THERMAL STRESSES IN THE WALL OF A PIPE OF BIOT NUMBER BI>0 UNDER CONDITIONS OF CO-CURRENT COOLING USING A FLUID MEDIUM

Pavel ÉLESZTŐS[1] – Ladislav ÉCSI[1]
Eniko T. ENIKOV[2] – András SZEKERES[3]

[1] Slovak University of Technology of Bratislava, SK, pavel.elesztos@stuba.sk
[2] University of Arizona, USA
[3] TU Budapest Muegyetem 5, HU

ABSTRACT

In this paper an analytical solution of a stress field generated by a non-stationary temperature field of an infinite long thick-walled pipe under conditions of co-current cooling using a fluid medium is presented. The analytical solution of the temperature field with some simplification of thermal process conditions has been found in the form of an infinite series [1]. Utilizing the derived formulas, the stress field of the body can be determined. The solution is applicable to all technological processes that employ heat exchange.

Keywords: thermal-stresses, co-current cooling/heating, granular material, drying, extrusion

INTRODUCTION

Thermal stresses generated by a non-stationary temperature field of a body play an important role in all technological processes that employ heat exchange. They directly affect the quality of the processed products as well as the costs of their production. The stress field of the body is usually determined by a sequentially coupled thermal-structural analysis [2], [3]. In this paper an analytical solution of a thick-walled pipe of infinite length under conditions of co-current cooling/heating is presented, assuming some simplifications of the thermal process conditions [1].

TEMPERATURE FIELD OF THE PIPE

When determining the temperature field of the tick-walled pipe/cylinder the following assumptions were made. The processed extruded pipe, outer radius R and inner radius R_1, (Figure 1) of an ideal cylindrical shape enters into the calculation at the origin of the coordinate system. The body of the pipe is uniformly heated and has an initial temperature of T_{s0}. There is a perfectly isolated co-axial cylindrical chamber around the pipe, where the co-current flow of the fluid medium takes place with one end of the chamber located at the origin of the coordinate system. The coolant‘s initial temperature is T_{f0} at the origin of the coordinate system and it is in direct contact with the extruded body of the pipe. Both the pipe and the fluid medium move linearly in an axial direction along the axis of the coordinate system, where due to ideal mixing the later temperature is constant at all points of any cross section of the chamber. No heat generation rate per unit volume is considered inside the solid phase and all thermo-mechanical properties of the pipe material and the coolant, such as c_f, c_s, λ_f, λ_s, the heat transfer coefficient α, the mass flow of the fluid M_f and the solid phase M_s are assumed to be constant in the analysis.

MATHEMATICAL FORMULATION AND SOLUTION OF THE PROBLEM

Using cylindrical coordinates, the conservation of energy inside the body of the pipe during the cooling/heating process can be expressed with the Fourier-Kirchhoff equation:

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \qquad (1)$$

At the beginning of the analysis the solid and the fluid phase are uniformly heated using different temperatures. Then the initial conditions take the forms:

$$t = 0,\ T_s = T_{s0},\ T_f = T_{f0}. \qquad (2)$$

Assuming that the inner surface of the pipe is perfectly insulated, the boundary condition on the inner radius can be written as

$$\left[\frac{\partial T_s}{\partial r}\right]_{r=R_1} = 0\,. \qquad (3)$$

Considering the heat transfer between the solid and the fluid, the boundary condition on the outer radius can be [4] expressed as

$$\alpha\left[T_f - (T_s)_{r=R}\right] = -\lambda_s\left[\frac{\partial T_s}{\partial r}\right]_{r=R}, \qquad (4)$$

where the surface temperature is also denoted as: $(T_s)_{r=R} = T_{sp}$. After introducing the average calorimetric temperature of the solid phase

$$T_{sc} = \frac{1}{1-\left(\frac{R_1}{R}\right)^2}\int_{\frac{R_1}{R}}^{1} 2\frac{r}{R}T_s\frac{dr}{R} = \frac{2}{\left(1-\rho_1^2\right)}\int_{\rho_1}^{1}\rho T_s d\rho\,, \qquad (5)$$

the conservation of energy between the pipe and the fluid under co-current cooling/heating can be expressed as

$$M_s c_s (T_{sc} - T_{s0}) = M_f c_f (T_{f0} - T_f). \qquad (6)$$

Let~~'s~~ us introduce the following dimensionless variables:

