INVESTIGATION OF A HARMONIC MOTION BY MEANS OF COMPUTER

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ABSTRACT

In this paper we want to present the various possibilities of the MATHEMATICA programme applications for learning linear, second order differential equations, describing some types of harmonic motion. The main advantage of this programme is that it enable students to effectively investigate and verify the properties of functions using their graphs. For example, we have presented particular solutions and their graphs satisfying the initial conditions. It was shown that mutual intersecting of the integral curves at the same „cross points" was preserved in all three cases which is the most natural initial value especially from the point of view of physical interpretation .
Key words: differential equations, damped oscillations, undamped oscillations, MATHEMATICA

INTRODUCTION

The main aim of our paper is to show some of the capabilities of the programme *MATHEMATICA* and its utilization in differential equation education, especially homogeneous linear equations of the second order with constant coefficients and their physical applications. We have used this programme at our faculty for several years not only in some subjects in basic bachelor programs, but also in several special subjects of advanced mathematics in master graduate programs. Students, who learned to work with this programme at the begining of their studies, use it for the elaboration of their individual work, presented in various competitions, bachelor work, diploma projects and others. We have many positive experiences ([1] - [9]) and reactions, not only from students.

Utilizing very good visual and graphic abilities, mainly in the higher versions of the programme *MATHEMATICA,* we can choose it for an unconventional procedure in solving different problems. For instance, if students have already mastered the theory of the methods used in the investigation of the real functions, properties and behaviour, then in the investigation of the behaviour of an individual function, we can start by drawing the graph of the given function and only then do we verify with standard methods the properties of the function, such as the position and value of local extrema, monotonicity, convexity, concavity, points of inflection, all kinds of asymptotes, and others that we see on the graph. It is considerably more effective and especially more visual than traditional procedures ([1]).

We want to demonstrate a similar procedure in solving differential equations of the second order with constant coefficients, which describe a simple harmonic motion. We will deal with the particular case of free damped oscillations.

The main advantage of this programe is that it will allow us, or more precisely, our students, to solve „effortlessly" a larger number of studied equations under certain initial conditions, and to draw graphs of the corresponding solutions. By means of these graphs we are able not only to verify well-known properties of the completed solutions, following on from the theoretical, or more precisely, physical nature of the given subject, but also find some more, and subsequently explain and prove them.

FREE DAMPED OSCILLATIONS

Differential equations of the form

$$y''+py'+qy=0 \qquad (1)$$

where $p, q>0$

describe a simple harmonic motion, which is a motion such that the initial acceleration of the motion is directly proportional to the displacement of the body from the static equilibrium position, the direction of the acceleration is opposite to this displacement and there is no external force acting on the system.

In this case, for $p>0$, the motion is said to be damped. It is the coefficient p that represents the damping of the system, induced, for example, by a friction and its value depends on the damping magnitude. We consider the most interesting case of so called **underdamping**, leading to damped oscillations, corresponding to such positive values of coefficients p and q, that it is valid $p^2-4q<0$. In this case the general solution of the equation (1) is of the form

$$y=e^{-\frac{p}{2}x}\left(c_1\cos(\omega x)+c_2\sin(\omega x)\right) \qquad (2)$$

where $\omega=\dfrac{\sqrt{4q-p^2}}{2}$

and c_1, c_2 are arbitrary constants.

In the other cases, if $p^2-4q>0$ (overdamping), or $p^2-4q=0$ (critical damping), the body does not oscillate.

We will therefore deal with particular solutions to the differential equation (1), satisfying some special initial conditions at the point $x=0$, at the begining of the investigated motion. We choose coefficients in our equation in such a way, that its general solution is as simple as possible and the damping of the system is not too strong, which is advantageous for the shape of particular solution graphs. These requirements are fulfilled, for example, by the following equation

$$y''+\frac{1}{5}y'+\frac{101}{100}y=0 \qquad (3)$$

with the general solution

$$y=e^{-\frac{1}{10}x}\left(c_1\cos x+c_2\sin x\right)$$

where c_1, c_2 are arbitrary constants.

We will consider three types of initial conditions, associated particular solutions, and integral curves.

A) Let us first consider the case of harmonic oscillations with the same initial displacement, that is to say, with the same initial value of the function, but with different initial velocities, in other words, with the different values of the derivative. Let the initial value of the function be 1 and initial values of the derivative -1, -0,5, 0, 0,5, 1.

