DESIGN OF VIBRATING MECHANICAL SYSTEMS VIA SYMMETRIC INVERSE PROBLEM

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ABSTRACT

This paper considers a symmetric positive definite inverse vibration problem for linear vibrating systems described by a vector differential equation with constant coefficient matrices and nonproportional damping. The inverse problem of interest here is that of determining real symmetric and positive definite coefficient matrices assumed to represent the mass normalized velocity and position coefficient matrices. The approach presented here extends the previous results to include noncommuting (or commuting) coefficient matrices which preserve eigenvalues, eigenvectors, and definiteness.
Key words: vibration, inverse eigenvalue problem, symmetric and positive definite matrices

INTRODUCTION

Here we consider linear lumped parameter systems which can be modeled by a vector differential equation in the second order form given by

$$M\,\ddot{q}(t) + D\,\dot{q}(t) + K\,q(t) = 0 \tag{1.1}$$

where $q(t)$ is an n vector of time-varying elements representing the displacement of the masses in a lumped mass model of some structure or device. The vectors $\dot{q}(t)$ and $\ddot{q}(t)$ represent the velocities and acceleration, respectively. The coefficients M, D and K are matrices of constant real elements representing the various physical parameters of mass, damping and stiffness. The matrices M, D and K could in general be asymmetric, however, here we are concerned with the symmetric case and the case in which M is a positive definite. Since M is positive definite and symmetric, and it has a matrix square root, with a symmetric, positive definite inverse denoted by $M^{-1/2}$. Let us then consider the transformation $q(t) = M^{-1/2}u(t)$. Substitution of this change of coordinates into Eq. (1.1) yields

$$\ddot{u}(t) + \widetilde{D}\,\dot{u}(t) + \widetilde{K}\,u(t) = 0 \tag{1.2}$$

where $\widetilde{K} = M^{-1/2}KM^{-1/2}$ and $\widetilde{D} = M^{-1/2}DM^{-1/2}$ are necessarily symmetric. The matrices $\widetilde{D}$ and $\widetilde{K}$ referred to here as the mass normalized damping and stiffness matrices. The eigenvalue problem of the system described by (1.2) is defined by

$$\left(\lambda^2 I + \lambda\,\widetilde{D} + \widetilde{K}\right)x = 0 \tag{1.3}$$

x is a nonzero vector of constants, called the eigenvector, and λ is a scalar, called the eigenvalue.

From the spectral theory of matrix polynomials it is well known that the solutions of the system (1.2) are intimately connected with the algebraic properties of the matrix polynomials (Gohberg et al., 1982) of the form

$$L(\lambda) = \lambda^2 I + \lambda \tilde{D} + \tilde{K} \tag{1.4}$$

Here a scalar λ and a nonzero vector x are again called an eigenvalue and associated (right) eigenvector of $L(\lambda)$ if $det\, L(\lambda) = 0$ and $L(\lambda)\, x = 0$, respectively. This forms an obvious connection between (1.3) and (1.4).

Previously, inverse spectral problems in vibration of lumped systems have been solved by Lancaster and Maroulas (1987), and Starek and Inman (1991, 1992, 1995 and 1997). The results presented here build on those of Lancaster and Maroulas (1987) and those of Starek and Inman (1992 and 1997). Lancaster and Maroulas have solved the inverse problem in vibration by means of the spectral theory of matrix polynomials. They defined Jordan pairs that determine a self-adjoint matrix polynomial. Starek and Inman (1992) have solved the inverse spectral problems in the state-space form. They have defined the conditions for given spectral and modal data under which the inverse formulas determine real symmetric coefficient matrices $\tilde{K}$ and $\tilde{D}$, but their state space solution requires that the given eigenvalues must all be complex valued and does not preserve given eigenvectors.
The goal of this paper is to derive conditions under which spectral and modal data determine real symmetric positive definite coefficient matrices $\tilde{D}$ and $\tilde{K}$, which do not necessarily compute. Symmetric systems are of particular interest in the eigenstructure assignment method of control, in the model updating problem of structural dynamics, and in fault detection problems for machine and structure diagnostics. Such inverse methods have been used in determining the condition of the bonding of the protective tiles to the space shuttle.

NON-NEGATIVE MATRIX POLYNOMIAL CONDITION-SOLUTION 1

A matrix polynomial is said to be non-negative (or positive) if for every $\lambda \in R$ and $x \in C^n$, $x^* L(\lambda) x \geq 0$ (or > 0). From the theory of matrix polynomials it is well known that since $\tilde{D}$ and $\tilde{K}$ are Hermitian, $L(\lambda)$ is a self-adjoint matrix polynomial and thus can be decomposed into a product of two linear factors, i.e., there are $n \times n$ complex valued matrices Z and T, such that $L(\lambda) = (I\lambda - T)(I\lambda - Z)$. The eigenvalues of Z and of T make up the eigenvalues of $L(\lambda)$. The eigenvectors of Z are also eigenvector $L(\lambda)$. The first of the above-mentioned result gives the relation between the eigenvectors of T and $L(\lambda)$ as follows

Theorem 1 (Lancaster and Maroulas).

