.

Graphically a vector is shown in Fig.1.1

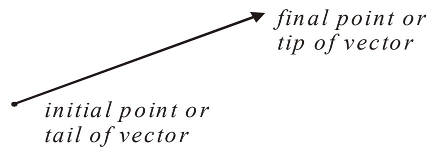

Fig. 1. 1

Some properties of vectors.

Addition of two or more vectors is the arithmetic sum of these vectors. For two vectors is the sum defined as

$$a + b = R \ , \qquad (1.1)$$

Where R is called resultant vector of the sum of vectors a and b .Note that there is valid a commutative law in addition of vectors. Graphically procedure of the addition of two vectors is shown in Figs.1.2 and 1.3

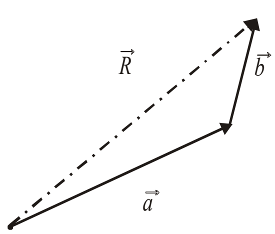

Fig. 1.2

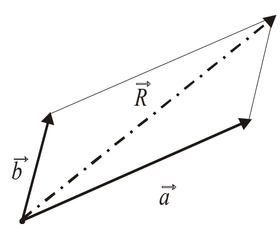

Fig. 1. 3

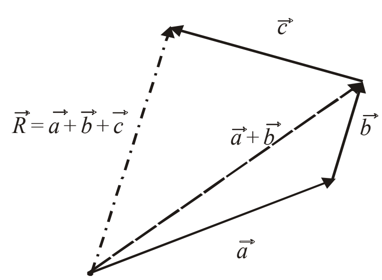

Fig. 1. 4

The graphical method of adding vectors can be extended to determine the sum of more than two vectors. For example the sum of three vectors, a , b and c is shown in Fig.1.4.This property is known as **associativity.** Mathematically is associativity of vector addition in the form

$$a + (b + c) = (a + b) + c \tag{1.2}$$

In the same way we can write the sum of several vectors

Equality of two vectors. Two vectors a and b are defined to be equal if they have the same magnitude and point in the same direction. That is

$$a = b \tag{1.3}$$

Only if *a*=*b* they act along parallel direction as is shown in Fig.1.5

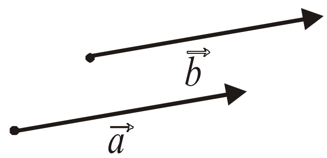

Fig. 1. 5

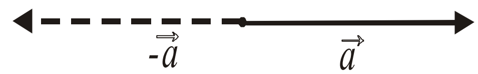

Fig. 1. 6

Negative of vector. Negative of vector $-a$, for example, is defined as the vector when added to vector a gives zeros vector. It means

$$a - a = 0 \tag{1.4}$$

The situation is graphically shown in Fig.1.6

Subtraction of two vectors. The operation of subtraction can by used in the definition of the negative of the vector as

$$a - b = a + (-b) = R \tag{1.5}$$

The situation is graphically shown in Fig.1.7

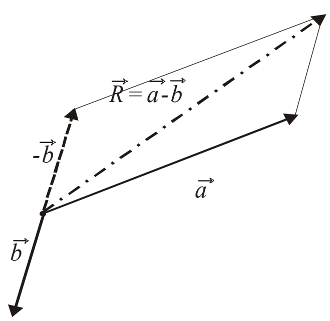

Fig. 1. 7

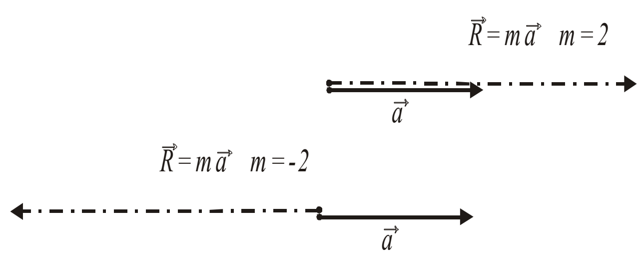

Fig. 1. 8

Multiplication of a vector by a scalar. Multiplication of vector a by a scalar *m* is defined as

$$R = ma \tag{1.6}$$

Resultant vector R is the positive vector if the scalar *m* has the positive value. It means that the resultant vector is in the same direction as is the direction of the vector a. Its magnitude equals *ma.* If the value of scalar *m* is less than zero, the resultant vector has the same magnitude but its direction is opposite the direction of vector a. The situation is graphically shown in Fig.1.8

Components of vector and unit vector. As a Cartesian coordinate system is introduced, vectors can be described by three numbers, called **components of the vector**. Vector a, for example, in Cartesian coordinate system in two dimensions is shown in Fig.1.9

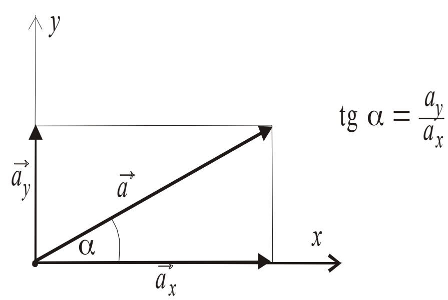

Fig. 1. 9

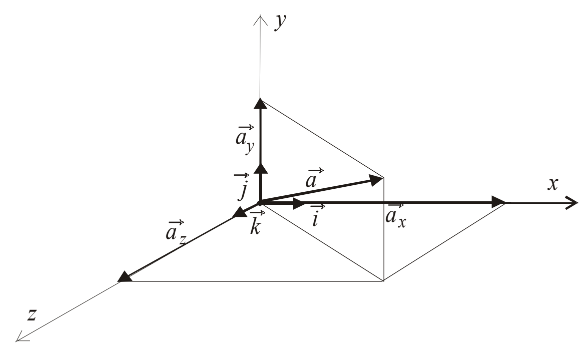

Fig. 1. 10

Where a_x, a_y are rectangular components of vector a and their values are given by