$Bi = \frac{\alpha R}{\lambda_s}$ Biot number,

$Fo = \frac{at}{R^2}$ Fourier number,

$m = \frac{M_s c_s}{M_f c_f}$ thermal capacitance ratio of the contact phases,

$\rho = \frac{r}{R}$ dimensionless coordinate, $\rho \in (\rho_1;1)$, (7)

$\Theta_s = \frac{T_s - T_{s0}}{T_{f0} - T_{s0}}$ relative temperature difference of the solid phase,

$\Theta_{sc} = \frac{T_{sc} - T_{s0}}{T_{f0} - T_{s0}}$ average calorimetric relative temperature difference of the solid phase,

$\Theta_{sp} = \dfrac{T_{sp} - T_{s0}}{T_{f0} - T_{s0}}$ surface relative temperature difference of the solid phase,

$\Theta_f = \dfrac{T_f - T_{s0}}{T_{f0} - T_{s0}}$ relative temperature difference of the fluid phase,

Let us also launch the following substitutions

$$V_0(k_i\rho) = Y_1(k_i\rho_1)J_0(k_i\rho) - J_1(k_i\rho_1)Y_0(k_i\rho), \tag{8}$$

$$V_1(k_i\rho) = Y_1(k_i\rho_1)J_1(k_i\rho) - J_1(k_i\rho_1)Y_1(k_i\rho), \tag{9}$$

$$D_i = \frac{2k_i\left(1-\rho_1^2\right)\left(V_1(k_i) - \rho_1 V_1(k_i\rho_1)\right)}{4m\left(V_1(k_i) - \rho_1 V_1(k_i\rho_1)\right)^2 + \left(1-\rho_1^2\right)k_i^2\left[V_0^2(k_i) + V_1^2(k_i) - \rho_1^2\left(V_0^2(k_i\rho_1) + V_1^2(k_i\rho_1)\right)\right]}, \tag{10}$$

then the relative temperature difference, the average calorimetric relative temperature difference of the solid phase and the relative temperature difference of the fluid phase can be determined using the equations below:

relative temperature difference of the solid phase:

$$\Theta_s = \frac{1}{1+m} - \sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} V_0(k_i\rho), \tag{11}$$

average calorimetric relative temperature difference of the solid phase:

$$\Theta_{sc} = \frac{1}{1+m} + \frac{2}{\left(1-\rho_1^2\right)} \sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \left[\frac{V_1(k_i) - \rho_1 V_1(k_i\rho_1)}{k_i}\right], \tag{12}$$

relative temperature difference of the fluid phase:

$$\Theta_f = \frac{1}{1+m} - \frac{2m}{\left(1-\rho_1^2\right)} \sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \left[\frac{V_1(k_i) - \rho_1 V_1(k_i\rho_1)}{k_i}\right], \tag{13}$$

where k_i for $i = 1, 2, ..., \infty$ are the roots of the following transcendental equation:

$$0 = -k_i^2 V_1(k_i) + \frac{2mBi}{\left(1-\rho_1^2\right)} V_1(k_i) + Bik_i V_0(k_i). \tag{14}$$

STRESS FIELD DETERMINATION OF THE THICK-WALLED CYLINDER/PIPE

In the determination of the thermal stresses we assumed that the pipe material was isotropic and its loading generated by the non-stationary temperature field of the body was elastic. We also assumed that the temperature field of the pipe was one-dimensional and that the temperature distribution in the radial direction adequately determined the loading of the body [1]. The temperature gradient and the corresponding loading in axial direction were ignored.

