MINKOWSKI SET OPERATIONS IN GEOMETRIC MODELLING OF CONTINUOUS RIEMANNIAN MANIFOLDS

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ABSTRACT

This paper presents some ideas about the application of Minkowski set operations of point sets in the Euclidean space $\mathbf{E}^{n}$, *to geometric modelling of continuous Riemannian manifolds. The definition of the basic concepts and properties of the Minkowski sum and products are presented, together with several examples, namely models of curves and surfaces in* $\mathbf{E}^3$ *obtained using this method. The approach presented utilises the formalism of the Creative geometric space and presents general formulae for vector representations of resulting surfaces determined by operands defined by vector equations. Minkowski operations are applied in CAD, robotics and in offsetting theory connected to the elaboration of materials, on a large scale.*
Key words: Riemannian manifold, Minkowski set operations, Minkowski sum, Minkowski product, geometric modelling of curves and surfaces.

INTRODUCTION

Minkowski operations of two point sets are binary geometric operations defined on point subsets of the Euclidean space $\mathbf{E}^n$, which can be interpreted in different ways. The Minkowski sum was introduced in 1903 by Hermann Minkowski. The most common interpretation is represented by means of vector sums of the position vectors of all points in the given sets – point subsets of the space $\mathbf{E}^n$. The next most frequently used interpretation of the Minkowski sum is a continuous movement of one set operand on the boundary of the other one without any change of orientation.
A different approach can be applied to the definition of algorithm for calculations of the Minkowski product of two point sets in the Euclidean space $\mathbf{E}^n$. In some references one can find a definition based on the product of complex numbers, which can be extended to the space of quaternions and a special construction of an action defined on the sets of quaternions [1], [3], [4]. Another approach is represented by the outer, wedge product defined for two vectors in the higher dimensional vector spaces, referred to also as the tensor product of two vectors [5], which is an algebraic generalisation of the vector product of two vectors in the three dimensional vector space for the purposes of the exterior algebra, or the Grassmannian algebra [5].
Algorithms presented in this paper, for both sum and product of two continuous point subsets of the Euclidean space $\mathbf{E}^3$, are based on vector representations of geometric figures in the role of operands. The resulting manifold is a geometric figure in the space $\mathbf{E}^3$ determined by vector representation, with intrinsic geometric properties inherited from the determining operands. Applications of the Minkowski sum and product can be found in computer graphics, robotics, in finding algorithms for the dense placement of planar figures and the determination of offsets, in geometric modelling, in CAD, and many others.
Examples of the Minkowski sum and product of two space curves are presented, as the sum and product of two continuous point subsets of the Euclidean space $\mathbf{E}^3$, serving as a useful tool for surface modelling in CAGD. Algorithms are based on vector representations of operands, vector functions defined on unit intervals in real numbers $\mathbf{R}$, while the sum and product of the two given

continuous Riemannian manifolds is a continuous Riemannian manifold in the Euclidean space $\mathbf{E}^3$ determined by vector function in two variables defined on the unit square in $\mathbf{R}^2$.

DEFINITION AND PROPERTIES OF MINKOWSKI SUM

Let A and B be two point sets in the n-dimensional Euclidean space $\mathbf{E}^n$ with the related vector space V^n and the Cartesian orthogonal coordinate system (O; $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$) defined by the origin O and direction unit vectors $\mathbf{e}_i$ of the coordinate axes. Points in the Euclidean space $\mathbf{E}^n$ are represented by their positioning vectors from V^n

$$a = \overrightarrow{0a} = \left({}^1a, {}^2a, ..., {}^na\right) = {}^1a\mathbf{e}_1 + {}^2a\mathbf{e}_2 + ... + {}^na\mathbf{e}_n \, . \tag{2.1}$$

Definition 1.
The Minkowski sum of two point sets A and B in the Euclidean space $\mathbf{E}^n$ is a point set, which is the sum of all points from the set A with all points from the set B, i.e. the set of points

$$A \oplus B = \{a + b; a \in A, b \in B\}. \tag{2.2}$$

Definition 2.
The Minkowski sum of two point sets A and B in the Euclidean space $\mathbf{E}^n$ is the point set

$$A \oplus B = \bigcup_{b \in B} A^b \, , \tag{2.3}$$

where A^b is the set A translated by the vector b

$$A^b = \{a + b; a \in A\}. \tag{2.4}$$

The Minkowski sum is also called a binary dilatation of the set A by the set B. Minkowski sum is sometimes also denoted, not exactly and correctly, as a convolution of point sets A and B.
The basic properties of the Minkowski sum of point sets A, B, C, D in the Euclidean space $\mathbf{E}^n$, which are used most frequently for its calculations are listed in the following.