Let $L(\lambda) = (I\lambda - T)(I\lambda - Z)$, and assume that the set of eigenvalues of matrices T and Z make up disjoint parts of the spectrum of $L(\lambda)$, and let $Z = X_Z J_Z X_Z^{-1}$ where J_Z is the Jordan normal form of the matrix Z. Let $V = [X_Z, Y]$ and $\Lambda = diag[J_Z, J_T]$ be a Jordan pair for $L(\lambda)$. Then there is a nonsingular matrix X_T such that $T = X_T J_T X_T^{-1}$ where

$$X_T = Y J_T - Z Y \tag{2.1}$$

In theorem 1 we may set $T = Z = \left(X_Z^{-1}\right)^* \bar{J}_Z X_Z^*$ and it is easily seen that $X_T = \left(X_Z^{-1}\right)^* P$. Substituting $Z = X_Z J_Z X_Z^{-1}$ we get the following result (Lancaster and Maroulas (1987)).

Theorem 2.
Let $L(\lambda) = \left(I\lambda - Z^*\right)\left(I\lambda - Z\right)$. The set of eigenvalues of matrices Z^* and Z make up disjoint parts of the spectrum of $L(\lambda)$, and $Z = X_Z J_Z X_Z^{-1}$, where J_Z is a Jordan normal form. Let $\hat{Y}$ be the unique solution of

$$\hat{Y} J_Z^* - J_Z \hat{Y} = \left(X_Z^* X_Z\right)^{-1} \tag{2.2}$$

The goal here is to derive the conditions under which spectral and modal data determine real symmetric positive definite coefficient matrices $\widetilde{D}$ and $\widetilde{K}$ of Eq.(1.2) for the case where all eigenvalues are complex. Then there is a Jordan matrix Λ such that (the case when all eigenvalues are complex)

$$\Lambda = diag\left[J_Z, J_T\right] = diag\left[J_Z, \bar{J}_Z\right] \tag{2.3}$$

Where J_Z is the matrix with all its eigenvalues in the upper half of the complex plane. The modal matrix V is partitioned in a compatible way as (2.3), i.e

$$V = \left[X_Z,\ Y\right] = \left[X_Z, \bar{X}_Z\right] \tag{2.4}$$

To this end substitute $Y = \bar{X}_Z U$ into $\hat{Y} = X_Z^{-1} Y P$ and then substitute $\hat{Y}$ into (2.2). After some manipulation this results in

$$X_Z^* \left(\bar{Z} - Z\right) \bar{X}_Z U = I. \tag{2.5}$$

The question remains of how to choose eigenvalues and eigenvectors of a matrix Z such that (2.5) will be valid. The procedure for determining real symmetric matrices $\widetilde{D}$ and $\widetilde{K}$ is summarized as follows (using MATLAB, but other software could be used here):

a) Select the required eigenvalues of the system (they are give by the matrix J_Z).

b) Choose the orthonormal matrix C .

c) Use the MATLAB code $X = \mathrm{lyap}\left(E, F\right)$ for computing the matrix X which must be a positive definite matrix.

d) Use the MATLAB code $X_{Zr} = \mathrm{chol}\left(X\right)$ which produces an upper triangular X_{Zr} so that $X = X_{Zr}^T X_{Zr}$.

e) Determine the coefficient matrices and either by inverse formulas (Starek and Inman (1992))

$$\left[-\widetilde{K} - \widetilde{D}\right] = V \Lambda^2 X^{-1} \tag{2.6}$$

here the modal matrix V and the spectral matrix Λ are given by the Eq.(2.3) and (2.4).) and the

$$X = \begin{bmatrix} V \\ V\Lambda \end{bmatrix} \tag{2.7}$$

or from the formulas that follow from the definition of a nonnegative matrix polynomial $L(\lambda) = \left(I\lambda - Z^*\right)\left(I\lambda - Z\right)$ which is given by the Theorem 2, i.e.,

$$\widetilde{D} = -Z - Z^* \tag{2.8}$$

$$\widetilde{K} = Z^* Z \tag{2.9}$$

NON-NEGATIVE MATRIX POLYNOMIAL CONDITION-SOLUTION 2

From an examination of the above procedure it follows that the proposed solution designs a system, which has the desired eigenvalues. The disadvantage is that the solution is limited to choices such orthogonal matrices C, that the solution of Lyapun equation will give symmetric and positive definite matrix $X = X_r^T X_r$. So, the goal of the next part of the paper is to give a better solution.