The force equilibrium in both radial and axial directions using a cylindrical coordinate system can be formulated as (Fig. 1):

$$\frac{\partial(r\sigma_r)}{\partial r} - \sigma_t = \sigma_r - \sigma_t + r\frac{\partial\sigma_r}{\partial r} = 0 \tag{15}$$

$$r\frac{\partial\sigma_o}{\partial z}+\frac{\partial(r\tau_t)}{\partial r}=0 \quad . \tag{16}$$

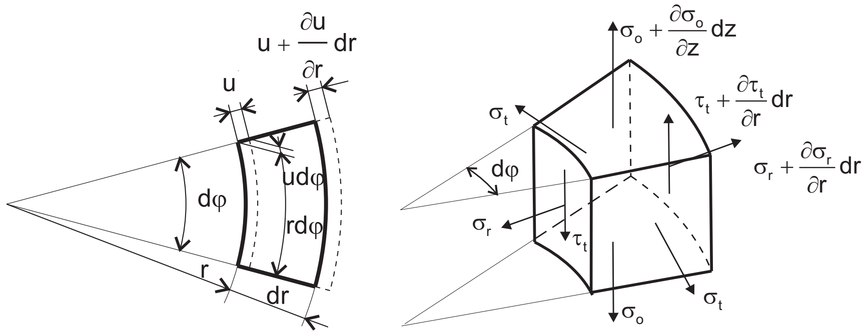

Fig. 1 *Displacement and stress components of the pipe element in cylindrical coordinates.*

Neglecting the shear component of the Cauchy stress tensor in equation (16) the force equilibrium in axial direction takes the following form

$$\frac{\partial\sigma_o}{\partial z}=0 \quad , \qquad \varepsilon_o=\text{const.}.$$

Combining eqns. (15) and (16) we then have

$$\frac{\sigma_t-\sigma_r}{r}=\frac{\partial\sigma_r}{\partial r} \quad .$$

(17) The one-dimensional temperature field defined earlier generates a spatial stress field (Fig. 2). After combining the radial, tangential and axial strain components,

$$\varepsilon_r=\frac{\partial u}{\partial r}\ , \quad \varepsilon_t=\frac{u}{r}\ , \quad \varepsilon_o=\frac{\partial w}{\partial z}\ , \tag{18}$$

the elastic constitutive equations of the corresponding components of the Cauchy stress tensor and equation (17)

$$\sigma_r=2G\left[\varepsilon_r+\frac{\upsilon}{1-2\upsilon}\left(\varepsilon_r+\varepsilon_t+\varepsilon_o\right)\right]-\frac{E}{1-2\upsilon}\vartheta\Delta T\Theta_s, \tag{19}$$

$$\sigma_t=2G\left[\varepsilon_t+\frac{\upsilon}{1-2\upsilon}\left(\varepsilon_r+\varepsilon_t+\varepsilon_o\right)\right]-\frac{E}{1-2\upsilon}\vartheta\Delta T\Theta_s, \tag{20}$$

$$\sigma_o=2G\left[\varepsilon_o+\frac{\upsilon}{1-2\upsilon}\left(\varepsilon_r+\varepsilon_t+\varepsilon_o\right)\right]-\frac{E}{1-2\upsilon}\vartheta\Delta T\Theta_s\ , \tag{21}$$

we arrive at the following partial differential equation

$$\frac{\partial\sigma_r}{\partial r}=2G\left[\left[\frac{1-\upsilon}{1-2\upsilon}\frac{\partial\varepsilon_r}{\partial r}+\frac{\upsilon}{1-2\upsilon}\left(\frac{\partial\varepsilon_r}{\partial r}+\frac{\partial\varepsilon_z}{\partial r}\right)\right]-\frac{\vartheta}{1-2\upsilon}\frac{E}{2G}\Delta T\frac{\partial\Theta_s}{\partial r}\right]. \tag{22}$$

Similarly, as in the case of the temperature field, dimensionless Cauchy stress tensor components and boundary conditions can be defined in terms of Fourier, Biot numbers and other dimensionless variables, such as dimensionless displacements, dimensionless coordinates and the thermal capacitance ratio of contact phases. Using the definitions (7)-(13) equation (18) can be rewritten as,

$$\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\frac{\partial\Theta_s}{\partial r}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}, \tag{23}$$

which after introducing the dimensionless radial displacement ξ and the dimensionless radial coordinate ρ, can be expressed with the following non-homogenous second order linear differential equation

$$\frac{\partial^2\xi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\xi}{\partial\rho}-\frac{\xi}{\rho^2}=\frac{1+\nu}{1-\nu}\vartheta\Delta T\frac{\partial\Theta_s}{\partial\rho} \quad . \tag{24}$$