PS1. The Minkowski sum of convex sets is also a convex set.

PS2. The Minkowski sum is a commutative operation, $A \oplus B = B \oplus A$.

PS3. The Minkowski sum is an associative operation, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

PS4. The following equation holds

$$(A \cup B) \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D) \cup (B \oplus C) \cup (B \oplus D). \tag{2.5}$$

PS5. The Minkowski sum of the union of two sets is the union of the Minkowski sums of the separate set operands. Let P_i and Q_j for $i, j \in N$ be the point sets in the Euclidean space $\mathbf{E}^n$. Then the following relation holds

$$\bigcup_i P_i \oplus \bigcup_j Q_j = \bigcup_{i,j} P_i \oplus Q_j \tag{2.6}$$

Property PS5 is very useful for determination of the Minkowski sum of non-convex point sets, particularly. These point sets must be firstly split and distributed into a finite number of disjunctive convex subsets whose unification forms the former non-convex sets. For all these subsets the Minkowski sums has to be calculated separately. Only after that it is possible to utilise property PS5 for calculation in order to obtain the Minkowski sum of the given non-convex point sets. This is finally represented as the union of the partial Minkowski sums calculated before.

DEFINITION AND PROPERTIES OF MINKOWSKI PRODUCT

Let A and B be two point sets in the n-dimensional Euclidean space $\mathbf{E}^n$ with the related vector space V^n represented by their positioning vectors a and b, as given in relation (2.1).

Definition 1.
The Minkowski product of point sets A and B in the Euclidean space $\mathbf{E}^n$ is a point set, which is the product of all points in the set A with all points in the set B, i. e. the set of points

$$A \otimes B = \{a \wedge b; a \in A, b \in B\} \tag{3.1}$$

where $a \wedge b$ is the outer wedge product of the positioning vectors of points a and b.

Outer wedge product of two vectors is a generalization of the well-known vector product of two vectors in the three-dimensional vector space to the spaces of higher dimensions. Several basic properties of the outer product of vectors in the vector space V^n are presented in the list below.

P1. The equality $\mathbf{u} \wedge \mathbf{u} = 0$ holds for an arbitrary vector $\mathbf{u} \in V$.

P2. The outer product is an associative operation, i.e. for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ the following relation holds

$$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}\,. \tag{3.2}$$

P3. The outer product is an anti-commutative (anti-symmetric) operation, i.e. $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$ holds for arbitrary two vectors $\mathbf{u}, \mathbf{v} \in V$.

The proof. $0 = (\mathbf{u} + \mathbf{v}) \wedge (\mathbf{u} + \mathbf{v}) = \mathbf{u} \wedge \mathbf{u} + \mathbf{u} \wedge \mathbf{v} + \mathbf{v} \wedge \mathbf{u} + \mathbf{v} \wedge \mathbf{v} = \mathbf{u} \wedge \mathbf{v} + \mathbf{v} \wedge \mathbf{u} \Rightarrow \mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

Property P3 can be generalised in the following way.

If $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_k$ are elements of the vector space V^n and σ is an arbitrary permutation of integers [1, ..., k], then the following holds

$$\mathbf{u}_{\sigma(1)} \wedge \mathbf{u}_{\sigma(2)} \wedge \ldots \wedge \mathbf{u}_{\sigma(k)} = \operatorname{sgn}(\sigma)\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \wedge \mathbf{u}_k\,, \tag{3.3}$$

where sgn(σ) is the sign of the permutation σ.

P4. For any vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$ and real numbers $c_1, c_2 \in R$ the following relations hold

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \wedge \mathbf{v} = c_1(\mathbf{u}_1 \wedge \mathbf{v}) + c_2(\mathbf{u}_2 \wedge \mathbf{v}) \tag{3.4}$$

$$\mathbf{u} \wedge (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1(\mathbf{u}_1 \wedge \mathbf{v}_1) + c_2(\mathbf{u} \wedge \mathbf{v}_2) \tag{3.5}$$

P5. Vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ are linearly independent, if the following inequality holds

$$\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \wedge \mathbf{u}_k \neq 0 \tag{3.6}$$

Several basic properties of the Minkowski product of point sets A, B, C, D in the Euclidean space $\mathbf{E}^n$ that are most frequently utilized in application for its calculations are listed below.