Thus the initial conditions have the form

$y_i(0)=1,\ y_i'(0)=-1+(i-1)0.5,\ \ i=1,2,\ldots 5$

With the aid of the programme *MATHEMATICA* we now find corresponding solutions and draw graphs of these five particular solutions, at an interval of $\langle 0, 8\pi \rangle$ in Fig.1.

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Table[DSolve[{y''[x] + (1/5) y'[x] + (101/100) y[x] == 0, y[0] == 1, y'[0] == i}, y[x], x],
 {i, -1, 1, 1/2}]
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$$\left\{\left\{\left\{y[x]\to \frac{1}{10}e^{-x/10}(10\,\mathrm{Cos}[x]-9\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to \frac{1}{5}e^{-x/10}(5\,\mathrm{Cos}[x]-2\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to \frac{1}{10}e^{-x/10}(10\,\mathrm{Cos}[x]+\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to \frac{1}{5}e^{-x/10}(5\,\mathrm{Cos}[x]+3\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to \frac{1}{10}e^{-x/10}(10\,\mathrm{Cos}[x]+11\,\mathrm{Sin}[x])\right\}\right\}\right\}$$

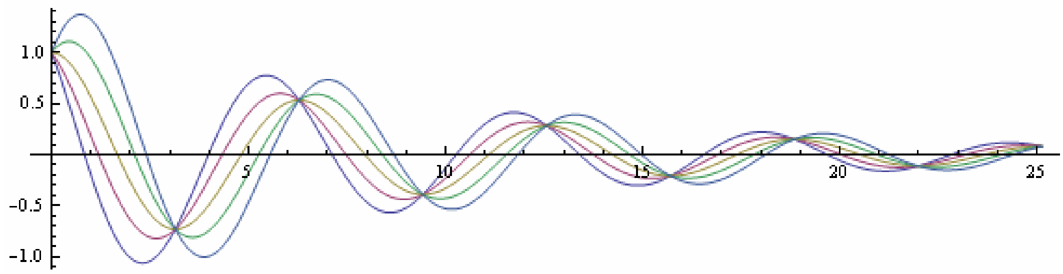

Fig.1 Harmonic oscillations with the same initial displacement and different initial velocities.

B) Now let us solve the case if the initial velocity of the motion, i.e. the initial value of the derivative is the same, for example 1, but with different initial displacements, i.e. different initial values of the function, for example -1, -0,5, 0, 0,5, 1. Therefore we have five innitial value problems of the type $y_i(0)=-1+(i-1)0.5,\ y_i'(0)=1,\ \ i=1,2,\ldots 5$.

We solve the problems and again draw graphs of the particular solutions obtained at an interval of $\langle 0, 8\pi \rangle$ in Fig.2.

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Table[DSolve[{y''[x] + (1/5) y'[x] + (101/100) y[x] == 0, y[0] == i, y'[0] == 1}, y[x], x],
 {i, -1, 1, 1/2}]
```

$$\left\{\left\{\left\{y[x]\to -\frac{1}{10}e^{-x/10}(10\,\mathrm{Cos}[x]-9\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to -\frac{1}{20}e^{-x/10}(10\,\mathrm{Cos}[x]-19\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to e^{-x/10}\mathrm{Sin}[x]\right\}\right\},\left\{\left\{y[x]\to \frac{1}{20}e^{-x/10}(10\,\mathrm{Cos}[x]+21\,\mathrm{Sin}[x])\right\}\right\},\left\{\left\{y[x]\to \frac{1}{10}e^{-x/10}(10\,\mathrm{Cos}[x]+11\,\mathrm{Sin}[x])\right\}\right\}\right\}$$

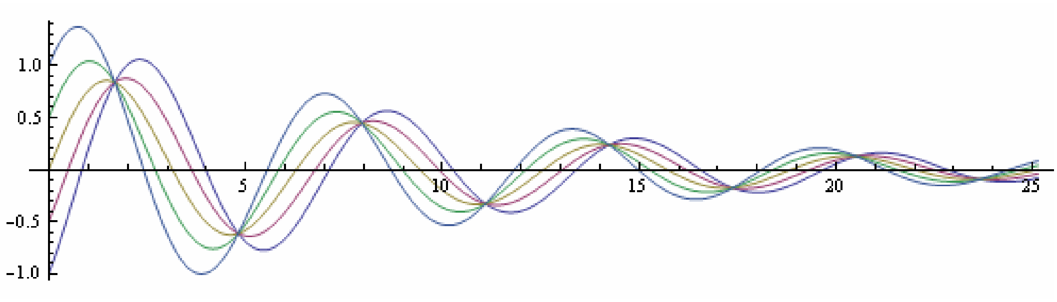

Fig.2 Harmonic oscillations with the same initial velocities and different initial displacements.

C) Finally we investigate the case when the initial displacement is equal to the initial velocity, which means, when the initial value of the function is the same as the initial value of its derivative. Let these common values be again -1, -0,5, 0, 0,5, 1, hence the initial conditions are

$$y_i(0) = y_i'(0) = -1 + (i-1)0.5, \quad i = 1, 2, \ldots 5$$

We solve the given equations and draw the corresponding graphs in Fig.3 .

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Table[DSolve[{y''[x] + (1/5) y'[x] + (101/100) y[x] == 0, y[0] == i, y'[0] == i}, y[x], x],
  {i, -1, 1, 1/2}]
```