The general solution of (24) using variation of parameters takes the following form

$$\xi=C_1\rho+\frac{C_2}{\rho}+\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\frac{1}{\rho}\int\rho\Theta_s d\rho\right] \quad . \tag{25}$$

After back substitution for the strain values we have

$$\varepsilon_r=\frac{d\xi}{d\rho}=C_1-\frac{C_2}{\rho^2}-\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\frac{1}{\rho^2}\int\rho\Theta_s d\rho-\Theta_s\right], \tag{26}$$

$$\varepsilon_t=\frac{\xi}{\rho}=C_1+\frac{C_2}{\rho^2}+\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\frac{1}{\rho^2}\int\rho\Theta_s d\rho\right] \quad . \tag{27}$$

Similarly, the constitutive relations defined by eqns. (19)-(21) can be combined in the forms:

$$\sigma_r=2G\left[\frac{1}{1-2\upsilon}C_1-\frac{C_2}{\rho^2}-\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\frac{1}{\rho^2}\int\rho\Theta_s d\rho\right]+\frac{\upsilon}{1-2\upsilon}\varepsilon_z\right], \tag{28}$$

$$\sigma_t=2G\left[C_1\frac{1}{1-2\upsilon}+\frac{C_2}{\rho^2}+\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\frac{1}{\rho^2}\int\rho\Theta_s d\rho-\Theta_s\right]+\frac{\upsilon}{1-2\upsilon}\varepsilon_z\right], \tag{29}$$

$$\sigma_z=2G\left[C_1\frac{2\upsilon}{1-2\upsilon}-\frac{1+\upsilon}{1-\upsilon}\vartheta\Delta T\left[\Theta_s\right]+\frac{1-\upsilon}{1-2\upsilon}\varepsilon_z\right]. \tag{30}$$

In order to determine the missing integration constants, equations (28)-(30) have to be supplemented with boundary conditions that meet the following requirements:

The radial component of the Cauchy stress tensor must be zero on the inner and outer surface of the pipe. Then we have

$$\rho = \rho_1 \quad \sigma_r = 0$$
$$\rho = 1 \quad \sigma_r = 0 \,. \tag{31}$$

The third boundary condition comes from the static force equilibrium in axial direction

$$\int_{\rho_1}^{1} \sigma_o \rho d\rho = 0 \quad . \tag{32}$$

After solving the aforementioned system of equations, we arrive at the following formulas for the Cauchy stress components

$$\sigma_r = \frac{E}{1-\upsilon} \vartheta \Delta T \left[\frac{\rho^2 - \rho_1^2}{2\rho^2} \Theta_{sc} - \frac{1}{\rho^2} \int \rho \Theta_s d\rho \right], \tag{33}$$

$$\sigma_t = \frac{E}{1-\upsilon} \vartheta \Delta T \left[\frac{\rho^2 + \rho_1^2}{2\rho^2} \Theta_{sc} + \frac{1}{\rho^2} \int \rho \Theta_s d\rho - \Theta_s \right] \quad , \tag{34}$$

$$\sigma_o = \frac{E}{1-\upsilon} \vartheta \Delta T \left[\Theta_{sc} - \Theta_s \right], \tag{35}$$

which can also be expressed in dimensionless forms using equations (7)-(11) and (12)

$$\psi_r = \frac{\sigma_r}{E} = \frac{1}{1-\upsilon} \vartheta \Delta T \left[\begin{array}{l} (-1) \dfrac{\rho^2 - \rho_1^2}{\left(1-\rho_1^2\right)} \dfrac{1}{\rho^2} \displaystyle\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \dfrac{V_1(k_i) - \rho_1 V_1(k_i \rho_1)}{k_i} \\ + \dfrac{1}{\rho^2} \displaystyle\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \dfrac{\rho V_1(k_i \rho) - \rho_1 V_1(k_i \rho_1)}{k_i} \end{array} \right], \tag{36}$$