Theorem 3.
A Jordan pair (*X, J*) corresponds to a self-adjoint monic matrix polynomial if, and only if, there is a $T \in A_I(J)$ (That denotes the subalgebra of invertible matrices that commutes with *J*) and a canonical matrix P_C such that

$$\begin{bmatrix} X \\ XJ \\ . \\ XJ^{n-1} \end{bmatrix} (TP_C T^*) X^* = \begin{bmatrix} 0 \\ 0 \\ . \\ I \end{bmatrix} \quad (3.1)$$

with *T= I* and *n=2* Eq.(3.1) takes the form

$$V_C P_C V_C^T + V_R P_R V_R^T + \overline{V}_C P_C V_C^T = 0 \quad (3.2)$$

$$V_C J_C P_C V_C^T + V_R J_R P_R V_R^T + \overline{V}_C \overline{J}_C P_C \overline{V}_C^T = 0 \quad (3.3)$$

To simplify the conditions (3.2) and (3.3) let the imaginary part of the modal matrix be presented as the product $V_i = V_r C$, where the $n \times n$ matrix C is real valued and nonsingular. We remind the reader that $V_C = V_r(I + iC) = X_C$ and the vibrating system is underdamped (real parts of modal and spectral data don't exist). Substituting that value into conditions (3.2) and (3.3) we yield

$$2V_r E V_r^T = 0 \quad (3.4)$$

$$2V_r F V_r^T = I \quad (3.5)$$

where *I* is the identity matrix and *E* and *F* are defined by

$$E = P_C - CP_C C^T \quad (3.6)$$

$$F = J_r P_C - CJ_i P_C - J_i P_C C^T - CJ_r P_C C^T \quad (3.7)$$

Note that Eq. (3.6) will be satisfied if *E=0.* This means, that the matrix *C* must be orthonormal. Upon futher examinations *F* must be symmetric. If J_i is chosen to be in the lower part of the complex plane, then the matrix *F* is seen to be positive definite. Consider a product of matrices resulting in matrix *L* defined by

$$L = 2V_r F V_r^T \quad (3.8)$$

Upon examination we can see that matrix *L* is symmetric and positive definite and so it has Cholesky decomposition i.e. there exists a nonsingular matrix *T* such that $L = T^T T$. Hence Eq. (3.8) becomes

$$T^T T = 2V_r F V_r^T \quad (3.9)$$

Premultiplying this expression by $(T^T)^{-1}$ and postmultiplyig by $(T)^{-1}$ yields

$$2(T^T)^{-1} V_r F (V_r)^T T^{-1} = I \quad (3.10)$$

A comparison of Eqs. (3.10) and (3.5) yields that if V_r is chosen to be

$$V_{rr} = (T^T)^{-1} V_r \quad (3.11)$$

then V_{rr} is a matrix of real parts of eigenvectors such that the choice of $\widetilde{D}$ and $\widetilde{K}$ given by Eqs. (2.8) and (2.9) will be symmetric and positive definite.

EXAMPLE

Consider the design of both proportional and a nonproportional symmetric positive definite inverse problem with two degrees of freedom. Let the system have the following eigenvalues

$$J_Z = \begin{bmatrix} -1-i & 0 \\ 0 & -1-3i \end{bmatrix}$$

Choose the orthogonal matrix C to be $C = \begin{bmatrix} .8944 & -.4472 \\ .4472 & .8944 \end{bmatrix}$, $\quad C_d = \begin{bmatrix} .9 & 0 \\ 0 & .75 \end{bmatrix}$.

and note that $CC^T = I$ as it should . Then compute

$$F = \begin{bmatrix} 1.7887 & -.8944 \\ -.8944 & 5.3663 \end{bmatrix} \qquad F_d = \begin{bmatrix} 1.61 & 0 \\ 0 & 4.0625 \end{bmatrix}$$

$$L = \begin{bmatrix} 3.3947 & .9472 \\ .9472 & 6.8997 \end{bmatrix} \qquad L_d = \begin{bmatrix} 3.9894 & 1.1501 \\ 1.1501 & 4.5286 \end{bmatrix}$$