PP1. The Minkowski product of convex sets is again a convex set.

PP2. The Minkowski product is a non-commutative operation, inequality holds

$$A \otimes B \neq B \otimes A. \tag{3.7}$$

PP3. The Minkowski product is an associative operation, there holds

$$(A \otimes B) \otimes C = A \otimes (B \otimes C). \tag{3.8}$$

PP4. The following relation hold

$$(A \cup B) \otimes (C \cup D) = (A \otimes C) \cup (A \otimes D) \cup (B \otimes C) \cup (B \otimes D). \tag{3.9}$$

MODELLING OF SURFACE PATCHES IN $\mathbf{E}^3$ AS MINKOWSKI SUMS AND PRODUCTS OF 2 CURVE SEGMENTS

Let there be defined two smooth regular curve segments k and h in the space $\mathbf{E}^3$ determined by their vector functions parametrized on the unit intervals in real numbers

$$\mathbf{r}(u) = (xr(u), yr(u), zr(u)), u \in \langle 0,1 \rangle \tag{4.1}$$

$$\mathbf{s}(v) = (xs(v), ys(v), zs(v)), v \in \langle 0,1 \rangle \tag{4.2}$$

The Minkowski sum of curves k and h is a patch of translation surface that can be created by translation of one curve alongside the other one

$$k \oplus h = \chi \tag{4.3}$$

and which is determined by vector function defined on the unit square in in $\mathbf{R}^2$ in the form

$$\mathbf{p}(u,v) = \mathbf{r}(u) + \mathbf{s}(v) = (xr(u) + xs(v), zr(u) + zs(v), yr(u) + ys(v)), (u,v) \in \langle 0,1 \rangle^2 \tag{4.4}$$

In fig. 1 on the left, a view of the patch of an elliptic-elliptic surface is presented, determined by vector equation

$$\mathbf{p}(u,v) = (m_1 + a_1 \cos 2\pi u + m_2 + a_2 \cos 2\pi v, n_2 + b_2 \sin 2\pi v, n_1 + b_1 \sin 2\pi v), (u,v) \in \langle 0,1 \rangle^2, \tag{4.5}$$

while operand ellipses in the coordinate planes xz and xy are determined by their vector equations

$$\mathbf{r}(u) = (m_1 + a_1 \cos 2\pi u, 0, n_1 + b_1 \sin 2\pi u), u \in \langle 0,1 \rangle \tag{4.6}$$

$$\mathbf{s}(v) = (m_2 + a_2 \cos 2\pi v, n_2 + b_2 \sin 2\pi v, 0), v \in \langle 0, 1 \rangle \tag{4.7}$$

On the right in Fig. 1, the parabolic-parabolic surface patch can be seen, determined by 2 parabolas located also in coordinate planes xz and xy

$$\mathbf{r}(u) = (a_1 u, 0, b_1 (u^2 - u)), u \in \langle 0,1 \rangle \tag{4.8}$$

$$\mathbf{s}(v) = (a_2 v, b_2 (v^2 - v), 0), v \in \langle 0, 1 \rangle \tag{4.9}$$

The vector equation of this surface patch appears in the following form

$$\mathbf{p}(u,v) = (a_1 u + a_2 v, b_2 (v^2 - v), b_1 (u^2 - u)), (u,v) \in \langle 0,1 \rangle^2 \tag{4.10}$$

Let k and h be curve segments in the space $\mathbf{E}^3$ determined by vector functions (4.1) and (4.2). Minkowski product of curves k and h is the surface patch

$$k \otimes h = \chi \tag{4.11}$$

determined by vector function defined on the unit square in $\mathbf{R}^2$, for $(u,v) \in \langle 0,1 \rangle^2$

$$\begin{gathered} \mathbf{p}(u,v) = \mathbf{r}(u) \times \mathbf{s}(v) = \\ (yr(u)zs(v) - zr(u)ys(v), xr(u) + zs(v) - zr(u)xs(v), xr(u)ys(v) - yr(u)xs(v)). \end{gathered} \tag{4.12}$$