$$\Psi_t = \frac{\sigma_r}{E} = \frac{1}{1-\upsilon} \vartheta \Delta T \left[\begin{array}{l} \dfrac{\rho^2 - \rho_1^2}{2\rho^2} \dfrac{1}{1+m} + (-1) \dfrac{\rho^2 + \rho_1^2}{\left(1-\rho_1^2\right)} \dfrac{1}{\rho^2} \displaystyle\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \dfrac{V_1(k_i) - \rho_1 V_1(k_i \rho_1)}{k_i} - \\ - \dfrac{1}{\rho^2} \displaystyle\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \dfrac{\rho V_1(k_i \rho) - \rho_1 V_1(k_i \rho_1)}{k_i} + \displaystyle\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} V_0(k_i \rho) \end{array} \right], \tag{37}$$

$$\Psi_o = \frac{\sigma_z}{E} = \frac{1}{1-\upsilon} \vartheta \Delta T \left[\sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} V_0(k_i \rho) - \frac{2}{\left(1-\rho_1^2\right)} \sum_{i=1}^{\infty} D_i e^{-k_i^2 Fo} \frac{V_1(k_i) - \rho_1 V_1(k_i \rho_1)}{k_i} \right]. \tag{38}$$

The parameters below were used in plotting the dimensionless Cauchy stress components:

$Bi = 2$

$m = 1$

$\rho_1 = 0.5$

$\Delta T = T_{f0} - T_{s0} = 100$ °C

$\vartheta = 1 \,.\, 10^{-5}$ °C^{-1}

$E = 10^6$ MPa

$\upsilon = 0.3$.

The problem was solved with the Mathemathica 5.1 commercial package using the first 30 roots of the transcendent equation (15) in the solution. Figure 2 shows the relative temperature difference of the solid phase versus Fourier number on the inner, outer and mean radius of the pipe ρ=0.75, as well as the relative temperature difference of the fluid phase. The graph implies that as the time passes ~~by~~, the temperature of the solid and the fluid phases equalize, and eventually approach a value at the thermal equilibrium state. Figure 3 depicts the dimensionless axial stress

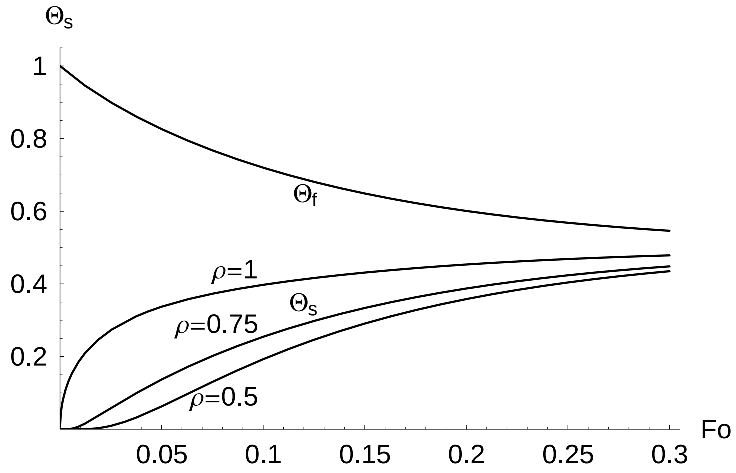

Fig. 2 *Relative temperature difference of the solid/fluid phase versus Fourier number on the inner, outer and mean radius of the pipe.*

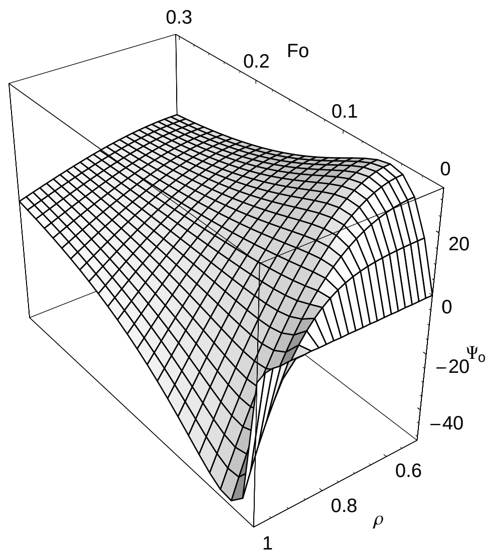

Fig. 3 *Dimensionless axial stress time history in term of Fo number over the pipe wall.*

time history in term of Fo number over the pipe wall $\rho \in (0.5;\ 1)$. Equation (32) implies that there is a significant increase in the axial stress value with opposite signs on the inner and outer surfaces at the beginning of the analysis. As the time passes ~~by~~, the temperature difference between the solid and the fluid phase vanishes. Figure 4 depicts the dimensionless axial Cauchy stress component versus Fourier number curves on the inner, outer and mean radius of the pipe. The same graph is drawn in figure 5 using the tangential component of the Cauchy stress tensor. The dimensionless radial stress time history in term of Fo number over the pipe wall $\rho \in (0.5;\ 1)$ can be seen in figure 6.