For computing correction matrices we use Eq. (3.9) and Cholesky decompositon. We yield

$$T = \begin{bmatrix} 1.8425 & .5141 \\ 0 & 2.5759 \end{bmatrix} \qquad T_d = \begin{bmatrix} 1.9973 & .5758 \\ 0 & 2.0487 \end{bmatrix}$$

so that X_Z becomes

$$X_Z = \begin{bmatrix} 0.4885 + 0.5370\mathrm{i} & 0.2238 - 0.0183\mathrm{i} \\ -0.2575 - 0.1288\mathrm{i} & 0.2271 + 0.3183\mathrm{i} \end{bmatrix}$$

The matrix $Z = X_Z J_Z X_Z^{-1}$ becomes

$$Z = \begin{bmatrix} -1.3979 - 1.1599\mathrm{i} & -0.7902 - 0.7378\mathrm{i} \\ -0.1160 - 0.7377\mathrm{i} & -0.6021 - 2.8401\mathrm{i} \end{bmatrix}$$

so

$$\widetilde{K} = Z^* Z = \begin{bmatrix} 3.8573 + 0.0\mathrm{i} & 4.1255 + 0.0\mathrm{i} \\ 4.1255 - 0.0\mathrm{i} & 9.5973 \end{bmatrix}$$

$$\widetilde{D} = -Z - Z^* = \begin{bmatrix} 2.7958 & 0.9062 + 0.0\mathrm{i} \\ 0.9062 + 0.0 & 1.2042 \end{bmatrix}$$

Thus, if the set desired eigenvalues are chosen to be complex with negative real parts, and the eigenvectors are complex valued the inverse algorithm presented here will produce symmetric and positive definite mass normalized damping and stiffness matrices.

Note regarding the choice of eigenvectors

If conditions (3.4) and (3.5) are fulfilled, then the chosen eigenvectors must be in the subspace spanned by the columns of Z. This subspace is of dimension n which is equal to the *rank* (Z). If we choose an eigenvector which lies precisely in the subspace spanned by columns of *Z***,** in that case the proposed method will preserve both eigenvalues and eigenvectors (the matrix $L=I$, Eq.3.8). In general, however, a desired $L \neq I$ eigenvector will not reside in the prescribed subspace and hence the matrix. In that case the eigenvectors will not be preserved.

For the case of $L \neq I$ it is possible to find a choice for an achievable eigenvector. This best possible eigenvector is the projection of the desired eigenvector onto the subspace spanned by the columns of Z.

CONCLUSIONS

This manuscript presents a solution to the inverse vibration problem for the case where the desired coefficient matrices are symmetric, and the resulting system contains the desired or specified eigenvalues and eigenvectors. This is an improvement over symmetric inverse problem solutions which do not preserve the eigenvectors. This is useful for vibrating systems where it is known in advance that the system described by the equations of motion should be symmetric. Furthermore, if the real part of the eigenvalues are all negative, the resulting inverse solution produces positive definite matrices. In this paper the condition (2.5) for given spectral and modal properties is defined. When this condition is fulfilled then inverse formula (2.6) or alternately Eqs. (2.8) and (2.9) determine real symmetric matrices for linear lumped parameter nonconservative systems. The synthesized system must be of simple structure (i.e., diagonal spectral matrix) and each mode of the system is underdamped. The system identified by this inverse vibration problem will have proportional damping if the eigenvectors are chosen to all be real. In general, however, nonproportional damping results.

The weak point of the proposed method is clearly the choice of the orthogonal matrix C in step b. This choice is somewhat arbitrary and little guidance is provided by the theory. The numerical methods work well for any size of problem, except that as the order becomes larger the choice of C becomes more illusive. Our approach is to choose a simple form of C, such as the identity matrix, keeping in mind that diagonal C will produce a proportionally damped system. Likewise, a nondiagonal C will produce a nonproportionally damped system with complex mode shapes. A sparse matrix is used in the example simply because it is the first level of complexity past a diagonal matrix. The nature of the matrix C, and precise methods for constructing C form the topics of future research.

REFERENCES

[1] Gohberg, I. et al.: Matrix Polynomials. Academic Press, 1982.

[2] Lancaster, P., Maroulas, J.: Inverse Eigenvalue Problems for Damped Vibrating Systems, Journal of Mathematical Analysis and Application, Vol.123, No.1, 1987, pp.238-261.

[3] Starek, L., Inman, D.J.: On the Inverse Vibration Problem with Rigid Body Modes", ASME Journal of Applied Mechanics, Vol.58, 1991, pp.1101-1104.

[4] Starek, L., Inman, D.J., and Kress A.: A Symmetric Inverse Vibration Problem", ASME Journal of Vibration and Acoustics, Vol.114, 1992, pp.564-568.

[5] Starek, L., Inman, D.J.: A Symmetric Inverse Vibration Problem for Non-proportional, Underdamped System. ASME Journal of Applied Mechanics, Vol.64, No.3, 1997, pp. 601-605.