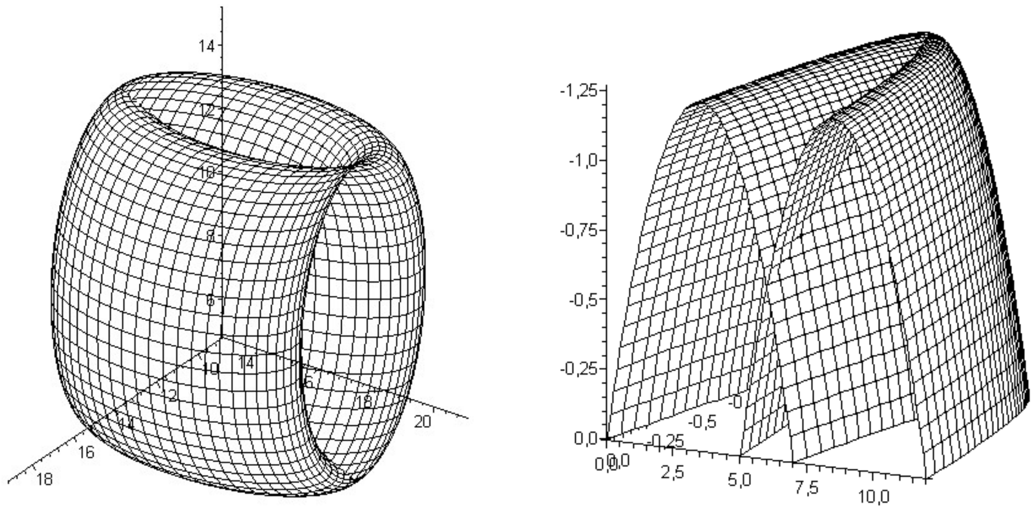

Fig. 1 Minkowski sum of two ellipses and two parabolas.

The surface patch illustrated on the left in Fig. 2 is the product of two ellipses located in parallel planes $z = c_1$, $z = c_2$, with axes parallel to the coordinate axes and semi-axes (a_1, b_1) and (a_2, b_2), respectively. Parametric representations of the surface patch defined on the unit squares is

$$\mathbf{p}(u,v) = \begin{pmatrix} c_2 b_1 \sin 2\pi u - c_1 b_2 \cos 2\pi v, \\ c_2 a_1 \cos 2\pi u - c_1 a_2 \sin 2\pi v, \\ a_1 b_2 \cos 2\pi u \cos 2\pi v - a_2 b_1 \sin 2\pi u \sin 2\pi v \end{pmatrix}, (u,v) \in \langle 0, 1 \rangle^2 \quad (4.13)$$

The product of two parabolas located in perpendicular planes xy and xz with axes parallel to the coordinate axes is surface patches parametrically represented on the unit squares in $\mathbf{R}^2$ by

$$\mathbf{p}(u,v) = \left(-b_1 b_2 (u^2 - u)(v^2 - v), a_2 b_1 v (u^2 - u), a_1 b_2 u (v^2 - v)\right), (u,v) \in \langle 0, 1 \rangle^2 \quad (4.14)$$

and illustrated on the right in Fig.2.

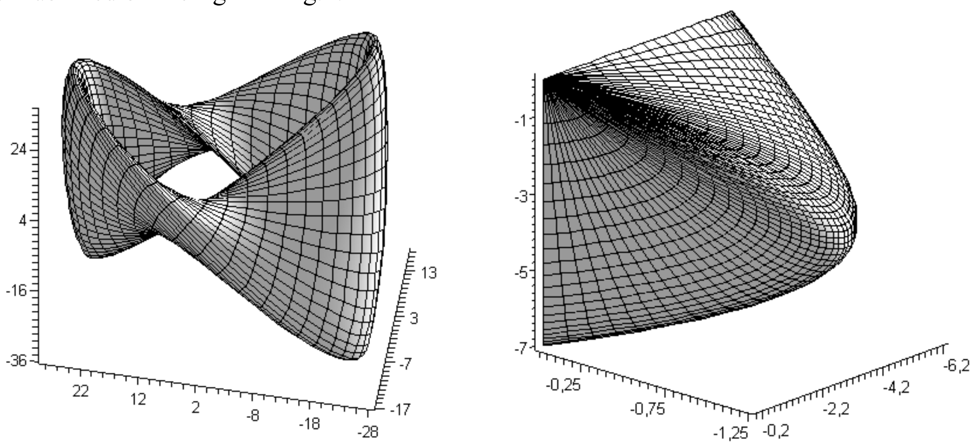

Fig. 2 Minkowski product of two ellipses and two parabolas.