$$a_x = a.\cos\alpha$$

$$a_y = a.\sin\alpha \qquad (1.7)$$

From these equations we can see that the angle between these components is given by

$$tg\alpha = \frac{a_y}{a_x} \qquad (1.8)$$

We can determine the magnitude of vector a using the Pythagoras'theorem. Therefore, we can see from Fig.1.9 that the magnitude of vector a can be written in the form

$$a = \sqrt{a_x^2 + a_y^2} \qquad (1.9)$$

A Cartesian coordinate system in three dimension starts with three perpendicular axes labelled *x,y* and *z* as (see Fig.1.10).**Such a coordinate system is called right-handed** and every vector can be described by three numbers, called components. Vectors can be related algebraically to their components with the help of unit vectors. We use the symbol i to represent the unit vector along the positive *x* axis, and j to represent the unit vector along the positive *y* axis and k to represent the unit vector along the positive *z* axis. Then vector a has the vector's components $a_x i$, $a_y j$ and $a_z k$. Note that all unit vectors have the magnitudes equal one:

$$|i| = |j| = |k| = 1$$

Using these unit vectors we can express any vector a algebraically in terms of its components. The relation is

$$a = a_x i + a_y j + a_z k \qquad (1.10)$$

Then the magnitude of this vector is given by the expression

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad (1.11)$$

Now we suppose we wish to add vector $a = a_x i + a_y j + a_z k$ to vector $b = b_x i + b_y j + b_z k$. The procedure for performing this sum is to simply add the *x*, *y* and *z* components separately. The resultant vector $R = a + b$ is therefore given by

$$R = (a_x + b_x)i + (a_y + b_y)j + (a_z + b_z)k$$

The product of two vectors

The scalar product

The scalar product (or dot product) between two vectors a and b is defined as

$$a.b = |a|.|b|.\cos\alpha \qquad (1.12)$$

Where: α is the angle between vectors a and b

$|a|$ is the magnitude of vector a

$|b|$ is the magnitude of vector b

As we can see from eq.(1.12) the scalar product of two vectors is just a number, it means the scalar. Note that these vectors need to have the same units. Graphically is dot product of two vectors shown in Fig.1.11

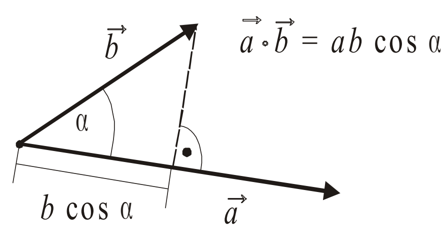

Fig. 1. 11

Properties of scalar product of two vectors:

1. The scalar product of two vectors equals the magnitude of vector a multiplied by the projection of vector b onto vector a and vice versa.

2. Scalar product is commutative:

$$a.b = b.a$$

3. If vector a is perpendicular to vector b then the angle α equals 90^0 and we have $a.b = 0$.

4. If these vectors have the same directions then the angle between them equals zero and the dot product has the value $a.b = ab$

5. By the definition the scalar product between unit vectors is equal

$i.i = j.j = k.k = 1$ and $i.j = i.k = j.k = 0$

6. Scalar product of two vectors can be expressed in Cartesian coordinate system by expression

$$a.b = a_x b_x + a_y b_y + a_z b_z$$

7. We can determine the angle between two vectors using the definition of scalar product as $\cos\alpha = \frac{a.b}{a.b}$.

The vector product

The vector product (or cross product) between two vectors is defined as a third vector, c , given by the expression

$$c = axb = |a|.|b|.\sin\alpha.c^0 \quad (1.13)$$

where c^0 is the unit vector, $|a||b|\sin\alpha$ is the magnitude of the resultant vector , c , and α is the angle included between vectors a and b .The unit vector is perpendicular to the plane formed by the vectors a and $\bar{b}$ and its sense is determined by the advance of **the right-hand screw** (or right hand rule) when turned from

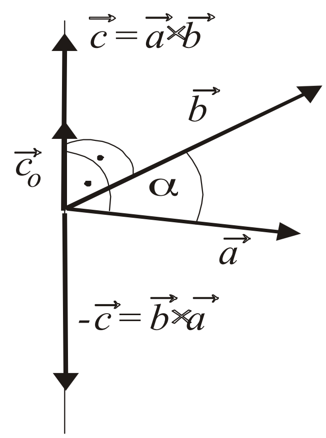

Fig. 1. 12

vector a to vector b through the angle α . The cross product is illustrated in Fig.1.12

Properties of cross product of two vectors:

1. The quantity $a.b.\sin\alpha$ is equal to the area of the parallelogram formed by vectors a and b

.2. The cross product is not commutative. From definition we have

$$axb = -bxa$$

3. If the vector a is perpendicular to vector b then $|axb| = a.b$

4. If vector a is parallel to vector b then $axb = 0$

5. By the definition of the vector product the vector product of unit vectors in Cartesian coordinate system equals

$$i \times i = j \times j = k \times k = 0$$

since the angles between these vectors is zero, i.e. $\sin 90^0 = 0$

On the other hand

$$i \times j = k,\ j \times i = -k,\ j \times k = i,\ k \times j = -i,\ k \times i = j,\ i \times k = -j$$

6. Only in Cartesian coordinate system can be cross product expressed in the following determinant form

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

or

$$a \times b = (a_y b_z - a_z b_y)i + (a_z b_x - a_x b_z)j + (a_x b_y - a_y b_x)k$$