$$\left\{\left\{\left\{y[x] \to -\frac{1}{10} e^{-x/10} (10\,\mathrm{Cos}[x] + 11\,\mathrm{Sin}[x])\right\}\right\},\right.$$
$$\left\{\left\{y[x] \to -\frac{1}{20} e^{-x/10} (10\,\mathrm{Cos}[x] + 11\,\mathrm{Sin}[x])\right\}\right\}, \{\{y[x] \to 0\}\},$$
$$\left.\left\{\left\{y[x] \to \frac{1}{20} e^{-x/10} (10\,\mathrm{Cos}[x] + 11\,\mathrm{Sin}[x])\right\}\right\}, \left\{\left\{y[x] \to \frac{1}{10} e^{-x/10} (10\,\mathrm{Cos}[x] + 11\,\mathrm{Sin}[x])\right\}\right\}\right\}$$

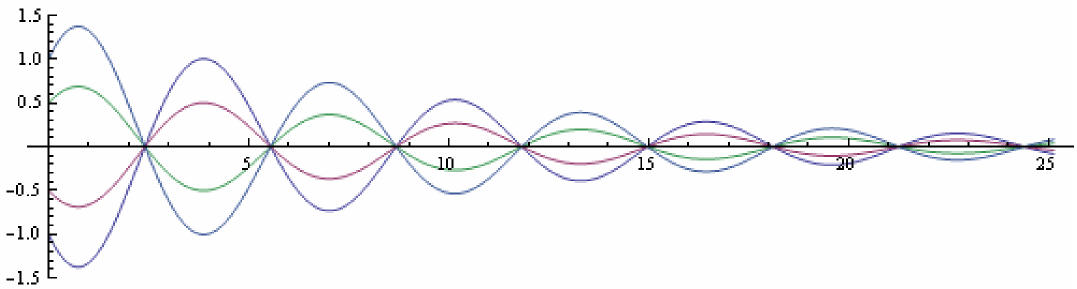

Fig.3 Harmonic oscillations when the initial displacement is equal to the initial velocity.

In all three figures in the preceding cases we observe an interesting phenomenon. The integral curves intersect at the same points.

In what follows we will show that it is not a special property of the individual differential equations we solved, but the same situation occurs in particular solutions of every differential equation (1), if $p^2 - 4q < 0$, satisfying any type of initial conditions described in A), B), or C). Moreover, we are able to find the exact position of these intersection points.

Let us return to the general equation (1), having a general solution of the form (2). Consider our three cases.

A) Let y_1, y_2 be two particular solutions, satisfying the initial conditions

$$y_1(0) = y_2(0) = s,\ s \in \mathbb{R}$$
$$y_1'(0) = v_1,\ y_2'(0) = v_2,\ v_1, v_2 \in \mathbb{R},\ v_1 \neq v_2$$

It can be easily derived that

$$y_i = e^{-\frac{p}{2}x}\left(s\,cos(\omega x) + \frac{1}{\omega}\left(\frac{ps}{2} + v_i\right) sin(\omega x)\right), \quad i = 1,2$$

If at a point x_0 their values are identical, $y_1(x_0) = y_2(x_0)$, it is clear, that it is possible only if

$$sin(\omega x_0) = 0 \ \Rightarrow\ x_0 = k\frac{\pi}{\omega}$$

where *k* is an integer. For the equation (3), with $\omega = 1$, it is true at integer multiples of π.