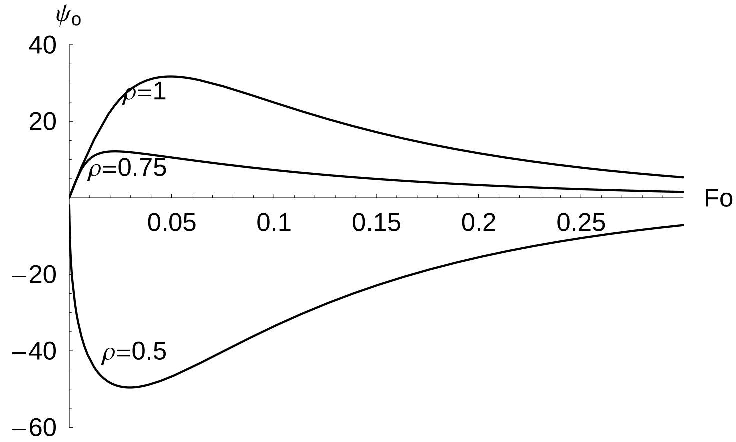

Fig. 4 *Dimensionless axial Cauchy stress component versus Fourier number curves* for *ρ=0.5, 0.75, 1.0* .

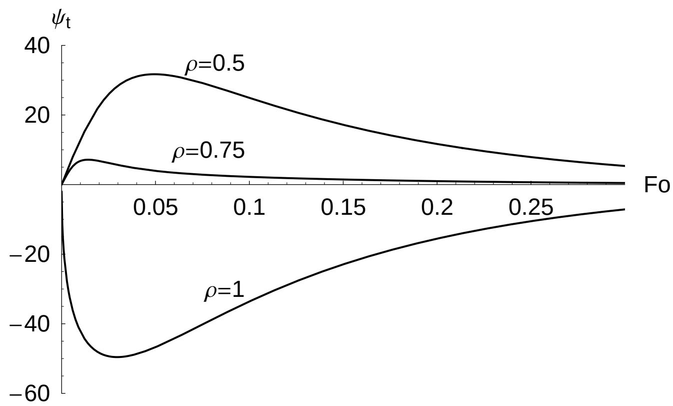

Fig. 5 *Dimensionless tangential Cauchy stress component versus Fourier number curves for ρ=0.5, 0.75, 1.0* .

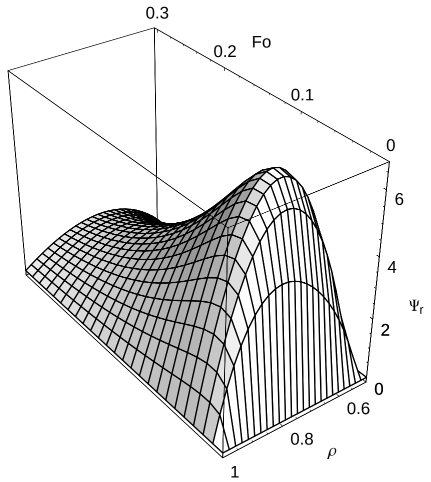

Fig. 6 *Dimensionless radial stress time history in term of Fo number over the pipe wall.*

CONCLUSION

In the presented paper thermal stresses generated by the non-stationary temperature field of an infinite cylinder at co-current cooling/heating were investigated. The analytical solution of the temperature field was determined earlier [1] in a dimensionless form, using dimensionless temperatures, deflections, coordinates, the Fourier number, the Biot number and the temperature capacitance ratio of the contact phases. The same dimensionless variables were employed in the dimensionless Cauchy stress component determination.

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