CONCLUSIONS

The Minkowski sum plays an important role in many algorithms of 2D and 3D computer graphics; it is also the basis of the sweep modelling of surfaces and solids. It is also used in determining the trajectory of rigid motion of object in-between obstacles, for calculations of optimal trajectory and for configurations of the active working space of a robot, defined as the set of reachable positions. In an easy model accepting only translation movement of an object in the plane, when the position of the object can be determined uniquely with respect to a given reference point, the working space is the Minkowski sum of the set of obstacles and the moving object. In computer aided control systems and technological solutions of material elaboration, the Numerical-Control machines programming is based on the principles of the Minkowski sum. Here the sum of the geometric model of the cutting tool and its trajectory, which is a planar curve, enables us to determine the form of the cutting edge on the material. The problems of determinating the equidistant to a given figure boundary, presented here, form a specific field of computer graphics, the offsetting theory. Interesting forms of surface patches that can be achieved as the Minkowski products of various curve segments in variegated superpositions, confirm this algorithm as a functional and robust tool in CAD, for industrial and art designers.
Comparative examples of the Minkowski sum and the product of two smooth curves in the Euclidean space, the shamrock curve and versiére, located in perpendicular planes, are illustrated in Fig. 3.

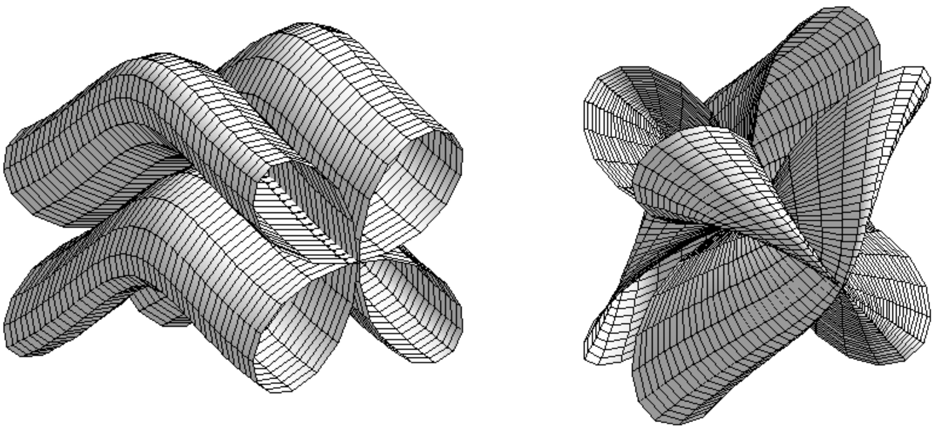

Fig. 3 Minkowski sum (on the left) and product (on the right) of shamrock curve and versiére.

REFERENCES

[1] Ghosh, P. K. A Unified Computational Framework for Minkowski Operations. In Comput. & Graphics, Vol. 17, No. 4, 1994.
[2] Smukler, M. Geometry, Topology and applications of the Minkowski Product and Action. Harvey Mudd College. Senior thesis, 2003.
[3] Tomičková, S.: Minkowski operations and their applications. PhD work. Plzeň, ZČU 2006.
[4] MacLane, S.; Birkhoff, G. Algebra, AMS Chelsea, 1999, ISBN 0-8218-1646-2.
[5] http://en.wikipedia.org/wiki/Wedge_product.
[6] Velichová, D.: Surface Modelling by Means of Minkowski Sum, Aplimat Journal of Applied Mathematics, STU in Bratislava, N°1/2009, Volume II, ISSN 1337-6365, pp. 165-173.
[7] Velichová, D.: Minkowski Product in Surface Modelling, Aplimat - Journal of Applied Mathematics, N°1/2010, Volume 3, STU in Bratislava, 2010, ISSN 1337-6365, pp. 277-286.