B) Let y_1, y_2 be two particular solutions, satisfying the initial conditions

$$y_1(0) = s_1,\ y_2(0) = s_2,\ s_1, s_2 \in \mathbb{R},\ s_1 \neq s_2$$
$$y_1'(0) = y_2'(0) = v,\ v \in \mathbb{R}$$

It can be simply proved that

$$y_i = e^{-\frac{p}{2}x}\left(s_i\,cos(\omega x) + \frac{1}{\omega}\left(\frac{ps_i}{2} + v\right) sin(\omega x)\right), \quad i = 1,2$$

If at a point x_0 $y_1(x_0) = y_2(x_0)$, it can be proved that it must be

$$tan(\omega x_0) = \frac{-2\omega}{p} \quad \Rightarrow \quad x_0 = \frac{1}{\omega}\left(arctan\left(\frac{-2\omega}{p}\right) + k\pi\right)$$

where *k* is an integer, therefore the particular solutions of (3) have the same values at points $x_0 = arctan(-10) + k\pi$.

C) Let y_1, y_2 be two particular solutions, satisfying the initial conditions

$$y_1(0) = y_1'(0) = a_1,\ a_1, a_2 \in \mathbb{R}$$
$$y_2(0) = y_2'(0) = a_2,\ a_1 \neq a_2$$

Then obviously

$$y_i = e^{-\frac{p}{2}x}\left(a_i\,cos(\omega x) + \frac{a_i}{\omega}\left(1 + \frac{p}{2}\right) sin(\omega x)\right), \quad i = 1,2$$

If at a point x_0 $y_1(x_0) = y_2(x_0)$, it is possible only if

$$tan(\omega x_0) = \frac{-2\omega}{2+p} \quad \Rightarrow \quad x_0 = \frac{1}{\omega}\left(arctan\left(\frac{-2\omega}{2+p}\right) + k\pi\right)$$

where k is an integer. It follows that in the case of the equation (3) it is at points $x_0 = arctan\left(\frac{-10}{11}\right) + k\pi$.

FREE UNDAMPED OSCILLATIONS

Undamped oscillations represent a special kind of a harmonic motion. They are described by a differential equation of the form

$$y'' + q\,y = 0 \qquad (4)$$

where $q > 0$

Therefore $p = 0$, i.e. it is supposed that the damping of the system is vanishing. Of course, this assumption is more or less theoretical, but it is accurate in practice, if damping of the system is so weak that it can be neglected. In fact, equation (4) is a specific type of the equation (1) and the same is true for its general solution, which is a specific type of the general solution (2) and it has the form

$$y = c_1\, cos\left(\sqrt{q}\, x\right) + c_2\, sin\left(\sqrt{q}\, x\right)$$

where c_1, c_2 are arbitrary constants.

What about the points of intersection of the undamped motion? It means, for equation (4), from the formulas derived for the damped motion, that substituting in these formulas $p = 0,\ \omega = \sqrt{q}$

(in case B we must take into consideration that $\lim\limits_{x \to -\infty} arctan\, x = -\frac{\pi}{2}$), we get the following formulas for the positions of the points of intersection in the case of undamped oscillations:

A) $x_0 = k \frac{\pi}{\sqrt{q}}$,

B) $x_0 = \frac{1}{\sqrt{q}}\left(2k - 1\right)\frac{\pi}{2}$,

C) $x_0 = \frac{1}{\sqrt{q}}\left(arctan\left(-\sqrt{q}\right) + k\pi\right)$,

where *k* is an integer.

CONCLUSION

Similar problems, resolved by a modification of the previous problem, dealing with other types of the harmonic motion are appropriate for skilled students' individual investigations. They can be solved as seminar works and presented at various student competitions, conferences and sessions.

For example, we have presented particular solutions and their graphs which satisfy the initial conditions at the point $x = 0$, which is the most natural initial value especially from the physical interpretation point of view. But the fact that this is not a necessary condition can be simply and quickly verified (again by means of the programming system *MATHEMATICA*).

The property we have investigated, mutual intersecting of integral curves at the same points, is maintained in all three cases A), B), C), and also when the corresponding three types of initial conditions are required at the point $x = a$, for an arbitrary real number *a*.

In the same way as for $x = 0$, in this more general case, we are able to quite easily find (even if it is numerically a little more complicated) the position of the points of intersection, depending of course on the value of *a*.

And the situation is analogical in the case of forced damped and undamped oscillations (see [3]), if there is an external force acting on the system. These forced oscillations are again described by linear differential equations of the second order, in a similar way to free oscillations, but in contrast to them, the equations are not homogeneous.

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